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Content

How the Digital Ecosystems are transforming the Banking Industry

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In the next 5 to 10 years the World will see the results of Digital Transformation, which will also disrupt the Global Banking Sector. Namely:

- Digital commerce spending is expected to reach $26.2 trillion annually by 2028, driven by e-Retail acceleration [1].
- Digital Banking Market size is estimated to exceed $13 trillion by 2032 [2].
- Global market for loyalty programs is anticipated to be $216 billion in 2022 [3].
- Assuming that companies are spending at least 2% on bonus points [4], the total retail market size with the loyalty programs is estimated at about $11 trillion.

The above figures indicate the size of the digital economy and the potential of creating digital ecosystems in various forms including eCommerce, Digital Banking and Loyalty Platforms.

The case of Kaspi Bank JSC (Kazakhstan) explains how digital ecosystems can transform the banking industry delivering value to both, banking customers and bank shareholders. Figure 1 presents the comparison of the Total assets and Market Capitalization for two leading banks in Kazakhstan: Kaspi Bank JSC and Halyk Bank JSC.

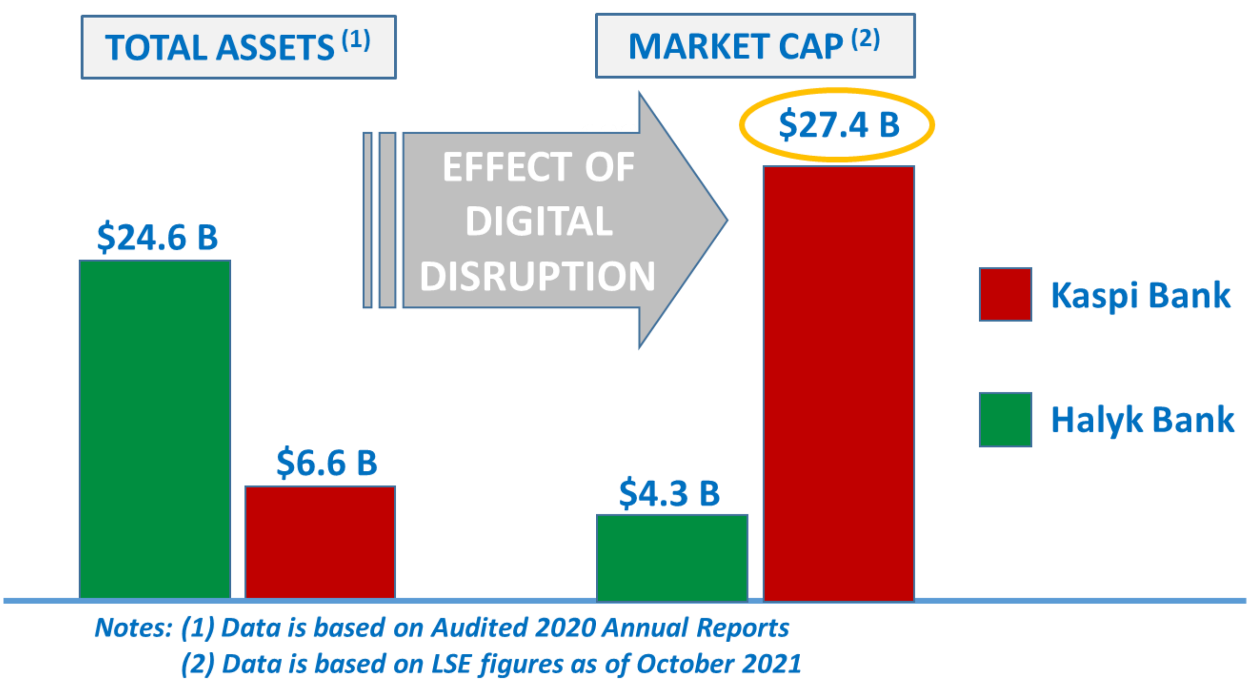

Fig. 1 Comparison of the Total Assets and Market Capitalization

The data on Figure 1 is based on the annual audited reports of year 2020 regarding Kaspi Bank JSC [5] and Halyk Bank JSC [6]. The market capitalization figures are based on the data of the London Stock Exchange (LSE) as of October 2021 for Kaspi Bank JSC [7] and Halyk Bank JSC [8].

As can be seen from Figure 1, the total assets of the Halyk Bank is about 3.7 times larger than that of Kaspi Bank, however the market capitalization figures show the rather opposite picture. Namely, the market cap of Kaspi Bank is about 6.4 times higher than that of Halyk Bank. This difference is a result of the digital disruption initiated by Kaspi Bank in the Kazakhstan's banking Sector.

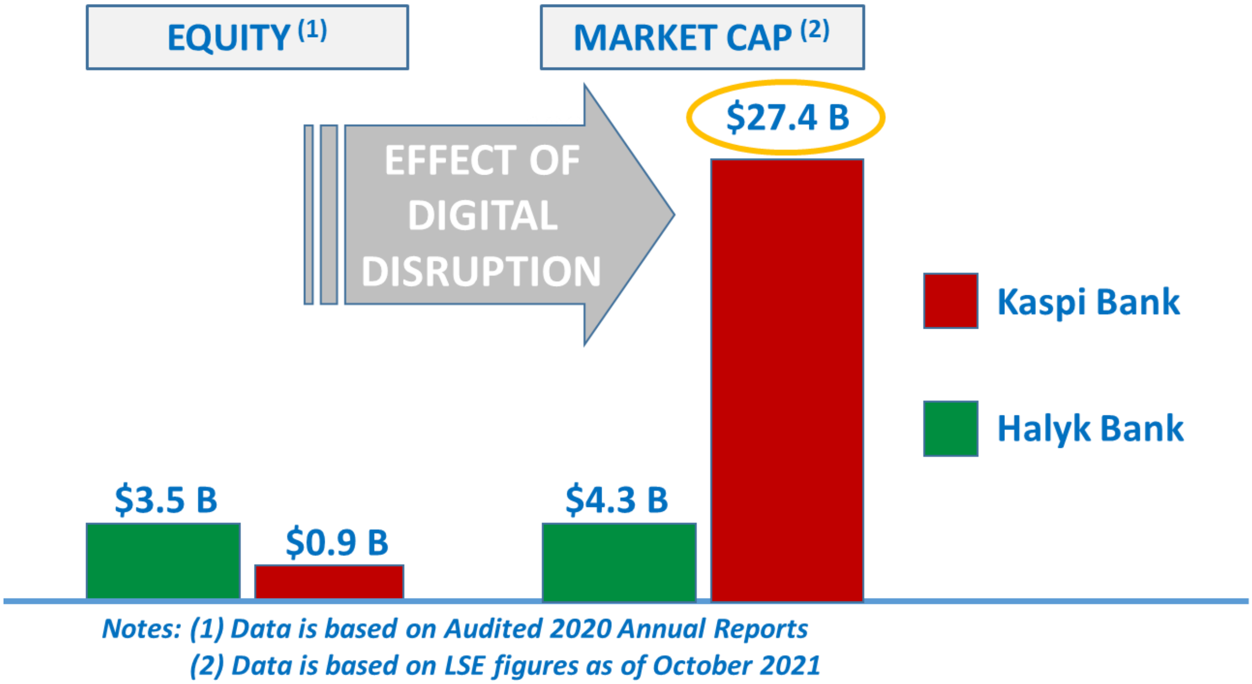

Fig. 2 Comparison of the Equity and Market Capitalization

The effect of digital disruption is even more evident when comparing the equity size with the market cap for these two banks. It can be seen from Figure 2 that the market cap of Halyk Bank is almost the same as its equity, while the market cap of Kaspi Bank is more than 30 times its equity – a truly remarkable result. This is the result of the market expectation of higher returns due to synergy effects of an innovative digital ecosystem set up by Kaspi Bank, which incorporates into the ecosystem the banking services with Fintech solutions, electronic marketplace and loyalty platform in association with a large number of merchants.

In order to evaluate the effect of digital ecosystem, it would be interesting to compare the present-day performance characteristics of these two banks with those some years ago, i.e. before the introduction of the digital ecosystem approach in banking. For comparison we have chosen the year 2014 and the results are presented in Figure 3. In 2014 the total assets of Halyk Bank was about 3 times higher than that of Kaspi Bank. The same can be said about the operating profits, i.e. the operating profit of Halyk Bank was about 3 times higher than that of Kaspi Bank. This indicates that $1 of assets had generated almost the same operating profit for both banks, thus displaying a similar operating performance of these banks in 2014.

However, the picture has changed drastically in 2020. Figure 3 indicates that during the period since 2014 until 2020 the size of the total assets of Halyk Bank increased at a higher rate than that of Kaspi Bank. More precisely, the total assets of Kaspi Bank in 2020 was only about 1/4th of that of Halyk Bank.

