



**Georgian
American
University
(GAU)**

**Applications of
Stochastic Processes
and Mathematical
Statistics to Financial
Economics and
Social Sciences IV**

**Tbilisi Science and
Innovation Festival
2019**

Conference Materials,
25-26 September, 2019,
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**Business School
Business Research
Scientific Center**

Georgian American University (GAU)

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SYMBOLIC PERCEPTION IN WORK-RELATED EXPERIENCE

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ABSTRACT

The paper presents the ideas related to organizational initiation and social perception in work-related experience. Symbolic perception is the process of interpretation and transforming the outside experience in metaphors, images and signs. Symbolic perception is the important part of the organizational initiation process in two dimensions – transition to new stage in career (new job, first job, new organization), and in transition to new state in career (new post, promotion, job enrichment). There are presented the first research results and analysis of the study, that will be continued further.

Key words: *organizational anthropology, social initiation, symbolic perception.*

“The *duende*...Where is the *duende*?
Through the empty archway a wind of the spirit enters,
blowing insistently over the heads of the dead,
in search of new landscapes and unknown accents:
a wind with the odor of a child’s saliva,
crushed grass, and medusa’s veil,
announcing the endless baptism of freshly created things”.

Federico García Lorca

Perception is the way of interpreting reality, and the significant factor with high impact on behavior. Perception is based on the filters and psychological programs one has. Symbol – is dual spiritual signal, where the form and the meaning exist in the interactive unity. Brain generates the signals faster than we realize them, creating a connection of neurons in a

response to the external situation and internal abilities. One formulates the behavioral strategy before realizes these signals, and before consciously understands the idea, thought or action. The time gap in perception can be described in the symbolic interactionist perspective. Symbol gets its power in interpretative situation, when there is a transition from the abstract to the real and back. A thing or phenomenon can become a symbol in interpretation. Thinking in images, metaphors and signs is realized in symbol when situation needs interpretation, and perception in slight difference with sensation is interpretative context. Symbolic perception is the space of imperative instinct and intention, as it expresses the inclination towards the object and main motivation for interpretation. In some sense, perception is filtered interpretation, which is, along with individual factors, significant foundation of behavior.

In social sciences there exists the idea of 'social transition' of the individual or group of people from one social state to another (from example from childhood into adulthood, from student into the employee etc.). In different societies there are special transition rituals, specifically initiations. In most of the initiations there is an indivisible principle – to go through the death and to transit into a new state. M. Eliade offered a bit different view, based not on the idea who one was and who one becomes, but on the process of initiation, what happens during: 1. age rituals – access to new knowledge, and 2. specialized initiations, where there is transformation of the human state (like to become a guru, master). Eliade states that learning the language of initiation means finding in the inevitable struggles of our own lives "certain types of real ordeals...the spiritual crises, the solitude and despair through which every human being must pass in order to attain to a responsible, genuine and creative life (Eliade, 2012, P.10). Organizational anthropology is not going far away from the content of these two processes: 1. when a person starts career, or moves to a new organization; and 2. when an employee is promoted, gets new post, like manager, team leader, vice president etc. Both are transition processes, constituting organizational initiation. In some organizations it aligned with a ritual, in others it is a part of orientation training programs.

Work experience behavioral patterns are realized in symbolic perception, as the moment of interpretation. It is the first day at work, or the first job, the new position, - this is the exact moment of symbolic interaction with the organization, as one experiences the moment of interpretation, transition of meanings and real social initiation experience. Symbol is the thread towards the Truth, and a prism of Universe cognition, as stated by K. Jaspers, Existence is comprehended in a symbol and by the symbol (Jaspers, 1991). This is the dual energy of symbolic perception. G. W. F. Hegel gives a dialectic interpretation of this phenomenon, as in symbols the truth because of the sensing element is yet vague and concealed, it is shown to the consciousness only in the form of a thought, its meaning is the thought itself (Hegel, 1970, P. 292). In the process of symbolic perception in organizational initiation, there happens the moment of realizing the action, idea, value. In symbolic tension it becomes real experience.

The study explores the symbolic perception of the work-related experience, asking respondents to describe their 'organizational initiation' in the fairy-tale form. The study is not completed yet, it needs some additional data and validation (see the Survey Form following).

Survey Form

Dear Friends and Colleagues, please, fill in this form for my research of triggers and imprinting in organizational life. Recall your first experience when you joined the company you work for now, or worked for before. If you would describe this experience in the fairy-tale format, please, answer the following questions.

1. Who is the main hero of your fairy-tale story? Describe the main hero (traits, behaviors, style) *
2. Who are the other heroes in your fairy-tale story? Describe their traits, behaviors, style. *
3. What happened with the main hero in the beginning? *
4. What made the main hero to transform him/her or his/her style? *
5. What is the end of your fairy-tale story? *
6. What is the morale of your fairy-tale story? *

7. What are the most important values you hold if you would be the main hero in your fairy-tale story? *

8. What was your position / job you associated this fairy-tale with? *

Thank you for your imagination and creative impact!

The answers varied in their imagination and context interpretation, as well reflected the nature of the socialization and perception experience respondents have or had in their orientation period in the workplace. In the first question about the hero, about 80% of respondents underlined the courage and at the same time freight of the main hero, as well as openness and readiness for learning. About 20% chose as the main hero a strong figure (associated with the Leader in the organization), e.g. Father or Beast (from 'Beauty and the Beast' tale), describing the hero as strong, rich, successful and protecting. These are the examples how respondent see themselves in their symbolic perception of work-related experience:

- young brother Biorn
- a little girl
- be me, a very shy person who was on the verge of a new life chapter
- my father
- Beast
- beautiful and sensitive princess
- Alice in Wonderland
- катушка шёлковых белых ниток
- тонкая гибкая и непреклонная березка

The second question included other characters of the fairy-tale story with their behavioural patterns and styles description. Respondents demonstrated the reverse in this question, i.e. those who have as the main hero someone not confident enough yet, choose as other characters confident and ready-to-help people, experienced, educated and calm. If the others were described as 'cruel', the main hero is looking for those who are 'helpful and supportive' (only about 10% of respondents). In one example, the hero and other heroes are not in a reverse, both are confident, calm and ready.

The third question was dedicated to the events in the beginning. Most of the answers metaphorically represent some problem, new issue to deal with, something not known or not lived through experience, e.g. 'the world is new', 'struggle, separation from family', 'curse', or something bad happened, 'the main hero's sister was taken to the deep forest by Hydra', etc. In all the beginnings there was a need of both hero and other characters who are more experienced, to participate in the saving campaign.

Question four had the most important meaning for the research, to see the reasons people find their transformations based on, like the symbolic transformational energy, more level of symbolic imprinting and symbolic triggers. Respondents include in this metaphoric challenge someone or something not met before in the story, e.g. family, new person, hero's partner, people like her, and others. About 20% of respondents underline that 'life' itself or situation itself made the hero to transform.

In the question five the respondents' endings vary. The hero and the partner defeated the 'monster and saved the girl', they solved the problem, and came out as real heroes. The hero is aware of who she is and what she wants, come back to real self and get the partner who accepts this real self. The hero overcame the moral obstacles. About 30% offer 'happy end', and 10% suggests a negative situation in the end, which strengthens the hero in what s/he is.

Questions six and seven were related to the morale and values of the fairy-tale stories. These two mostly copied each other, so the answers were in parallel, demonstrating the values and attitudes important for the respondents in the workplace, e.g. strength, unity, to be oneself and to accept oneself, follow your goals, stability, calmness, kindness, to become not a hero, but the other character in the fairy-tale story (to reverse). In the answers there are underlined the following significant values in symbolic perception of the work-related experience:

- fearlessness, family, skills
- to be the person you are
- stability and calmness
- kindness above all and being humble
- kindness, self-respect, do the job
- be totally like Belle
- core values, don't compromise on important things
- respective toward others
- never judge people by appearances, be flexible
- узор должен быть красивым, добрым
- вера в себя

In symbolic perception of work-related experience, transition process is reflected by the challenges happening with the hero, and the ways other characters of the story help him/her to win in the end. It is very close to the social initiation process, where there is a real or metaphoric challenge the person should go through, to win over oneself and over the situational factors. Moreover, with the values it is shown the motivation frame for the

employee, and what makes important meaning in the organization. The study will be continued, and by the elaborated criteria there would be developed recommendation for the orientation training programs for non-managerial position, and supportive coaching programs for managerial positions.

Symbolic perception as the interpretation process in organizational initiation, creates the space of dual experience, and understanding if the inner psychological factors that influence work-related experience, and the effectiveness of orientation programs.

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Service Employees' Performance Problematics

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ABSTRACT

Performance management is on the brink of a huge change. The old systems of the past, where performance goals were set once a year and re-examined at the year-end review, are beginning to die. Companies are recognizing that these systems generate a lot of work and aren't tied to employee engagement and performance. In fact, they often require employees and managers to set year-long goals that become stale in the months that pass between their actual review and their setting – they can also de-motivate employees, when they feel that the process is threatening or not relevant to them. This is why we're on the brink of a great change in performance management – making it modern, engaging, transparent – tying into many other changes in how employees are managed and exposed to learning in a digitally transformed world.

“Performance management is the continuous process of improving performance by setting individual and team goals which are aligned to the strategic goals of the organisation, planning performance to achieve the goals, reviewing and assessing progress, and developing the knowledge, skills and abilities of people.”

M. Armstrong

Performance management is the continuing process of communication between managers and employees, with the mutual goal of accomplishing the strategic goals of the company. It is the foundation for employee performance and engagement in any organization that wants its employees to reach their maximum potential and boost their productivity and success. The process- often combining both oral and written components, is happening throughout the year and usually results in an annual performance evaluation. This entire communication process involves defining clear specific expectations, setting objectives, establishing goals, providing ongoing feedback and examining results. Next generation performance management isn't about setting annual goals, but rather personalized goals for employees, changing as dynamically as the business does. This not only fits with the dynamic changes in the business environment but also gives employees real-time feedback on how they are performing, letting them self-correct as well as giving them a sense of accomplishment.

Overseeing performance and providing feedback is not an isolated event, focused in an annual performance review. It is an ongoing process that takes place throughout the year. The Performance Management process is a cycle, with discussions varying year-to-year based on changing objectives.

The cycle includes Planning, Checking-In, and Review.

- To begin the planning process, you and your employee review overall expectations, which includes collaborating on the development of performance objectives. Individual development goals are also updated. You then develop a performance plan that directs the employee's efforts toward achieving specific results to support organizational excellence and employee success.
- Goals and objectives are discussed throughout the year, during check-in meetings. This provides a framework to ensure employees achieve results through coaching and mutual feedback.
- At the end of the performance period, you review the employee's performance against expected objectives, as well as the means used and behaviors demonstrated in achieving those objectives. Together, you establish new objectives for the next performance period.

Most organizations already have a performance management system, but if you are planning to implement a performance management system in your organization or to improve the existing one, here's a list of common reasons for the failure of performance management system that you might need to avoid.

1. The Process Is Not Structured

One reason why performance management fails is that the process is not properly structured. It is not a one-time process and needs to be repeated more often. It is not possible if you don't have a well-designed structure for performance management. Because the main reason behind having a performance management process in an organization is to improve overall performance of the employees in the organization.

So, a well-structured performance management system helps the employees understand the organizational strategies better and work towards achieving their goals with more motivation.

2. Goals Aren't In The Picture

In many organizations, employee goals aren't in the picture when the performance review process is carried out. Sometimes performance management also fails because wrong goals are chosen to optimize the performance of the employees. In short, goals are important to keep employees motivated and perform better.

3. Lack Of Communication

Sometimes, managers aren't able to communicate what they expect from their employees. It is important for them to communicate properly with the employees about the benefits of performance management. Employees should be provided with necessary resources to improve themselves constantly and should know the importance of continuous learning and a performance management system.

4. Overemphasizing Recent Performances

This is one of the most common mistakes that managers/ HRs make. Performance management processes are plagued with various biases in general. Recency bias is one of them. This is an

unconscious bias since part of the problem can be attributed to memory and the way the mind makes associations. But it is a dangerous bias all the same. Very simply, it is because recency bias can make or break a performance review. So overemphasizing recent performance can result to failure of the performance management system.

5. Annual Performance Evaluation

Performance evaluation is valuable only when it is carried out on a regular basis. Annual performance review is not enough for any organization. Some managers claim that it consumes a major portion of their time. But if performance evaluation is carried out annually, employees have to wait for a year to give or receive feedback; which is not good for the organization's productivity.