In contrast, the operating profit of Kaspi Bank had increased at a higher rate than that of Halyk bank and reached 82% of the operating profit generated by the latter. To summarize, $1/4^{th}$ of assets (in case of Kaspi Bank) had generated about 82% of the operating profit of Halyk Bank. I.e. $1 of Kaspi Bank's assets generated 3.02 times more operating profit than $1 of Halyk Bank's assets (refer to Figure 3).

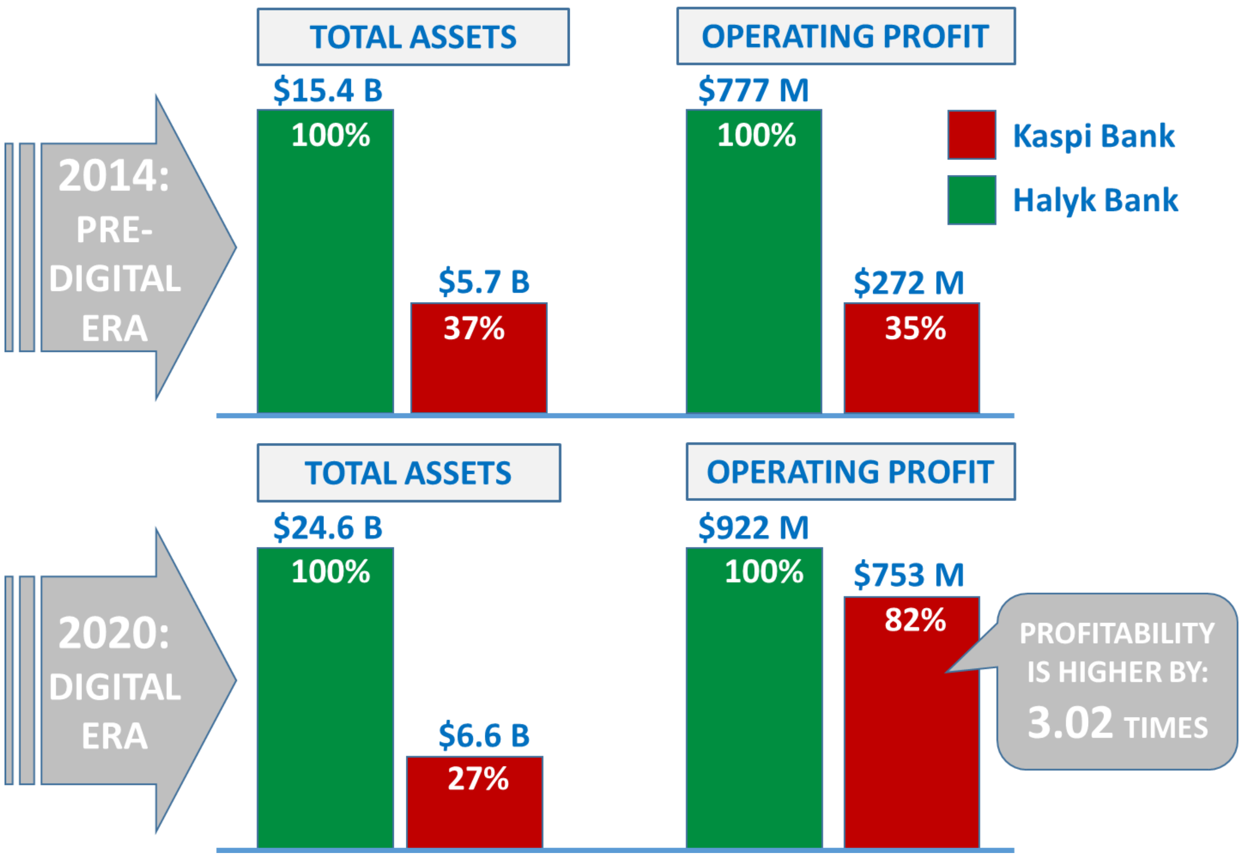

Fig. 3 Comparison of the Total Assets and Operating Profit in 2014 and 2020

It is interesting to find out the causes of such phenomenal performance. For this reason, we need to compare the operating profits and asset turnovers for these 2 banks. The results of the comparison are presented in Figure 4. Namely, 3.02 times more operating profit per $1 of assets generated by Kaspi Bank in comparison to Halyk Bank is attributed to two factors: (a) Higher operating profit and (b) Higher asset turnover. More specifically, the operating profit of Kaspi Bank is about 50% of its revenues, which is some 1.32 times higher than that of Halyk Bank. This is a typical result for digital banks, which operate mainly by using online channels that are much cheaper in comparison to their brick-and-mortar counterparts.

An even more staggering picture emerges when comparing the figures reflecting the asset turnovers. Namely, the asset turnover of Kaspi Bank is 2.28 times higher than that of Halyk Bank. The multiplication of these two parameters, i.e. 1.32 times higher operating profit and 2.28 times higher asset turnover gives 3.02 times more operating profit per $1 of assets as indicated in the above.

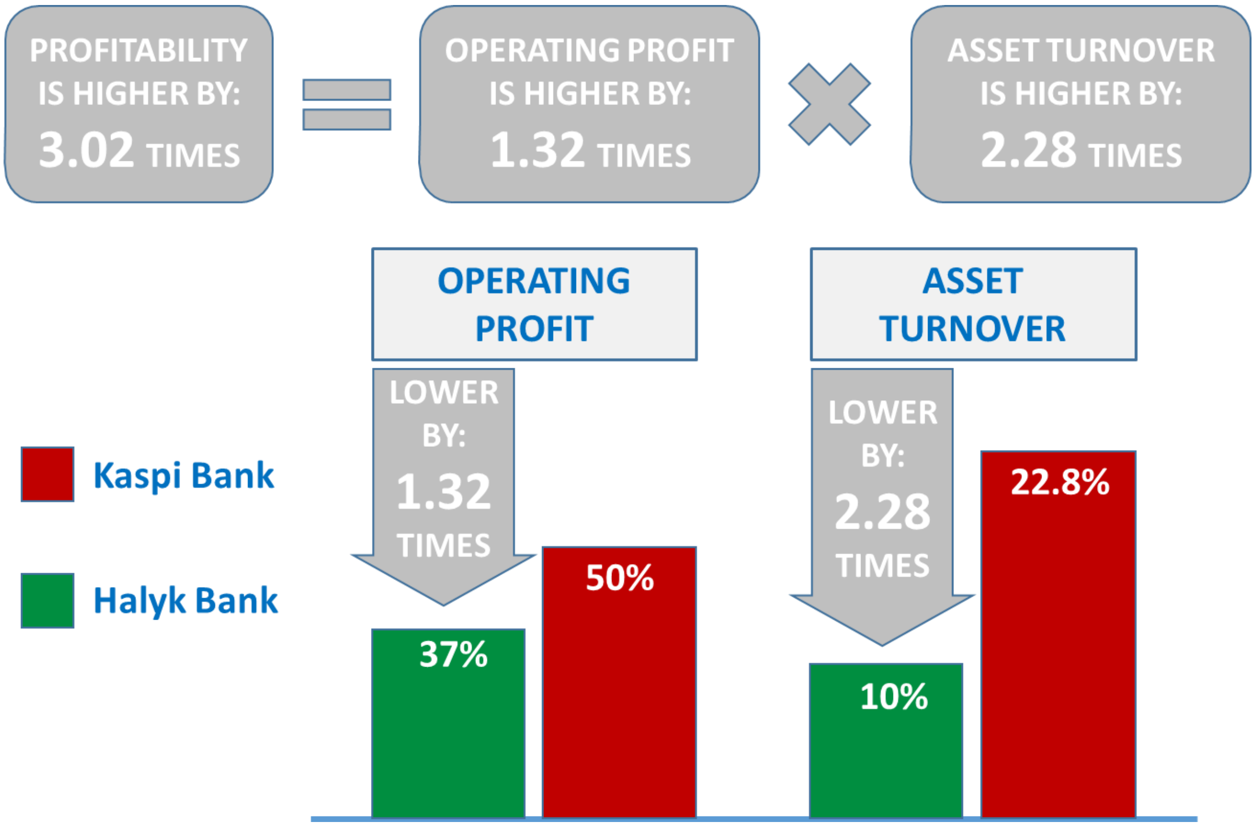

Fig. 4 Comparison of the Operating Profit and Asset Turnover

The next step in our analysis will be the identification of the reasons behind the 2.28 times higher asset turnover of Kaspi Bank in comparison to Halyk Bank. This analysis is presented in Figure 5, which gives the breakdown of the asset turnover in terms of interest revenues and fees & commissions.

The analysis of the financial results presented in Figure 5 shows that the interest revenue per asset of Kaspi Bank is 1.63 times higher than that of Halyk Bank. This can be logically attributed to the more retail-oriented portfolio of Kaspi Bank and to a lesser accent on mortgage loans that are traditionally characterized with lower APRs.