So, annual performance evaluation is one reason why your performance management system could fail.

6. No Recognition/ Rewards

Appreciation and recognition are very important to keep your employees inspired and to drive productivity. A performance management system that doesn't include recognition and rewards for employee performance tends to fail more easily. It is important to keep track of employee performance and appreciate their good work.

Service performance in the public sector is a topic that receives increasing attention. Customer demands are changing rapidly as well as the services that they require. In the public sector, consumers participate in the service delivery process and services are provided from public stakeholder and the funding is mainly provided by public resources. Public services can refer to justice, fairness and equity, they are not only about efficiency and effectiveness.

Customer Service Qualifications

A suitable candidate for a career in customer service should have strong listening, communication and phone skills, and should excel at [problem-solving](#), documenting information, [resolving conflict](#), analyzing information, building relationships, and multi-tasking. Many customer service jobs are entry-level, with the potential for growth in supervisory and management positions.

Most customer service positions require a high school diploma and familiarity with computers. Many will provide on-the-job training. For management positions, a college degree or experience is required, and excellent [communication skills](#) and customer orientation are necessary. In a customer service position, you will interact with customers to provide information, handle complaints, and process orders.

1. **Client Relations Associate:** Client relations staff build and maintain relationships with their organization's most important customers. This is a hands-on position working with assigned clients to make sure that they are satisfied with the company's services and products. Your job is to ensure client satisfaction and to work with team members, other company departments, and external vendors to make sure customer needs are met.
 2. **Client Services Coordinator:** Client services coordinators are responsible for ensuring tasks are completed. They may handle order processing, setting up and coordinating services, tracking deliveries and installations, and resolving customer issues. This job requires solid phone skills and the ability to juggle several tasks simultaneously.
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1. **Customer Service Representative:** A customer service representative is the catch-all job title for many [different roles in customer service](#). Customer service representatives interact with customers to process orders, provide information about an organization's products and services, and resolve issues. Almost every organization provides some level of customer service, and jobs are plentiful.
 2. **Front Desk Associate or Receptionist:** If you have strong [interpersonal skills](#) and the ability to handle in-person and phone communications, a front desk position is a good job option. In a corporate setting, you will likely work a 40-hour week. A position at a hotel, resort, or other hospitality facility will probably require additional hours working nights, weekends, and holidays.

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Organizational Challenges of Talent Management in Private Sector

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Abstract

This paper explores talent management and challenges connected to this topic in Georgian private sector companies. Private business sector in Georgia plays a critical role in country and economic development. For the given moment, our country doesn't have as much experience in making business as other developed countries. That's why it's not often flexible to implement newest managerial approaches. "Talent management" is a new term in the field of management. This is the approach and understanding, that an employee is the driving force for modern organization. It's different from the old approach which concerns only hiring staff, paying salary and in case of leave finding someone else. Instead, in the new reality, organization attracts and tries to be liked by the best candidates, takes care of them after hiring and tries to flourish and nourish their talents, makes everything to retain the talented people as long as possible, get the best results from them and respectively, gain competitive advantage on the market. During the observation on this approach in Georgia, several types of challenges and problems have emerged. These are the main topics discussed in this paper.

Key words: Talent, organization, management, challenges

Talent Management

The term “Talent Management” is new to the field of management. For the given moment, there is no one agreed definition of this term in scientific literature. Scientists are trying to differentiate talent management and human resource management from each other. However, the main difference from these two concepts still seems to be the approach that talent management discusses human resources as people with unique skills and abilities and uses instruments needed to attract and retain them in the company. The goal of talent management is to maximize the intrinsic resources and talents of employees and direct them towards the goals of the organization, which respectively, should affect the success of the company. This discussion shows us as well the strategic nature of talent management. Strategic mission for every organization is to attain specific goals, which can't be realized without employee involvement and their strong professional skills. From this point of view, we can say that managing talents properly inside the organization can be critically important even, on the level of strategic planning.

In order to increase the effectiveness of a firm several resources can be used to achieve this, which includes money, men and machines. Of these resources, the most important of them all is the people (Kehinde, 2012).

In 1997 McKinsey & Co started a large-scale research which showed that many organizations were involved in the war on talent, where they were fighting to obtain the best employees and use them to increase or maintain their competitive advantage.

During the research, McKinsey & Co studied 77 large organizations from various industries in order to describe the extent of the war for talent. They surveyed about 400 corporate officers and 6000 executives and the findings were verified and supported by the case studies of 20 companies which were regarded to have a good talent profile.

The surveys resulted that companies were experiencing a considerable shortage of the highly competent workforce. From surveyed 6000 executive managers only 23 percent of them indicated that their companies attract the best of the talent pool, out of which just about 10 percent were retained in the organization (Invosights, Just Another WordPress Site).

This same research conducted in later years, did not show any better results. Nowadays, world managers, including Georgian ones face the challenge of staffing their companies with the best employees. The next part of this paper discusses the issues which hinders managers to attain desired results in effective talent management. Issues and problems are discussed in regard of Georgian business environment, however identical issues can be faced in business realities of other countries.

Organizational Challenges of Talent Management

The first issue which is vague in this field concerns the decision about, group of people who should be considered as talents and managed using the recommended talent management tools. In this issue managers are split into two. One point of view is, that every employee can be potentially talented and respectively, all the managing methods should be used to reveal and develop their skills and abilities, another point of view states, that employees holding key positions in an organization should be regarded as talents and all the management tools and methods should be directed towards them.

Let's discuss both approaches and challenges connected to them. At first point, let's suppose that potentially, with proper management every employee can be developed and become important talent for an organization, but manager considers that it's impossible. In this case, beliefs of the manager is harmful for company, because there's a high risk of losing many potentially strong employees, because manager couldn't notice the best talents of person in the begging and considers that it's impossible to develop them later. From another point of view, let's suppose that potentially, it's impossible in reality to develop the skills and talents needed by the company in every employee, at the same time manager considers, that it's possible. In this case the decision of manager, is still harmful for company, because there's a high risk of losing the precious time and resources of an organization. Manager hires person, who can't develop and display their talents in the future.

Talent management often is not a whole process. Usually, it should begin with candidate attraction/selection and finish after their leave. Often, after hiring talented people, company's managers consider that they have done the main task as long as hired professional employee. However, in reality the process is not over, it continues and becomes more important during the period of employee work in company. This is the practice of developing employee talents continuously, fit them to the company needs and most importantly, retain them inside the firm.

Let's suppose, managers have acknowledged all the above discussed points. As well, they acknowledge that the whole process is strongly connected to a number of expenses and time. Here comes the time to make decision: who should be the beneficiaries of the whole process of talent management? Whole workforce or key employees? Mostly, second alternative is chosen. However, here exists the risk of losing potentially talented employee, who would be beneficial for company, even to substitute key positioned employees in the future. At the same time, it's important to consider the amount of expenses and time incurred in case for choosing first alternative.

While discussing issues and problems of talent management, it's necessary to mention employee retention. Many organizations do not have any organizational plan to retain employees, not even for talented ones. They don't have diverse programs. For example, in Georgia employee benefits usually include insurance, corporate phone/communication service, excursion/corporate events financing, basic trainings. Everyone would agree, that this package is quite standard and basic in fact, for every small and medium organization and doesn't represent important motivational tool to retain the best employees and stop them from considering the offers of competitors. Of course, the field of business and the nature of position held by particular employee is important in this case.

In Georgian business environment, word "Talent" often is not defined by managers. Here I mean that, it's not considered to be technical term on the professional level. "Talented", "Skilled" these words can be used to characterize particular person, but there is no practice of frame, method or technique to manage them as such employees.

Amidst all this, it's necessary to mention that there's a considerable shortage of professional workforce on the labor market. Organizations are all the time trying to take away from competitors the best talents, people who have been identified as experts and professionals in their field. Question is, why do we have such problems? Is this because of poor education system, which can't supply the market with skilled people? Of course, this can be the case, but not only can it result such shortage. We can as well assume, that companies do not contribute to this. They don't train, develop and create highly qualified employees that would help first of all them, and as well increase number of professionals in Georgia.

Finally, we can conclude that managing talents is one of the most important factors of management. Involvement into this practice, should be not only from the HR or talent managers, but from the top management of the company. Managing talent is strategically important for company, this is strategic decision. It can benefit not only particular company, but as well contribute to increase the number of professionals on the labor market of the country. Which in turn, will help emergence and increase of new businesses, innovation, research and science in the country.

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<https://invosights.wordpress.com/2011/11/23/war-for-talent-the-mckinsey-survey/>

Bilateral Tariffs and Exchange Rate Under International Competition

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Abstract

This paper explores the gain maximization problem of two nations engaging in non-cooperative bilateral trade. Volume of commodities exchanged determines the demand each nation has over the counter party's currency. However, each nation can manipulate this quantity by imposing a tariff on imported commodities. We determine the optimal tariffs to be imposed under the Nash's sense.

1 The Model

Let us assume two nations exchange N different commodities for which the demand and prices are known. For the domestic and foreign nations, annual demand and corresponding prices measured in national currency are $d_1, \dots, d_N, p_1, \dots, p_N$ and $d_1^*, \dots, d_N^*, p_1^*, \dots, p_N^*$ respectively. If we take x as an exchange rate of a unit of foreign currency in terms of domestic currency units, then the domestic and foreign nations' demand for foreign currency are given by

$$D(x) := \frac{1}{C_N} \sum_{k=1}^N \bar{E} \left(p_k^* d_k, \frac{p_k}{p_k^*} > x \right) \quad (1)$$

and

$$D^*(x) := \frac{1}{C_N^*} \sum_{k=1}^N \bar{E} \left(p_k d_k^*, \frac{p_k}{p_k^*} < x \right) \quad (2)$$

respectively, where $C_N = \sum_{k=1}^N \bar{E}(p_k^* d_k)$, $C_N^* = \sum_{k=1}^N \bar{E}(p_k d_k^*)$ and $\bar{E}$ is the mathematical expectation under $\bar{P}$ on a probability space $(\bar{\Omega}, \bar{F}, \bar{\Omega})$. If we introduce the extended probability space (Ω, F, P) , where

$$\Omega = \bar{\Omega} \times \{1, \dots, N\}, \quad P(A, k) = \frac{1}{N} \bar{P}(A), \quad A \in \bar{F}$$

and define random variables p, p^*, d, d^* by

$$\begin{aligned} p(\omega, k) &= p_k(\omega), \quad p^*(\omega, k) = p_k^*(\omega), \\ d(\omega, k) &= d_k(\omega), \quad d^*(\omega, k) = d_k^*(\omega), \end{aligned}$$

then (1),(2) can be rewritten as probability distribution functions

$$D(x) = E\left(p^* d, \frac{p}{p^*} > x\right), \quad D^*(x) = E\left(p d^*, \frac{p}{p^*} < x\right). \quad (3)$$

which indicate that the domestic nation will import the commodity if $\frac{p}{p^*} > x$ and the foreign nation will import if $\frac{p}{p^*} < x$. Since x is the value of a unit of foreign currency in terms of the domestic currency units, increasing the exchange rate makes foreign commodities more expensive for the domestic nation and the domestic commodities less expensive for the foreign nation. Therefore, D is a decreasing function of x and D^* is an increasing function of x . These functions have the following properties

$$D(0) = 1, D(\infty) = 0, D^*(0) = 0, D^*(\infty) = 1.$$

For an exchange rate x , solving the equation

$$xD(x) = D^*(x) \quad (4)$$

for x yields the equilibrium rate $x = e$. This equation determines the equilibrium exchange rate when both nations practice an unrestricted free trade policy. Left side of the equation is the foreign currency demand of a domestic nation and the right side is the foreign currency demand of a foreign nation, both measured in domestic currency units.