However, the more prominent picture emerges from comparing the revenues per asset generated by fees & commissions (including membership fees) paid by both, customers and partner merchants (for using the electronic marketplace incorporated into a unified digital ecosystem of Kaspi Bank). Namely, fees & commissions income per asset of Kaspi Bank is staggering 3.84 times higher than that of Halyk Bank. This is the result of cooperation between Kaspi Bank and a large network of merchants, which are incorporated into the Kaspi Bank's digital ecosystem. Such cooperation offers an additional and cost-effective channel to merchants to sell their products through Kaspi Bank's platform. At the end, an ultimate winner is a customer, who receives full eCommerce and banking services through Kaspi Bank's portal in a convenient and comfortable manner.

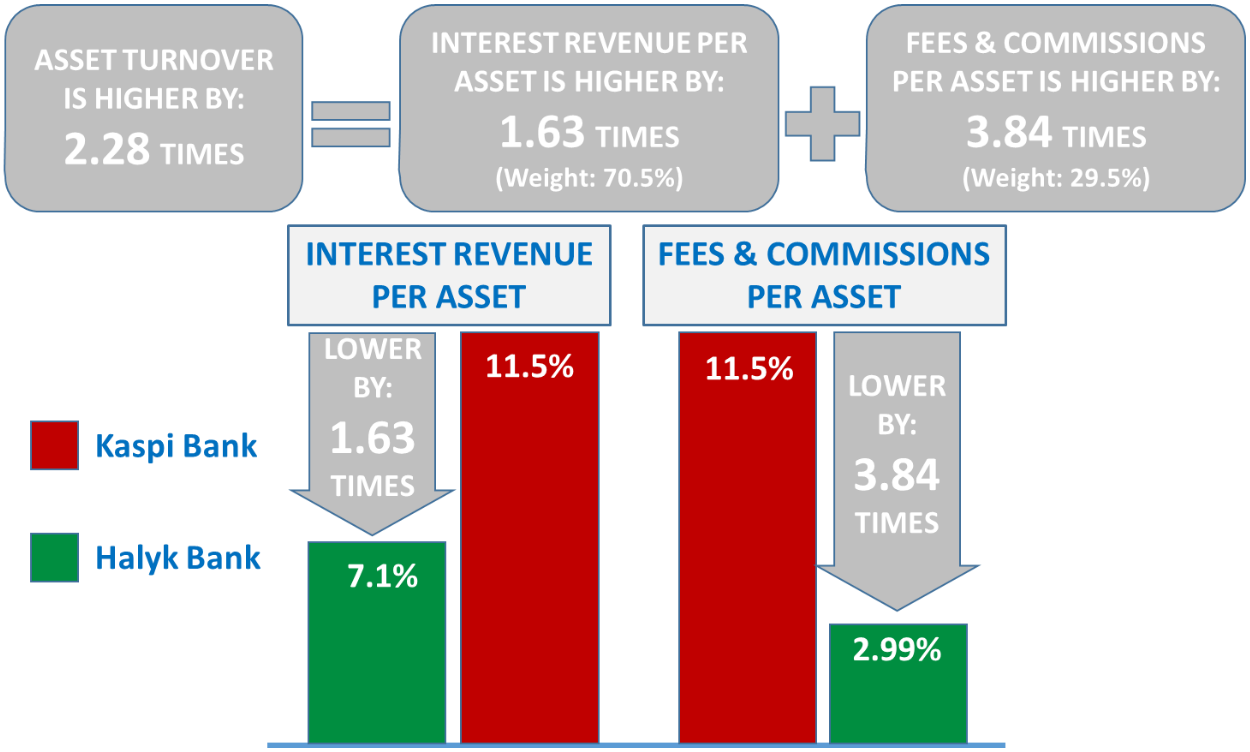

Fig. 5 Breakdown of the asset turnover into interest revenues and fees & commissions

Even more amazing is the fact that the revenue of Kaspi Bank generated from fees & commissions is almost exactly equal to its interest income (as shown in Figure 5), which is an extremely rare case in the banking industry. This fact indicates on the effectiveness of cross-sectoral cooperation and the synergy effects derived from such cooperation. The result is the high profitability of the ecosystem delivering full-scale digital services to large segments of customers in a convenient and comfortable manner.

	CAC/Initial Investment	Number of Customers	Transaction Origination	Operating Cost	CLV/ Profitability
Loyalty Aggregators	Low	High	Mid	Mid	Low
eCommerce Platforms	Mid	Mid	High	High	Mid
Embedded Finance	Mid	Low	Low	Low	High

Fig. 6 Benefits of a Digital Ecosystem Model

To analyze the reasons behind such a phenomenal performance of Kaspi Bank, let's refer to Figure 6, which presents the idea behind an ecosystem approach and the benefits it brings to its stakeholders. The business rationale of such an approach derives from the fact that none of the ecosystem components, i.e. Loyalty Aggregator, eCommerce Platform and Embedded Finance offering digital banking services, will not be sustainable in the long run, if operated independently. However, a unified digital ecosystem will incorporate all the elements of a sustainable business model. More specifically:

- Loyalty Aggregators will require low initial investment to acquire a large customer base, as the existing channels of participating merchants will be used to reach a vast number of customers. Hence, the number of customers will be high, and the Customer Acquisition Cost (CAC) will be low.
- eCommerce Platform will originate many transactions, as it is convenient for customers to shop and accumulate bonus points at one place. Moreover, a high operating costs of eCommerce business, associated with the logistics and delivery of goods, will be mitigated by introducing a platform approach, i.e. the ecosystem will only aggregate the information from the various eCommerce players without taking ownership of the title on the goods to be delivered.
- The ecosystem will incorporate embedded finances by offering digital banking services to its customers. By this, the Customer Lifetime Value (CLV) of the ecosystem will increase drastically, making it a sustainable business in the long run.

Summary

It can be concluded that an ecosystem approach offers a comfortable solution to modern day customers oriented at receiving digital services in their mobile phones. At the same time such approach results in a sustainable business model derived from a large customer base and low customer acquisition cost (due to participating merchants), large potential to originate eCommerce transactions, lower operating costs due to a platform approach and a high customer lifetime value due to digital banking services, which are offered to customers through an embedded finance. It only remains to see which companies will move swiftly to introduce the ecosystem approach, which has already been well-tested by an innovative Kaspi Bank JSC in Kazakhstan.

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A Generalization of the von Bertalanffy growth Model using the BSDE Approach

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Abstract. The generalized von Bertalanffy growth model with random extremal length is expressed as a unique solution of a Backward Stochastic Differential Equation.

Keywords: Bertalanffy growth model, Levy Process, Brownian Motion, Backward Equation.

1 Introduction

We shall use stochastic exponentials and Backward Stochastic Differential Equation (BSDE) approach to generalize the fish growth deterministic model of von Bertalanffy [4], which is most commonly used as a descriptive model of size-at age data.

The von Bertalanffy model, which is written for the case of decreasing growth with age is a differential equation with a linearly decreasing growth rate, or

$$\frac{dL}{dt} = K(L_\infty - L_t), \quad L_\infty > 0 \quad K > 0, \tag{1}$$

with initial condition $L(t_0) = L_0$. The solution to differential equation (1) is

$$L_t = L_\infty[1 - e^{-K(t-t_0)}] + L_0 e^{-K(t-t_0)}, \tag{2}$$

where

L_∞ - is the upper bound for the variable under study, that can only be reached after infinity time,

K - is the curvature parameter, or von Bertalanffy growth rate, that determines the speed with which the fish attains L_∞.

t_0 - determines the time at which the fish has a size equal to zero and could be negative.

For simplicity we assume that $L_0 = 0$ and $t_0 = 0$ which means that an individual would have been of length 0 at age 0. This results in the "LVB" growth model

$$L_t = L_\infty[1 - e^{-Kt}]. \tag{3}$$

Several stochastic growth models are available in the literature. Some individual-based stochastic models of growth (see, e.g., [1], [2]) are proposed using stochastic differential equations of the type

$$L_t = L_0 + \int_0^t a(s, L_s)ds + \int_0^t \sigma(s, L_s)dW_s,$$

where L_t is the size at time t, $a(t, L_t)$ characterizes the deterministic intrinsic growth of the individual, $\sigma(t, L_t)$ gives the magnitude of the random fluctuations and W_t is a Brownian Motion.

In Russo et al [3] the growth model of fish (and other animals) as a solution of linear stochastic differential equation driven by a Levy process with positive jumps (a subordinator) was proposed, the unique solution of which is the stochastic exponential of the Levy process. This model admits a certain number of desirable features and it is the first stochastic model with increasing paths, giving more realistic stochastic model of individual growth.

The model proposed in Russo et al [3] is given by the process Y_t, which is obtained as the solution of the stochastic differential equation (SDE)

$$dY_t = (L_\infty - Y_{t-})dX_t \tag{4}$$

with initial condition $L_0 = 0$, where X_t is a subordinator. Note that a subordinator is a Levy process with increasing paths.