Now suppose the domestic and foreign governments impose the following tariffs on imported commodities: $1 - \theta$ and $1 - \theta^*$. Then the domestic nation will import the commodity if $\frac{p\theta}{p^*} > x$, and the foreign nation will import if $\frac{p^*\theta^*}{p} > \frac{1}{x}$. Taking tariffs into account, the demand functions (3) now become

$$D\left(\frac{x}{\theta}\right) = E\left(p^* d \mathbf{1}_{\{\theta p > x p^*\}}\right), \quad D^*(x \theta^*) = E\left(p d^* \mathbf{1}_{\{\theta^* p^* x > p\}}\right). \quad (5)$$

So the relation (4) is rewritten as

$$x D\left(\frac{x}{\theta}\right) = D^*(\theta^* x) \quad (6)$$

from which it is clear that the equilibrium exchange rate $x = e$ now depends on θ and θ^* . Equation (6) always has the solution $e = 0$, $\frac{1}{e} = 0$, which does not carry any useful economic sense. Such conditions would restrict the involvement of both nations in trade. To rule out this possibility, we claim $\frac{1}{M} \leq e \leq M$, for some large number M and $\frac{1}{M} \leq \theta \leq 1$, $\frac{1}{M} \leq \theta^* \leq 1$.

Since the ultimate goal of both nations is to set the tariffs unilaterally which will maximize their gain from trade, we have to find the Nash equilibrium point, the pair $(\hat{\theta}, \hat{\theta}^*)$. The gain functions of each nation are given by

$$\begin{aligned} G(e, \theta, \theta^*) &= E\left(p d, \frac{p}{p^*} < \frac{\theta}{e}\right) - E\left(p d^*, \frac{p}{p^*} > \frac{1}{e \theta^*}\right) \\ &= E\left(\frac{p}{p^*} \mathbf{1}_{(\frac{p}{p^*} > \frac{e}{\theta})} p^* d\right) - E\left(p d^* \mathbf{1}_{(\frac{p}{p^*} > \frac{e}{\theta})}\right) \\ &= - \int_{e/\theta}^{\infty} y D'(y) dy - D^*(\theta^* e), \end{aligned} \quad (7)$$

and

$$\begin{aligned} G^*(e, \theta, \theta^*) &= E(p^* d^*, \frac{p}{p^*} < \theta^* e) - E(p^* d, \frac{p}{p^*} > \frac{e}{\theta}) \\ &= \int_{\frac{1}{\theta^* e}}^{\infty} \frac{1}{y} D^{*'}\left(\frac{1}{y}\right) dy - D\left(\frac{e}{\theta}\right), \end{aligned} \quad (8)$$

respectively. Since the equilibrium exchange rate is the function of tariffs, we have $e = e(\theta, \theta^*)$. The Nash equilibrium point is found from the system of equations

$$\frac{\partial}{\partial \theta} G(e, \theta, \theta^*) = 0, \quad (9)$$

$$\frac{\partial}{\partial \theta^*} G^*(e, \theta, \theta^*) = 0 \quad (10)$$

Given the currency demand functions $D(x)$ and $D^*(x)$, solution to the system of equations (9),(10) leads to yet another system of equations

$$D\left(\frac{e}{\theta}\right) = \theta^*(1 - \theta)D^{*'}(\theta^*e), \quad (11)$$

$$D\left(\frac{e}{\theta}\right) = \frac{e}{\theta}(\theta^* - 1)D'\left(\frac{e}{\theta}\right) \quad (12)$$

At this point, if the demand functions for foreign currency of each nation are known, from (11) and (12) the Nash equilibrium pair $(\hat{\theta}, \hat{\theta}^*)$ can be found. Ultimately putting these values in (6) and solving for x will result in the equilibrium triple $(\hat{e}, \hat{\theta}, \hat{\theta}^*)$. Hence the triple satisfy

$$\hat{e}D\left(\frac{\hat{e}}{\hat{\theta}}\right) = D^*(\hat{\theta}^*), \quad (13)$$

$$D\left(\frac{\hat{e}}{\hat{\theta}}\right) = \hat{\theta}^*(1 - \hat{\theta})D^{*'}(\hat{\theta}^*\hat{e}), \quad (14)$$

$$D\left(\frac{\hat{e}}{\hat{\theta}}\right) = \frac{\hat{e}}{\hat{\theta}}(\hat{\theta}^* - 1)D'\left(\frac{\hat{e}}{\hat{\theta}}\right). \quad (15)$$

Obviously, one should check whether the extremum points given by (11) and (12) are really maximums. Differentiating the derivatives of the gain functions once again and checking the signs for the equilibrium points serves this purpose. So the following inequalities must hold

$$\frac{\partial^2}{\partial \theta^2} G(\hat{e}, \hat{\theta}, \hat{\theta}^*) < 0, \quad (16)$$

$$\frac{\partial^2}{\partial \theta^{*2}} G^*(\hat{e}, \hat{\theta}, \hat{\theta}^*) < 0 \quad (17)$$

Hence we can formulate our main result:

If pair $(\hat{\theta}, \hat{\theta}^) \in (\frac{1}{M}, 1)^2$ is a unique solution of equations (13), (14), (15), (16), (17), then it is the Nash equilibrium of the game.*

2 Symmetry of Economies

The demand functions differ from nation to nation. Specifically, two nations are said to be economically symmetric if

$$D(x) = D^*\left(\frac{1}{x}\right) \quad (18)$$

We consider symmetric and asymmetric cases separately.

Symmetric case:

Here we consider one more symmetric case. Suppose $D(x) = D^*\left(\frac{1}{x}\right) = (1 - \alpha x)^+$, $\alpha < 1$.

The Nash equilibrium point is $(\hat{e}, \hat{\theta}, \hat{\theta}^*) = (1, \frac{2\alpha}{1+\alpha}, \frac{2\alpha}{1+\alpha})$. Similarly, given any value $\alpha < 1$ which defines the shapes of the demand functions, the equilibrium point will occur at the same tariffs for both nations.

Asymmetric case:

Now we generalize the problem to a more common asymmetric case. Suppose $D(x) = \exp(-\delta x)$, $D^*(x) = (\alpha x \exp(\beta x)) \wedge 1$. Then solving (6) yields the equilibrium exchange rate

$$e = \frac{-\theta \ln(\alpha \theta^*)}{\theta \theta^* \beta + \delta}$$

The Nash equilibrium condition (12),(13) gives

$$\theta = \frac{\delta(\theta^* - 1) \ln(\alpha \theta^*) - \delta}{\theta^* \beta}$$

and

$$\beta \theta^* (\theta^* - 1) = (\theta^* \beta - \delta(\theta^* - 1) \ln(\alpha \theta^*) + \delta)(\theta^* - (\theta^* - 1) \ln(\alpha \theta^*)).$$

Specifically, if $\alpha = 0.1, \beta = 2, \delta = 2.5$ The Nash equilibrium point is $(\hat{e}, \hat{\theta}, \hat{\theta}^*) = (0.2039, 0.20, 0.61)$.