If the process X cannot make jumps larger than 1 (which is natural to assume in this context), then solution of this equation is

$$L_t = L_\infty(1 - \mathcal{E}_t(-X)), \tag{5}$$

where $\mathcal{E}_t(-X)$ is the stochastic exponent of the process $-X$ and the extreme length L_∞ is assumed to be a constant. Note that L_t defined by this model is increasing process and it coincides with the von Bertalanffy growth curve when X is a deterministic subordinator $X_t = kt$.

This approach (as all existing) has a drawback as a growth model, since the asymptotic length of the fish is assumed to be a constant. This implies that the variation of fish length tends to zero, which is not realistic, as it would imply that all individuals should reach the same limiting size. In order to overcome this problem it seems natural to assume that the extremal size of a fish is itself a random variable, thus accounting for the individual variability. Therefore, it is natural to use Backward SDE's (instead of the forward SDEs) with the random boundary condition at the end equal to the asymptotic length of a fish.

To generalize the von Bertalanffy model when the extreme length L_∞ is a random variable, let first consider the simple case and only assume that L_∞ is a bounded random variable measurable with respect to $F^W_\infty = \vee_{t\geq 0} F^W_t$, where W is a Brownian Motion and $(F^W_t, t \geq 0)$ is the filtration generated by W.

We write this model as a solution of the Backward Stochastic Differential equation (BSDE)

$$Y_t = \int_0^t Y_s \frac{Ke^{-Ks}}{1 - e^{-Ks}} ds + \int_0^t Z_s dW_s, \tag{6}$$

with the boundary condition

$$Y_\infty = \lim_{t\to\infty} Y_t = L_\infty. \tag{7}$$

The solution process to equation (6)-(7) is

$$L_t = E(L_\infty | F^W_t)[1 - e^{-Kt}]. \tag{8}$$

More exactly the solution of (6)-(7) is a pair (Y_t, Z_t)

$$Y_t = L_t, \qquad Z_t = \varphi_t(1 - e^{-Kt}),$$

where L_t is defined by (8) and φ_t is the integrand from the integral representation of the martingale

$$E(L_\infty|F_t^W) = EL_\infty + \int_0^t \varphi_s dW_s,$$

which can be immediately verified by the integration by part formula.

Note that, since (8) implies

$$EL_t = EL_\infty[1 - e^{-Kt}],$$

the expectation of L_t follows the Von Bertalanffy-type pattern with L_∞ replaced by EL_∞.

Remark that, if in (8) instead of exponential distribution function $1-e^{-Kt}$ we shall take general continuous distribution function $G(t)$, then the process $L_t = E(L_\infty|F_t^W)G(t)$ will satisfy the BSDE

$$Y_t = \int_0^t \frac{Y_s}{G(s)} dG(s) + \int_0^t Z_s dW_s, \tag{9}$$

with the same boundary condition (7).

We shall generalize expression (5) (see Theorem 1) assuming that L_∞ is a random variable and consider this variable as a boundary condition at infinity of a BSDE for L_t driven by a subordinator X and a Brownian Motion W, independent of X. The linear BSDEs derived in the paper differ from classical cases by considering not integrable coefficients on the infinite time interval. Under additional assumption that the extreme size L_∞ of a fish is a random variable measurable with respect to the σ-algebra F_∞^W generated by the Brownian Motion W, i.e., when two sources of randomness, the random individual variability (related with L_∞) and the environmental randomness (related with the process X_t), are independent, the BSDE takes simpler and more natural form (see Corollary 1).

2 The main results.

Let $X = (X_t, t \geq 0)$ be a Levy process with affine process $\alpha t, \alpha > 0$, with zero Brownian part and with positive jumps (a subordinator). Let

$W = (W_t, t \geq 0)$ be a Brownian Motion. Suppose that X and W are independent processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $F = (F_t, t \geq 0)$ generated by W and X and let

$$F_\infty = \vee_{t\geq 0} F_t.$$

Suppose that in the model (3) or (5) L_∞ is an integrable F_∞-measurable random variable. Then L_t should be a random process and if we assume that the process L_t is adapted with respect to the filtration F_t, from (5) we obtain that

$$L_t = E(L_\infty|F_t)[1 - \mathcal{E}_t(-X)]. \tag{10}$$

Let consider the growth model given by expression (10) and let's see what equation this process satisfies.

Let μ be the measure of jumps of the process X and let $\tilde{\mu}$ be its compensator. Note that in our case the compensator is of the form

$$\tilde{\mu}(ds, dx) = \nu(dx)ds,$$

where $\nu(dx)$ is the Levy measure on $R_+ =]0, \infty[$ with $\int_{R_+}(1 \wedge x^2)\nu(dx) < \infty$.

We recall that the Levy process is a cadlag process with stationary independent increaments, hence all jumps of X are totally inaccesible.

Denote by H the expression

$$H(s,x) = \frac{x\mathcal{E}_{s-}(-X)}{1 + (x-1)\mathcal{E}_{s-}(-X)}.$$

Let consider the following linear BSDE (backward stochastic differential equation)

$$Y_t = \int_0^t \int_{R_+} (Y_s + K(s,x))H(s,x)\nu(dx)ds + \alpha \int_0^t Y_s \frac{\mathcal{E}_{s-}(-X)}{1 - \mathcal{E}_{s-}(-X)} ds +$$

$$+ \int_0^t \int_{R_+} K(s,x)(\mu - \tilde{\mu})(dx, ds) + \int_0^t Z_s dW_s, \tag{11}$$

with the boundary condition

$$Y_\infty = \lim_{t\to\infty} Y_t = L_\infty. \tag{12}$$

Definition. Let $\mathcal{V}$ be the class of cadlag processes $(Y_t, t \geq 0)$, such that the family of random variables $(Y_\tau, \tau \in \mathcal{T})$ is uniformly integrable, where $\mathcal{T}$ is the set of stopping times.

Theorem 1. Let X be a Levy process with increasing paths (a subordinator) and let $\Delta X_t < 1$ for all $t \geq 0$. Assume that L_∞ is an integrable F_∞-measurable random variable.

Then the process

$$L_t = E(L_\infty|F_t)[1 - \mathcal{E}_t(-X)] \quad t \geq 0 \tag{13}$$

is the unique solution of the BSDE (11)-(12) in the class $\mathcal{V}$.

Proof. The boundary condition follows from the Levy theorem, since $\lim_{t\to\infty} \mathcal{E}_t(-X) = 0$ and

$$\lim_{t\to\infty} L_t = \lim_{t\to\infty} E(L_\infty|F_t) \lim_{t\to\infty}(1 - \mathcal{E}_t(-X)) = L_\infty.$$

It follows from (13) that the process L_t is a special semimartingale and let

$$L_t = A_t + M_t, \qquad A_0 = 0, M_0 = 0 \tag{14}$$

be the canonical decomposition, where A is the predictable process of finite variation and M is a local martingale, which by the integral representation property can be expressed as

$$M_t = \int_0^t \int_{R_+} K(s,x)(\mu - \tilde{\mu})dxds + \int_0^t Z_s dW_s \tag{15}$$

for some predictable Z and K with

$$\int_0^t Z_s^2 ds < \infty, \quad \int_0^t \int_{R_+} K^2(s,x)\nu(dx)ds < \infty \quad a.s.$$

First note that, since $\Delta\mathcal{E}_t(-X) = -\mathcal{E}_{t-}(-X)\Delta X_t$, we have for any $0 < r < t$

$$\frac{1}{1 - \mathcal{E}_t(-X)} - \frac{1}{1 - \mathcal{E}_r(-X)} =$$

$$= \sum_{r<s\leq t} \left(\frac{1}{1 - \mathcal{E}_{s-}(-X)} - \frac{1}{1 - \mathcal{E}_{s-}(-X)} \right) - \int_r^t \frac{\alpha\mathcal{E}_s(-X)}{(1 - \mathcal{E}_s(-X))^2} ds =$$

$$= -\sum_{r<s\le t} \frac{\mathcal{E}_{s-}(-X)\Delta X_s}{\big(1-\mathcal{E}_{s-}(-X)\big)\big(1-\mathcal{E}_{s-}(-X)+\mathcal{E}_{s-}(-X)\Delta X_s\big)} - \int_r^t \frac{\alpha\mathcal{E}_s(-X)}{(1-\mathcal{E}_s(-X))^2}ds =$$

$$= -\int_r^t \int_{R_+} \frac{\mathcal{E}_{s-}(-X)x}{\big(1-\mathcal{E}_{s-}(-X)\big)\big(1-\mathcal{E}_{s-}(-X)+\mathcal{E}_{s-}(-X)x\big)}\mu(dxds) - \int_r^t \frac{\alpha\mathcal{E}_s(-X)}{(1-\mathcal{E}_s(-X))^2}ds$$

$$= -\int_r^t \int_{R_+} \frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)}\mu(ds,dx) - \int_r^t \frac{\alpha\mathcal{E}_s(-X)}{(1-\mathcal{E}_s(-X))^2}ds. \qquad (16)$$

By the Itô formula and (16) for any $r > 0$

$$\frac{L_t}{1-\mathcal{E}_t(-X)} - \frac{L_r}{1-\mathcal{E}_r(-X)} = \qquad (17)$$

$$= \int_r^t \frac{1}{1-\mathcal{E}_{s-}(-X)}dA_s + \int_r^t \frac{1}{1-\mathcal{E}_{s-}(-X)}dM_s +$$

$$+ \int_r^t L_{s-}d(1-\mathcal{E}_{s-}(-X))^{-1}) + [L,(1-\mathcal{E}(-X))^{-1}]_t - [L,(1-\mathcal{E}(-X))^{-1}]_r =$$

$$= \int_r^t \frac{1}{1-\mathcal{E}_{s-}(-X)}dA_s + \int_r^t \frac{1}{1-\mathcal{E}_{s-}(-X)}dM_s - \int_r^t \frac{\alpha L_s\mathcal{E}_{s-}(-X)}{(1-\mathcal{E}_{s-}(-X))^2}ds$$

$$- \int_r^t \int_{R_+} L_{s-}\frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)}\mu(ds,dx) - \int_r^t \int_{R_+} K(s,x)\frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)}\mu(ds,dx)$$

$$- \int_r^t \int_{R_+} \Delta A_s \frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)}\mu(ds,dx).$$

Since A is predictable and all jumps of X are totally inaccessible

$$\int_r^t \int_{R_+} \Delta A_s \frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)}\mu(ds,dx) = 0$$

and if we isolate in (17) the martingale part, we obtain that

$$\frac{L_t}{1-\mathcal{E}_t(-X)} - \frac{L_r}{1-\mathcal{E}_r(-X)} = \qquad (18)$$

$$= \int_r^t \frac{1}{1-\mathcal{E}_{s-}(-X)}dA_s - \int_r^t \frac{\alpha L_s\mathcal{E}_{s-}(-X)}{(1-\mathcal{E}_{s-}(-X))^2}ds =$$

$$- \int_r^t \int_{R_+} (L_s + K(s,x))\frac{H(s,x)}{1-\mathcal{E}_{s-}(-X)}\nu(dx)ds \ + \ \text{local martingale.}$$

Since by (13) the left-hand side of (18) is a martingale on the interval $[r, \infty]$ for any $r > 0$, the bounded variation part in (18) should be equal to zero. Therefore,

$$A_t - A_r = \alpha \int_r^t L_s \frac{\mathcal{E}_s(-X)}{1 - \mathcal{E}_{s-}(-X)} ds + \tag{19}$$

$$+ \int_r^t \int_{R_+} (L_s + K(s,x)) H(s,x) \nu(dx) ds$$

for any $r > 0$. Since $A_0 = 0$ the process A is a cadlag process of finite variation, passing to the limit as r tends to 0 we obtain from (19) that

$$A_t = \int_0^t \int_{R_+} (L_s + K(s,x)) H(s,x) \nu(dx) ds + \alpha \int_0^t L_s \frac{\mathcal{E}_s(-X)}{1 - \mathcal{E}_s(-X)} ds \tag{20}$$

which, together with (14)-(15), implies that L_t satisfies (11).

It is evident that $L \in \mathcal{V}$, since

$$0 \leq L_t \leq E(L_\infty | F_t)$$

and the family $(E(L_\infty | F_\tau), \tau \in \mathcal{T}))$ is uniformly integrable.

The proof of the uniqueness. Let Y_t be a solution of (11)-(12) from the class $\mathcal{V}$. Then it follows from (11) (after tedious application of the Itô formula) that the process

$$M_t = Y_t / (1 - \mathcal{E}_t(-X)) \tag{21}$$

is a local martingale. Since X is a subordinator and $0 \leq \Delta X_t < 1$ for all $t \geq 0$ we have that

$$\frac{1}{1 - \mathcal{E}_t(-X)} = \frac{1}{1 - e^{-\alpha t} \Pi_{s \leq t}(1 - \Delta X_s)} \leq \frac{1}{1 - e^{-\alpha t}}.$$

Therefore, from (21)

$$|M_t| \leq \frac{|Y_t|}{1 - e^{-\alpha t}}$$

and since $Y \in \mathcal{V}$, the process $(M_t, t \geq r)$ will be a uniformly integrable martingale for any $r > 0$. This implies that $M = (M_t, t \geq r)$ can be represented as $M_t = E(\eta | F_t)$ for some $\mathcal{F}$ measurable integrable random variable η, hence

$$Y_t / (1 - \mathcal{E}_t(-X)) = E(\eta | F_t), \quad t \geq r. \tag{22}$$

By the boundary condition (12) and the Levy theorem, passing to the limit as $t \to \infty$ in (22) we obtain that

$$\eta = Y_\infty = L_\infty,$$

which by arbitrariness of $r > 0$ and right-continuity of Y and M_t implies that $Y_t = L_t = E(L_\infty|F_t)[1 - \mathcal{E}_t(-X)]$ for any $t \geq 0$.

Remark 1. If X_t is a deterministic subordinator, i.e. if $X_t = \alpha t, \alpha > 0$, then equation (11) coincides with equation (6) from the introduction.

Remark 2. Note that in this model the solution L_t is no more an increasing process, but the expectation EL_t is an increasing function. Indeed, from the Itô formula

$$L_t = E(L_\infty|F_t^W)[1 - \mathcal{E}_t(-X)] = \tag{23}$$

$$\int_0^t E(L_\infty|F_s)\mathcal{E}_{s-}(-X)dX_s + \text{martingale}.$$

Since X is a Levy process with positive jumps, $X_t - EX_t$ is a martingale and EX_t is increasing. Therefore, it follows from (23) that

$$EL_t = E\int_0^t E(L_\infty|F_s)\mathcal{E}_{s-}(-X)dEX_s = \int_0^t EL_\infty\mathcal{E}_{s-}(-X)dEX_s,$$

which implies that EL_t is an increasing function.

Now suppose that the extreme size L_∞ is an integrable F_∞^W-measurable random variable. So, we assume that two sources of randomness, the random individual variability (related with L_∞) and the environmental randomness (related with the process X_t), are independent, which is natural to assume.

Under these conditions

$$E(L_\infty|F_t) = E(L_\infty|F_t^W),$$

$$L_t = E(L_\infty|F_t^W)[1 - \mathcal{E}_t(-X)]. \tag{24}$$

and the BSDE for the process L_t will be essentially simpler.

Denote by m the average size of jump of the process X

$$m \equiv \int_{R_+} x\nu(dx).$$

Since $0 < \Delta X_t < 1, t \geq 0$ and $\nu(dx)$ is the Levy measure

$$\int_{R_+} x^2 \nu(dx) < \infty,$$

hence m is finite.

Corollary 1. Let X be a Levy process with increasing paths and let $\Delta X_t < 1$ for all $t \geq 0$. Assume that L_∞ is an integrable F_∞^W-measurable random variable and the processes W and X are independent.

Then the process

$$L_t = E(L_\infty | F_t^W)[1 - \mathcal{E}_t(-X)] \quad t \geq 0$$

is the unique solution of the BSDE

$$Y_t = (\alpha + m) \int_0^t Y_s \frac{\mathcal{E}_s(-X)}{(1 - \mathcal{E}_s(-X))} ds + \int_0^t Z_s dW_s +$$

$$+ \int_0^t \int_{R_+} K(s,x)(\mu - \tilde{\mu})(dx, ds), \tag{25}$$

with the boundary condition

$$Y_\infty = \lim_{t \to \infty} Y_t = L_\infty. \tag{26}$$

Proof. It follows from (17), that the purely discontinuous martingale part of the martingale $L_t / 1 - \mathcal{E}_t(-X)$ is equal to

$$\int_0^t \int_{R_+} \frac{1}{1 - \mathcal{E}_{s-}(-X)} K(s,x)(\mu - \tilde{\mu})(ds, dx) - \int_0^t \int_{R_+} L_{s-} \frac{H(s,x)}{1 - \mathcal{E}_{s-}(-X)} (\mu - \tilde{\mu})(ds, dx)$$

$$- \int_0^t \int_{R_+} K(s,x) \frac{H(s,x)}{1 - \mathcal{E}_{s-}(-X)} (\mu - \tilde{\mu})(ds, dx).$$

Since $L_t/(1 - \mathcal{E}_t(-X)) = E(L_\infty | F_t^W)$ is a continuous martingale, we have that $\nu(dx)ds$ - a.e.

$$\frac{1}{1 - \mathcal{E}_{s-}(-X)} K(s,x) - L_{s-} \frac{H(s,x)}{1 - \mathcal{E}_{s-}(-X)} - K(s,x) \frac{H(s,x)}{1 - \mathcal{E}_{s-}(-X)} = 0.$$

This implies that

$$K(s,x) = L_s \frac{H(s,x)}{1 - H(s,x)}, \qquad \nu(dx)ds - \ a.e. \tag{27}$$

and substituting this expression in (20), we obtain that

$$A_t = \int_0^t \int_{R_+} L_s \frac{H(s,x)}{1-H(s,x)} \nu(dx)ds + \alpha \int_0^t L_s \frac{\mathcal{E}_s(-X)}{1-\mathcal{E}_s(-X)} ds. \tag{28}$$

By definition of H

$$\frac{H(s,x)}{1-H(s,x)} = \frac{x\mathcal{E}(-X)}{1-\mathcal{E}(-X)},$$

therefore

$$A_t = \int_0^t \int_{R_+} L_s \frac{x\mathcal{E}_s(-X)}{1-\mathcal{E}_s(-X)} \nu(dx)ds + \alpha \int_0^t L_s \frac{\mathcal{E}_s(-X)}{1-\mathcal{E}_s(-X)} ds =$$

$$= (\alpha + m) \int_0^t L_s \frac{\mathcal{E}_s(-X)}{1-\mathcal{E}_s(-X)} ds$$

which implies that L_t satisfies (25).