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Stochastic Models in Marketing and Management

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Abstract

In this paper we are describing few marketing and management problems which require further research due to changed market conditions and stochastic nature of the consumer demand. We are presenting different pricing strategies, showing the problems associated with each pricing models and revealing further study requirements. Also considering one of the examples of advertising effectiveness measuring models which require optimization and parameter estimation.

Introduction

In the article we are considering several marketing and management problem which requires further study. Fast changing competitive environment and innovative framework of contemporary business forces us to use more advance method of valuation of these problems. Development of ecommerce and enterprise systems in last decade makes available a huge data which create an opportunity to relatively easier reveal mutual relationships between market factors which influences customers and firms' behavior. In most case in the literature these problems are presented as deterministic. But study of these problems requires in most cases stochastic approach due to uncertain nature of the consumer market.

We can name few most challenging problems of marketing and management which should be studied as probabilistic models rather than deterministic: customer values and demand, market segmentation, initial pricing of product and services, price discounts and price differentiation [1], effectiveness of promotional and advertising campaigns [2], [3], [4], queuing and inventory models [5], [6], etc.

Pricing

Product price as one of the components of 4P marketing mix require dedicated attention. Could be named at least four different strategies of initial pricing of the products and services. Historically most common pricing is cost-plus pricing which based on return on all costs. Regardless that it's fails to capture

relationship between sales volume and costs and considers costs as “fixed” amount, cost-plus pricing even today is most common pricing approach strategy.

Many companies recognize necessity to incorporate customer value into price strategy and use so called customer-driven pricing model which based on customer value estimation. Unfortunately, in practice to achieve short term sales objectives marketers frequently misuse this pricing method and instead of determining the real worth of the product or service estimating how much customers are ready to pay for it.

Frequently price policies based on competition environment of the firm and used to gain desired market share. Regardless that share-driven pricing model’s price cuts can achieve in short terms market share gain, it can be easily matched by rivals. And, as a consequence, firm employed this pricing strategy can simply loose its margins without significant changes of market share.

Strategic pricing approach in contrast of previous pricing models consider price setting as part of the firm’s strategy and tries to achieve sustainable profitability objectives of the firm. It based on three principles: value-based, proactive, and profit-driven. Strategic pricing model includes estimation the customer value of the product and service, anticipates competitive threats (new competitors entry to the market or reverse auctions) and business economic conditions (recession), and ensures that price management based on profit optimization for the firm and deal with uncertain nature of demand.

In general pricing consists of estimation of demand curve and determining the willingness to pay for customers. We can define it as function $w(x)$ as a willingness to pay distribution across the population.

Then, for any given price range $p_2 > p_1 > 0$:

$$\int_{p_1}^{p_2} w(x) dx$$

Fraction of the population that are willing to pay

Where $0 \leq w(x) \leq 1$

Let $D = d(0)$ be the maximum demand. Then $d(p)$ can be defined as

$$d(p) = D \int_p^{\infty} w(x) dx$$

We can take the derivative and obtain

$$d'(p) = -Dw(x)$$

$$w(x) = -d'(p)/d(0)$$

For general case maximum profit can be obtained by choosing the optimal price along estimated demand curve. The model will be further complicated by adding different price optimization, bundling the product or service, price discounts and differentiation across different market segments, and considering customer values as stochastic model. Separately can be considered revenue management and overbooking policies.

Advertising

Another important problem in marketing is to measure the effectiveness of advertising. The few models are suggested: the Adstock model, model propose by G. Lilien, P. Kotler, S. Moorthy in their book Marketing Decision Models. Last one can be described by the following function:

$$Q_t = \alpha + \lambda Q_{t-1} + b \ln(A_t) + c \max(0, \Delta A_t)$$

Where

Q_t – period t sales

A_t - period t advertising

ΔA_t - percentage increase in advertising for period t compared to period $t - 1$

b, c, α , and λ are parameters that should be estimated.

With frequent changes of competitive marketing environment and development of e-commerce and social media models of estimation of advertising effectiveness becoming more complicated and require further study.

Conclusion

Finally we can conclude that due to rapid development of enterprise technologies and data analysis method, contemporary market trends and significant increase of e-commerce business and social media popularity, and frequently changing and uncertain business environment standard marketing and management problems require further study and modification of the mathematical and computational models.

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Martingale Method of Solving Lobachevsky's Functional Equation

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Abstract. The aim of this paper is to give a probabilistic (martingale) method to find the general measurable solution of the Lobachevsky functional equation. We show that to find general solution of this equation is equivalent to establish that a space-transformation of a Brownian Motion by suitable function is a martingale. This method can be applied for Cauchy's, Jensen's, Pexider's and other functional equations.

1 Introduction

We consider the Lobachevsky functional equation

$$f(x)f(y) = f^2\left(\frac{x+y}{2}\right), \quad \text{for all } x, y \in R, \quad (1)$$

where $f = (f(x), x \in R)$ is a real valued function (see, e.g. Aczel [1] about this and other functional equations and related results). It was shown by Neamtu [5] that if $f, f(0) \neq 0$ is a solution of (1) bounded on a neighborhood $(-r, r)$ of zero, then $f(x) = f(0)e^{\lambda x}$, for some $\lambda \in R$.

We give a martingale method to find the general measurable solution of the Lobachevsky functional equation. We don't require the boundedness on $(-r, r)$ for solutions of (1), but consider measurable solutions. We show that

f is a measurable solution of functional equation (1) if and only if the process $M_t = \ln f(0)f(W_t), t \geq 0$ is a martingale.

To this end we are using two facts from probability theory:

The first one is the Bernstein theorem ([3]) (see also [6] for definitive form) according to which if X and Y are independent random variables such that the random variables $X + Y$ and $X - Y$ are also independent, then X and Y admit normal distribution. Bernstein's theorem was used by S. Smirnov [7] to show that any measurable solution of Cauchy's functional equation is locally integrable. We use this idea from [7] to show the integrability of the transformed process of Brownian motion $f(W_t)$.

The second assertion we used is that if the transformed process $g(W_t)$ is a martingale, then the function g is linear

$$g(x) = ax + b, \quad \text{for some constants } a \in R, b \in R.$$

This fact follows from results of [2] or [4], where the semimartingale functions of Brownian motion are studied.

2 The proof of the main result

First we mention some simple properties of equation (1) which will be used in what follows. It is obvious and well known (see [5]) that a solution of (1) is either everywhere or nowhere 0 and if $f(0) \neq 0$ then

$$\text{sign} f(x) = \text{sign} f(0). \quad (2)$$

Indeed, if $f(x_0) = 0$ for some $x_0 \in R$, then $f^2(x) = f(x_0)f(2x - x_0) = 0$ for all $x \in R$ and if $f(0) \neq 0$ it follows from (1) by taking $y = 0$ that

$$f(0)f(x) = f^2\left(\frac{x}{2}\right) > 0, \quad (3)$$

which implies (2).

It is easy to see that the function g defined by $g(x) = \ln \frac{f(x)}{f(0)}$ is odd, since for $y = -x$ we have $f(x)f(-x) = f^2(0)$, which is equivalent to $\frac{f(x)}{f(0)} \frac{f(-x)}{f(0)} = 1$ and implies (since $f(x)/f(0) > 0$ for all $x \in R$) that

$$\ln \frac{f(x)}{f(0)} + \ln \frac{f(-x)}{f(0)} = 0. \quad (4)$$

Let $W = (W_t, t \geq 0)$ be a standard Brownian Motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and denote by $F = (\mathcal{F}_t, t \geq 0)$ the filtration generated by the Brownian Motion W . Assume that $\mathcal{F}_t$ is completed by sets from $\mathcal{F}$ having P -measure zero. As it is well known, all martingales with respect to such filtration are continuous. We consider martingales only with respect to this filtration.

Theorem 1. Let $(f(x), x \in R)$ be a measurable function and $f(0) \neq 0$. Then the following assertions are equivalent:

- a) the function f is a solution of the functional equation (1).
- b) The process $M_t = \ln f(0)f(W_t), t \geq 0$ is a martingale.
- c) $f(x) = f(0)e^{\lambda x}$ for some constant $\lambda \in R$.

Proof. a) $\rightarrow$ b). Assume that f is a measurable solution of (1) with $f(0) \neq 0$. Let us show that the process $(\ln f(0)f(W_t), t \geq 0)$ is a martingale. Let first show that

$$E|\ln f(0)f(W_t)| < \infty$$

for all $t \geq 0$. Let

$$X = f(0)f(W_t) \quad \text{and} \quad Y = f(0)f(B_t),$$

where B_t is a Brownian motion independent of W_t . It follows from (1) and (3) that

$$XY = f^2(0)f(W_t)f(B_t) = f^2(0)f\left(\frac{W_t + B_t}{2}\right) = f^3(0)f(W_t + B_t). \quad (5)$$

On the other hand, substituting $x = W_t - B_t, y = B_t$ in (1) we have from (3) that

$$f(W_t - B_t)f(B_t) = f^2\left(\frac{W_t}{2}\right) = f(0)f(W_t),$$

which implies that

$$\frac{X}{Y} = \frac{f(W_t)}{f(B_t)} = \frac{f(W_t - B_t)}{f(0)}. \quad (6)$$

Since $W_t + B_t$ and $W_t - B_t$ are independent, equations (5) and (6) imply that the random variables XY and $\frac{X}{Y}$ will be also independent. Therefore, it follows from Bernstein's theorem that $X = (f(0)f(W_t))$ (and $Y = f(0)f(B_t)$) will have the lognormal distribution and $\ln f(0)f(W_t)$ admits the normal distribution, hence $\ln f(0)f(W_t)$ is integrable for any $t \geq 0$.

Let us show now the martingale equality.

Substituting $x = W_t - W_s, y = W_s$ in (1) we have

$$f(W_t - W_s)f(W_s) = f^2\left(\frac{W_t}{2}\right) = f(0)f(W_t). \quad (7)$$

Multiplying both parts of (7) by $f^2(0)$ and taking logarithms we obtain that

$$\begin{aligned} \ln f(0)f(W_t - W_s) + \ln f(0)f(W_s) &= \\ &= \ln f^3(0)f(W_t) = \ln f^2(0) + \ln f(0)f(W_t) \end{aligned} \quad (8)$$

which implies that

$$\begin{aligned} \ln f(0)f(W_t) - \ln f(0)f(W_s) &= \ln f(0)f(W_t - W_s) - \ln f^2(0) = \\ &= \ln \frac{f(W_t - W_s)}{f(0)}. \end{aligned} \quad (9)$$

By independent increment property of the Brownian motion $\ln f(W_t - W_s)$ is independent of F_s and taking conditional expectation in (9) we have that $P - a.s.$

$$\begin{aligned} E(\ln f(0)f(W_t) - \ln f(0)f(W_s)/F_s) &= E\left(\ln \frac{f(W_t - W_s)}{f(0)} / F_s\right) = \\ &= E \ln \frac{f(W_t - W_s)}{f(0)}. \end{aligned} \quad (10)$$

But $E \ln \frac{f(W_t - W_s)}{f(0)} = 0$ since by equality (4) the function $\ln \frac{f(x)}{f(0)}$ is odd and $W_t - W_s$ is symmetrically distributed.

Thus, for any $s, t, s \leq t$

$$E(\ln f(0)f(W_t) - \ln f(0)f(W_s)/F_s) = 0; \quad P - a.s.$$

and the process $(\ln f(0)f(W_t), t \geq 0)$ is a martingale.

$b) \rightarrow c)$ Now let us assume that process $\ln f(0)f(W_t)$ is a martingale and $f(0) \neq 0$. This implies that the function $\ln f(0)f(x)$ is linear

$$\ln f(0)f(x) = \lambda x + c, \quad (11)$$

for some constants $\lambda \in R$ and $c \in R$. Hence

$$f(0)f(x) = \exp\{\lambda x + c\} = e^c e^{\lambda x},$$

which implies (taking $x = 0$ in this equality) that $e^c = f^2(0)$ and since $f(0) \neq 0$, we obtain that $f(x) = f(0)e^\lambda$.

c) $\rightarrow$ a) It is easy to verify that the function $f(x) = f(0)e^\lambda$ satisfies equation (1).