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Stochastic Volatility Model with Small Randomness. Construction of CULAN Estimators

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Abstract

CULAN (consistent uniformly linear asymptotically normal) estimators is one of the most important class of estimators in robust statistics. Construction of such estimators for stochastic volatility model with small randomness is a goal of the present paper.

Key words and phrases: Stochastic volatility, small randomness, CULAN estimators.

MSC 2010: 62F12, 62F35.

1 Introduction

Consider the stochastic volatility model described by the following system of SDE:

$$\begin{aligned} dX_t &= X_t\, dR_t, \quad X_0 > 0, \\ dR_t &= \mu_t(R_t, Y_t)\, dt + \sigma_t\, dw_t^R, \quad R_0 = 0, \\ \sigma_t^2 &= f(Y_t), \\ dY_t &= a(t, Y_t; \alpha)\, dt + \varepsilon\, dw_t^\sigma, \quad Y_0 = 0, \end{aligned} \tag{1.1}$$

where $w = (w^R, w^\sigma)$ is a standard two-dimensional Wiener process, defined on complete probability space $(\Omega, \mathcal{F}, P)$, $F^w = (\mathcal{F}_t^w)_{0\le t\le T}$ is the P-augmentation of the natural filtration $\mathcal{F}_t^w = \sigma(w_s,\ 0 \le s \le t)$, $0 \le t \le T$, generated by w, $f(\cdot)$ is a continuous one-to-one positive locally bounded function (e.g., $f(x) = e^x$), $\alpha = (\alpha_1, \dots, \alpha_m)$, $m \ge 1$, is a vector of unknown parameters, and ε, $0 < \varepsilon < 1$, is a small number. Assume that the system (1.1) has an unique strong solution.

Suppose that the sample path $(y_s)_{0\le s\le t}$ comes from the observations of process $(\widetilde{Y}_s)_{0\le s\le t}$ with distribution $\widetilde{P}_\alpha^\varepsilon$ from the shrinking contamination neighborhood of the distribution P_α^ε of the basic process $Y = (Y_s)_{0\le s\le t}$. That is,

$$\frac{d\widetilde{P}_\alpha^\varepsilon}{dP_\alpha^\varepsilon} \mid \mathcal{F}_t^w = \mathcal{E}_t(\varepsilon N^\varepsilon), \tag{1.2}$$

where $N^\varepsilon = (N_s^\varepsilon)_{0\le s\le t}$ is a P_α^ε-square integrable martingale, $\mathcal{E}_t(M)$ is the Dolean exponential of martingale M.

In the diffusion-type processes framework (1.2) represents the Huber gross error model (as it explain in Remark 2.3). The model of type (1.2) of contamination of measures for statistical models with filtration was suggested by Lazrieva and Toronjadze [1].

In Section 2, we study the problem of construction of robust estimators for contamination model (1.2).

In subsection 2.1, we give a description of the basic model and definition of consistent uniformly linear asymptotically normal (CULAN) estimators, connected with basic model (Definition 2.1).

In subsection 2.2, we introduce a notion of shrinking contamination neighborhood, described in terms of contamination of nominal distributions, which naturally leads to the class of alternative measures (see (2.18) and (2.19)).

In the same subsection, we study the asymptotic behaviour of CULAN estimators under alternative measures (Proposition 2.2), which is the basis for the formulation of the optimization problem.

In subsection 2.3, the optimization problem is solved which leads to construction of optimal B-robust estimator (Theorem 2.1).

2 Construction of CULAN estimators

2.1 Basic model

The basic model of observations is described by the SDE

$$dY_s = a(s, Y; \alpha)\, ds + \varepsilon\, dw_s, \quad Y_0 = 0, \quad 0 \le s \le t, \tag{2.1}$$

where t is a fixed number, $w = (w_s)_{0\le s\le t}$ is a standard Wiener process defined on the filtered probability space $(\Omega, \mathcal{F}, F = (\mathcal{F}_s)_{0\le s\le t}, P)$ satisfying the usual conditions, $\alpha = (\alpha_1, \dots, \alpha_m)$, $m \ge 1$, is an unknown parameter to be estimated, $\alpha \in \mathcal{A} \subset R^m$, $\mathcal{A}$ is an open subset of R^m, ε, $0 < \varepsilon \ll 1$, is small parameter (index of series). In our further considerations all limits correspond to $\varepsilon \to 0$.

Denote by $(C_t, \mathcal{B}_t)$ a measurable space of continuous on $[0, t]$ functions $x = (x_s)_{0\le s\le t}$ with σ-algebra $\mathcal{B}_t = \sigma(x : x_s,\ s \le t)$. Put $\mathcal{B}_s = \sigma(x : x_u,\ u \le s)$.

Assume that for each $\alpha \in \mathcal{A}$, the drift coefficient $a(s, x; \alpha)$, $0 \le s \le t$, $x \in C_t$ is a known nonanticipative (i.e., $\mathcal{B}_s$-measurable for each s, $0 \le s \le t$) functional satisfying the functional Lipshitz and linear growth conditions **L**:

L
$$|a(s, x^1; \alpha) - a(s, x^2; \alpha)| \le L_1 \int_0^s |x_u^1 - x_u^2|\, dk_u + L_2 |x_s^1 - x_s^2|,$$
$$|a(s, x; \alpha)| \le L_1 \int_0^s (1 + |x_u|)\, dk_u + L_2 (1 + |x_s|),$$

where L_1 and L_2 are constants, which do not depend on α, $k = (k(s))_{0\le s\le t}$ is a non-decreasing right-continuous function, $0 \le k(s) \le k_0$, $0 < k_0 < \infty$, $x^1, x^2 \in C_t$.

Then, as it is well known (see, e.g., Lipster and Shiryaev [2]), for each $\alpha \in \mathcal{A}$, the equation (2.1) has an unique strong solution $Y^\varepsilon(\alpha) = (Y_s^\varepsilon(\alpha))_{0\le s\le t}$ and, in addition (see Kutoyants [3]),

$$\sup_{0\le s\le t} |Y_s^\varepsilon(\alpha) - Y_s^0(\alpha)| \le C\varepsilon \sup_{0\le s\le t} |w_s| \quad P\text{-a.s.},$$

with some constant $C = C(L_1, L_2, k_0, t)$, where $Y^0(\alpha) = (Y_s^0(\alpha))_{0\le s\le t}$ is the solution of the following nonperturbated differential equation

$$dY_s = a(s, Y; \alpha)\, ds, \quad Y_0 = 0. \tag{2.2}$$

Change of initial problem of estimation of parameter α by the equivalent one, when the observations are modelled according to the following SDE

$$dX_s = a_\varepsilon(s, X; \alpha)\, ds + dw_s, \quad X_0 = 0, \tag{2.3}$$

where $a_\varepsilon(s, x; \alpha) = \frac{1}{\varepsilon} a(s, \varepsilon x; \alpha)$, $0 \le s \le t$, $x \in C_t$, $\alpha \in \mathcal{A}$.

It it clear that if $X^\varepsilon(\alpha) = (X_s^\varepsilon(\alpha))_{0 \le s \le t}$ is the solution os SDE (2.3), then for each $s \in [0, t]$, $\varepsilon X_s^\varepsilon(\alpha) = Y_s^\varepsilon(\alpha)$.

Denote by P_α^ε the distribution of process $X^\varepsilon(\alpha)$ on the space $(C_t, \mathcal{B}_t)$,i.e., P_α^ε is the probability measure on $(C_t, \mathcal{B}_t)$, induced by the process $X^\varepsilon(\alpha)$. Let P^w be a Wiener measure on $(C_t, \mathcal{B}_t)$. Denote $X = (X_s)_{0 \le s \le t}$ a coordinate process on $(C_t, \mathcal{B}_t)$, that is, $X_s(x) = x_s$, $x \in C_t$.

The conditions $\mathbf{L}$ guarantee that for each $\alpha \in \mathcal{A}$, the measures P_α^ε and P^w are equivalent ($P_\alpha^\varepsilon \sim P^w$), and if we denote $z_s^{\alpha,\varepsilon} = \frac{dP_\alpha^\varepsilon}{dP^w} \mid \mathcal{B}_s$ the density process (likelihood ratio process), then

$$z_s^{\alpha,\varepsilon}(X) = \mathcal{E}_s(a_\varepsilon(\alpha) \cdot X) := \exp\left\{ \int_0^s a_\varepsilon(u, X; \alpha)\, dX_u - \frac{1}{2} \int_0^s a_\varepsilon^2(u, X; \alpha)\, du \right\}.$$

Introduce a class Ψ of R^m-valued nonanticipative functionals ψ, $\psi : [0, t] \times C_t \times \mathcal{A} \to R^m$ such that for each $\alpha \in \mathcal{A}$ and $\varepsilon > 0$,

$$1) \quad E_\alpha \int_0^t |\psi(s, X; \alpha)|^2 ds < \infty, \tag{2.4}$$

$$2) \quad \int_0^t |\psi(s, Y^0(\alpha); \alpha)|^2 ds < \infty, \tag{2.5}$$

3) uniformly in α on each compact $K \subset \mathcal{A}$,

$$P_\alpha^\varepsilon - \lim_{\varepsilon \to 0} \int_0^t |\psi(s, \varepsilon X; \alpha) - \psi(s, Y^0(\alpha); \alpha)|^2 ds = 0, \tag{2.6}$$

where $|\cdot|$ is an Euclidean norm in R^m, $P_\alpha^\varepsilon - \lim_{\varepsilon \to 0} \zeta_\varepsilon = \zeta$ denotes the convergence $P_\alpha^\varepsilon\{|\zeta_\varepsilon - \zeta| > \rho\} \to 0$, as $\varepsilon \to 0$, for all ρ, $\rho > 0$.

Assume that for each $s \in [0, t]$ and $x \in C_t$, the functional $a(s, x; \alpha)$is differentiable in α and gradient $\dot{a} = \left(\frac{\partial}{\partial \alpha_1} a, \ldots, \frac{\partial}{\partial \alpha_m} a\right)'$ belongs to Ψ ($\dot{a} \in \Psi$), where the sign "$'$" denotes a transposition.

Then the Fisher information process

$$I_s^\varepsilon(X; \alpha) := \int_0^s \dot{a}_\varepsilon(u, X; \alpha)[\dot{a}_\varepsilon(u, X; \alpha)]' du, \quad 0 \le s \le t,$$

is well-defined, and moreover, uniformly in α on each compact,

$$P_\alpha^\varepsilon - \lim_{\varepsilon\to 0} \varepsilon^2 I_t^\varepsilon(\alpha) = I_t^0(\alpha), \tag{2.7}$$

where

$$I_t^0(\alpha) := \int_0^t \dot{a}(s, Y^0(\alpha); \alpha)[\dot{a}(s, Y^0(\alpha); \alpha)]' \, ds.$$

For each $\psi \in \Psi$, introduce the functional $\psi_\varepsilon(s, x; \alpha) := \frac{1}{\varepsilon}\psi(s, \varepsilon x; \alpha)$ and matrices $\Gamma_{t\varepsilon}^\psi(\alpha)$ and $\gamma_{t\varepsilon}^\psi(\alpha)$:

$$\Gamma_{t\varepsilon}^\psi(X, \alpha) : = \int_0^t \psi_\varepsilon(s, X; \alpha)[\psi_\varepsilon(s, X; \alpha)]' \, ds, \tag{2.8}$$

$$\gamma_{t\varepsilon}^\psi(X, \alpha) : = \int_0^t \psi_\varepsilon(s, X; \alpha)[\dot{a}_\varepsilon(s, X; \alpha)]' \, ds. \tag{2.9}$$

Then from (2.6) it follows that uniformly in α on each compact,

$$P_\alpha^\varepsilon - \lim_{\varepsilon\to 0} \varepsilon^2 \Gamma_{t\varepsilon}^\psi(\alpha) = \Gamma_{t0}^\psi(\alpha), \tag{2.10}$$

$$P_\alpha^\varepsilon - \lim_{\varepsilon\to 0} \varepsilon^2 \gamma_{t\varepsilon}^\psi(\alpha) = \gamma_{t0}^\psi(\alpha), \tag{2.11}$$

where the matrices $\Gamma_{t0}^\psi(\alpha)$ and $\gamma_{t0}^\psi(\alpha)$ are defined as follows:

$$\Gamma_{t0}^\psi(\alpha) = \int_0^t \psi(s, Y^0(\alpha); \alpha)[\psi(s, Y^0(\alpha); \alpha)]' \, ds, \tag{2.12}$$

$$\gamma_{t0}^\psi(\alpha) = \int_0^t \psi(s, Y^0(\alpha); \alpha)[\dot{a}(s, Y^0(\alpha); \alpha)]' \, ds. \tag{2.13}$$

Note that, by virtue of (2.4), (2.5) and $\dot{a} \in \Psi$, matrices given by (2.8), (2.9), (2.12) and (2.13) are well defined.

Denote by Ψ_0 the subset of Ψ such that for each $\psi \in \Psi_0$ and $\alpha \in \mathcal{A}$, $\operatorname{rank} \Gamma_{t0}^\psi(\alpha) = m$ and $\operatorname{rank} \gamma_{t0}^\psi(\alpha) = m$.

Assume that $\dot{a} \in \Psi_0$.

For each $\psi \in \Psi_0$, define a P_α^ε-square integrable martingale $L_t^{\psi,\varepsilon}(\alpha)$ as follows:

$$L_t^{\psi,\varepsilon}(X; \alpha) := \int_0^t \psi_\varepsilon(u, X; \alpha)(dX_u - a_\varepsilon(u, X; \alpha)du). \tag{2.14}$$

Now we give a definition of CULAN M-estimators.

Definition 2.1. An estimator $(\alpha_t^{\psi,\varepsilon})_{\varepsilon>0} = (\alpha_{1,t}^{\psi,\varepsilon}, \dots, \alpha_{m,t}^{\psi,\varepsilon})'_{\varepsilon>0}$, $\psi \in \Psi_0$, is called consistent uniformly linear asymptotically normal (CULAN) if it admits the following expansion:

$$\alpha_t^{\psi,\varepsilon} = \alpha + [\gamma_{t0}^{\psi}(\alpha)]^{-1}\varepsilon^2 L_t^{\psi,\varepsilon}(\alpha) + r_{t\varepsilon}^{\psi}(\alpha), \tag{2.15}$$

where uniformly in α on each compact,

$$P_\alpha^\varepsilon - \lim_{\varepsilon\to 0} \varepsilon^{-1} r_{t\varepsilon}^{\psi}(\alpha) = 0. \tag{2.16}$$

It is well known (see Kutoyants [3]) that under the above conditions, uniformly in α on each compact,

$$\mathcal{L}\{\varepsilon^{-1}(\alpha_t^{\psi,\varepsilon} - \alpha) \mid P_\alpha^\varepsilon\} \xrightarrow{w} N(0, V_t(\psi;\alpha)),$$

with

$$V_t(\psi;\alpha) := [\gamma_{t0}^{\psi}(\alpha)]^{-1}\Gamma_{t0}^{\psi}(\alpha)([\gamma_{t0}^{\psi}(\alpha)]^{-1})', \tag{2.17}$$

where $\mathcal{L}(\zeta \mid P)$ denotes the distribution of random vector ζ, calculated under measure P, symbol "$\xrightarrow{w}$" denotes the weak convergence of measures, $N(0, V_t(\psi;\alpha))$ is a distribution of Gaussian vector with zero mean and covariance matrix $V_t(\psi;\alpha)$.

Remark 2.1. In context of diffusion type processes, the M-estimator $(\alpha_t^{\psi,\varepsilon})_{\varepsilon>0}$ is defined as a solution of the following stochastic equation:

$$L_t^{\psi,\varepsilon}(X;\alpha) = 0,$$

where $L_t^{\psi,\varepsilon}(X;\alpha)$ is defined by (2.14), $\psi \in \Psi_0$.

The asymptotic theory of M-estimators for general statistical models with filtration is developed in Toronjadze [4]. Namely, the problem of existence and global asymptotic behaviour of solutions is studied. In particular, the conditions of regularity and ergodicity type are established under which M-estimators have a CULAN property.