Remark. Since $f(0) = 0$ implies that $f(x) = 0$ for all $x \in R$ and $f(x) = 0$ is a solution of (1),

$$f(x) = \alpha e^{\lambda x}, \quad \text{for some constants } \alpha \in R, \lambda \in R$$

will be the most general solution of (1). □

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Uniform integrability of the exponential martingales

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Abstract We establish necessary and sufficient condition for the uniform integrability of the exponential martingales in case of continuous filtration.

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Local martingale.

1. Introduction. Let us introduce a basic probability space $(\Omega, \mathcal{F}, P)$ and continuous filtration $(\mathcal{F}_t)_{0 \leq t \leq \infty}$, which means that every local martingale is continuous. For example filtration generated by the standard Brownian motion is continuous. Let $\mathcal{F}_\infty$ be the smallest σ -Algebra containing all $\mathcal{F}_t$ for $t < \infty$. Let $M = (M_t)_{t \geq 0}$ be a local martingale on the stochastic interval $[[0; T]]$, where T is a stopping time. Denote by $\mathcal{E}(M)$ the stochastic exponential of a local martingale M :

$$\mathcal{E}_t(M) = \exp\{M_t - \frac{1}{2}\langle M \rangle_t\}.$$

For a given local martingale M , the associated stochastic exponential $\mathcal{E}(M)$ is a local martingale, but not necessarily a true martingale. To know whether $\mathcal{E}(M)$ is a true martingale is important for many applications, e.g., when Girsanov's theorem is applied to perform a change of measure.

It is well-known that exponential martingales play an essential role in various questions concerning the absolute continuity of probability laws in stochastic processes. A. A. Novikov [13] showed that $\mathcal{E}(M)$ is a uniformly integrable martingale if $e^{\frac{1}{2}\langle M \rangle_\infty} \in L_1$ and that the constant $\frac{1}{2}$ can not be improved. In 1979 Kazamaki [10] proved that $\sup_\tau E e^{\frac{1}{2}M_\tau} < \infty$ is sufficient for uniform integrability of $\mathcal{E}(M)$. Then in 1994 Kazamaki [11] generalized his assertion introducing mixed Novikov-Kazamaki condition using constant $a \neq 1$ and lower functions (Kazamaki [11], p.19, Theorem 1.12). In 2013 J. Ruf [14] generalized mixed Novikov-Kazamaki criterion introducing general function of local martingale and its quadratic variation. In [4] and [3] the mixed Novikov-Kazamaki criterion is generalized using predictable process a_s instead of the constant a . A similar question in the exponential semi-martingale framework, in particular, for affine processes, has also attracted attention in Kallsen and Muhle-Kabre [8] and in Kallsen and Shiryaev [9]. In [8] a weak sufficient criterion and in [9] sufficient criterion in terms of cumulant process is given for uniform integrability of $\mathcal{E}(M)$.

The necessary and sufficient conditions for the uniform integrability of $\mathcal{E}(M)$ were provided in Mayerhofer, Muhle-Kabre and Smirnov [12] by considering the case when the initial martingale M represents one component of a multivariate affine process, and in Blei and Engelbert [1] and Engelbert and Senf [5] for the exponential local martingales associated with a strong Markov continuous local martingale. In [12] deterministic necessary and sufficient conditions is provided in terms of the parameters of the initial martingale M . In [1], the case of a strong Markov continuous local martingale M is studied and the deterministic criterion is expressed in terms of speed measure of M . In [5], the case of a general continuous local martingale M is considered and the condition of uniform integrability of $\mathcal{E}(M)$ is given in terms of time-change that turns M into a (possible stopped) Brownian motion. In [7] Yu. M. Kabanov, R. Sh. Liptser, A. N. Shiryaev showed that if the measure Q is locally absolutely continuous w.r.t. the measure P , then for absolute continuity of Q w.r.t. P necessary and sufficient is that $Q\{\langle M \rangle_T < \infty\} = 1$. For the treatment of the related questions of a local absolute continuity of

measures on filtered spaces see also Jacod and Shiryaev [6] and Cheridito, Filipovic and Yor [2]. We establish a necessary and sufficient conditions for the uniform integrability of the stochastic exponential $\mathcal{E}(\mathcal{M})$ in terms of the basic measure P .

In the next section we formulate the main result of this paper (**Theorem 1**).

2. The main result. In the following theorem we weakened condition $|a_s - 1| \geq \varepsilon > 0$ imposed in [4] and [3], which enable us to obtain new type necessary and sufficient condition:

Theorem 1. For the uniform integrability of the stochastic exponential $\mathcal{E}(M)$, it is necessary and sufficient, that there exists a predictable, M -integrable process a_s such that:

$$(i) \quad \sup_{\tau \leq T} E \exp \left\{ \int_0^\tau a_s dM_s + \int_0^\tau \left(\frac{1}{2} - a_s \right) d\langle M \rangle_s \right\} < \infty$$

where the sup is taken over all stopping times $\tau \leq T$;

$$(ii) \quad f(\langle M \rangle_s) \leq (a_s - 1)^2 \text{ for some function } f \geq 0 \text{ with } \int_0^\infty f(x) dx = \infty.$$

Notice that because $\mathcal{E}(M)$ is a supermartingale, $E\mathcal{E}_\tau(M) \leq 1$ for any stopping time $\tau \leq T$. So condition (i) of Theorem 1 is automatically satisfied when $a_s \equiv 1$. This means that condition (i) is not sufficient when a_s is quite close to 1. Accordingly, condition (ii) of Theorem 1 gives us an exact degree of proximity of a_s to 1.

Remark 1. Novikov's [13] condition $Ee^{\frac{1}{2}\langle M \rangle_T} < \infty$ and Kazamaki's [10] criterion $\sup_\tau Ee^{\frac{1}{2}M_\tau} < \infty$ are particular cases of Theorem 1 taking $a_s \equiv 0$, $f(x) \equiv 1$ and $a_s \equiv \frac{1}{2}$, $f(x) \equiv \frac{1}{4}$ respectively. Applying Theorem 1 for $a_s \equiv a \neq 1$ and $f(x) \equiv (1 - a)^2$ we obtain the mixed Novikov-Kazamaki's condition:

$$\sup_{\tau \leq T} Ee^{aM_\tau + (\frac{1}{2} - a)\langle M \rangle_\tau} < \infty.$$

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