For our model, in case when $\mathcal{A} = R^m$, the sufficient conditions for CULAN property take the form:

(1) for all s, $0 \le s \le t$, and $x \in C_t$, the functionals $\psi(s,x;\alpha)$ and $\dot{a}(s,x;\alpha)$ are twice continuously differentiable in α with bounded derivatives satisfying the functional Lipshitz conditions with constants, which do not depend on α.

(2) the equation (w.r.t. y)

$$\Delta_t(\alpha, y) := \int_0^t \psi(s, Y^0(\alpha); y)(a(s, Y^0(\alpha); \alpha) - a(s, Y^0(\alpha); y))\, ds = 0$$

has a unique solution $y = \alpha$.

The MLE is a special case of M-estimators when $\psi = \dot{a}$.

Remark 2.2. According to (2.7), the asymptotic covariance matrix of MLE $(\widehat{a}_t^\varepsilon)_{\varepsilon>0}$ is $[I_t^0(\alpha)]^{-1}$. By the usual technique one can show that for each $\alpha \in \mathcal{A}$ and $\psi \in \Psi_0$, $[I_t^0(\alpha)]^{-1} \le V_t(\psi; \alpha)$, see (2.17), where for two symmetric matrices B and C the relation $B \le C$ means that the matrix $C - B$ is nonnegative definite.

Thus, the MLE has a minimal covariance matrix among all M-estimators.

2.2 Shrinking contamination neighborhoods

In this subsection, we give a notion of a contamination of the basic model (2.3), described in terms of shrinking neighborhoods of basic measures $\{P_\alpha^\varepsilon, \alpha \in \mathcal{A}, \varepsilon > 0\}$, which is an analog of the Huber gross error model (see, e.g., Hampel et al. [5] and, also, Remark 2.3 below).

Let $\mathcal{H}$ be a family of bounded nonanticipative functional $h : [0, t] \times C_t \times \mathcal{A} \to R^1$ such that for all $s \in [0, t]$ and $\alpha \in \mathcal{A}$, the functional $h(s, x; \alpha)$ is continuous at the point $x_0 = Y^0(\alpha)$.

Let for each $h \in \mathcal{H}$, $\alpha \in \mathcal{A}$ and $\varepsilon > 0$, $P_\alpha^{\varepsilon,h}$ be a measure on $(C_t, \mathcal{B}_t)$ such that

$$\begin{aligned} &1) \quad P_\alpha^{\varepsilon,h} \sim P_\alpha^\varepsilon, \\ &2) \quad \frac{dP_\alpha^{\varepsilon,h}}{dP_\alpha^\varepsilon} = \mathcal{E}_t(\varepsilon N_\alpha^{\varepsilon,h}) \end{aligned} \tag{2.18}$$

where

$$3) \quad N_{\alpha,s}^{\varepsilon,h} := \int_0^s h_\varepsilon(u, X; \alpha)(dX_u - a_\varepsilon(u, X; \alpha)du), \tag{2.19}$$

with $h_\varepsilon(s, x; \alpha) := \frac{1}{\varepsilon} h(s, \varepsilon x; \alpha)$, $0 \le s \le t$, $x \in C_t$.

Denote by $\mathbf{P}_\alpha^{\varepsilon,\mathcal{H}}$ a class of measures $P_\alpha^{\varepsilon,h}$, $h \in \mathcal{H}$, that is,

$$\mathbf{P}_\alpha^{\varepsilon,\mathcal{H}} = \{P_\alpha^{\varepsilon,h};\ h \in \mathcal{H}\}.$$

We call $(\mathbf{P}_\alpha^{\varepsilon,\mathcal{H}})_{\varepsilon>0}$ a shrinking contamination neighborhoods of the basic measures $(P_\alpha^\varepsilon)_{\varepsilon>0}$, and the element $(P_\alpha^{\varepsilon,h})_{\varepsilon>0}$ of these neighborhoods are called alternative measures (or, simply, alternative).

Obviously, for each $h \in \mathcal{H}$ and $\alpha \in \mathcal{A}$, the process $N_\alpha^{\varepsilon,h} = (N_{\alpha,s}^{\varepsilon,h})_{0\le s\le t}$ defined by (2.19) is a P_α^ε-square integrable martingale. Since under measure P_α^ε the process $\overline{w} = (\overline{w}_s)_{0\le s\le t}$ defined as

$$\overline{w}_s := X_s - \int_0^s a_\varepsilon(u, X; \alpha)\, du, \quad 0 \le s \le t,$$

is a Wiener process, by virtue of the Girsanov Theorem the process $\widetilde{w} := \overline{w} + \langle \overline{w}, \varepsilon N_\alpha^{\varepsilon,h} \rangle$ is a Wiener process under changed measure $P_\alpha^{\varepsilon,h}$. But by the definition,

$$\widetilde{w}_s = X_s - \int_0^s (a_\varepsilon(u, X; \alpha) + \varepsilon h_\varepsilon(u; X; \alpha))\, du,$$

and hence one can conclude that $P_\alpha^{\varepsilon,h}$ is a weak solution of SDE

$$dX_s = (a_\varepsilon(s, X; \alpha) + \varepsilon h_\varepsilon(s, X; \alpha))ds + dw_s, \quad X_0 = 0.$$

This SDE can be viewed as a “small” perturbation of the basic model (2.3).

Remark 2.3. 1) In the case of i.i.d. observations $X_1, X_2, \ldots, X_n$, $n \ge 1$, the Huber gross error model in shrinking setting is defined as follows:

$$f^{n,h}(x; \alpha) := (1 - \varepsilon_n) f(x; \alpha) + \varepsilon_n h(x; \alpha),$$

where $f(x, \alpha)$ is a basic (core) density of distribution of r.v. X_i (w.r.t. some dominating measure μ), $h(x, \alpha)$ is a contaminating density, $f^{n,h}(x; \alpha)$ is a contaminated density, $\varepsilon_n = O(n^{-1/2})$. If we denote by P_α^n and $P_\alpha^{n,h}$ the measures on $(R^n, \mathcal{B}(R^n))$, generated by $f(x; \alpha)$ and $f^{n,h}(x; \alpha)$, respectively, then

$$\frac{dP_\alpha^{n,h}}{dP_\alpha^n} = \prod_{i=1}^n \frac{f^{n,h}(X_i; \alpha)}{f(X_i; \alpha)} = \prod_{i=1}^n (1 + \varepsilon_n H(X_i; \alpha)) = \mathcal{E}_n(\varepsilon_n \cdot N_\alpha^{n,h}),$$

where $H = \frac{h-f}{f}$, $N_\alpha^{n,h} = (N_{\alpha,m}^{n,h})_{1\le m\le n}$, $N_{\alpha,m}^{n,h} = \sum_{i=1}^m H(X_i; \alpha)$, $N_\alpha^{n,h}$ is a P_α^n-martingale, $\mathcal{E}_n(\varepsilon_n N_\alpha^{n,h}) = \prod_{i=1}^n (1 + \varepsilon_n \Delta N_{\alpha,i}^{n,h})$ is the Dolean exponential in discrete time case.

Thus

$$\frac{dP_\alpha^{n,h}}{dP_\alpha^n} = \mathcal{E}_n(\varepsilon_n \cdot N_\alpha^{n,n}) \tag{2.20}$$

and the relation (2.18) is a direct analog of (2.20).

2) The concept of shrinking contamination neighborhoods, expressed in the form of (2.18), was proposed in Lazrieva and Toronjadze [1] for more general situation, concerning with the contamination areas for semimartingale statistical models with filtration. □

In the remainder of this subsection, we study the asymptotic properties of CULAN estimators under alternatives.

For this aim, we first consider the problem of contiguity of measures $(P_\alpha^{\varepsilon,h})_{\varepsilon>0}$ to $(P_\alpha^\varepsilon)_{\varepsilon>0}$.

Let $(\varepsilon_n)_{n\geq 1}$, $\varepsilon_n \downarrow 0$, and $(\alpha_n)_{n\geq 1}$, $\alpha_n \in K$, $K \subset \mathcal{A}$ is a compact, be arbitrary sequences.

Proposition 2.1. *For each $h \in \mathcal{H}$, the sequence of measures $(P_{\alpha_n}^{\varepsilon_n,h})$ is contiguous to sequence of measures $(P_{\alpha_n}^{\varepsilon_n})$, i.e.,*

$$(P_{\alpha_n}^{\varepsilon_n,h}) \lhd (P_{\alpha_n}^{\varepsilon_n}).$$

Proof. From the predictable criteria of contiguity (see, e.g., Jacod and Shiryaev [6]), it follows that we have to verify the relation

$$\lim_{N\to\infty} \limsup_{n\to\infty} P_{\alpha_n}^{\varepsilon_n,h}\Big\{h_t^n\Big(\frac{1}{2}\Big) > N\Big\} = 0, \tag{2.21}$$

where $h_t^n(\frac{1}{2}) = (h_s^n(\frac{1}{2}))_{o\leq s\leq t}$ is the Hellinger process of order $\frac{1}{2}$.

By the definition of Hellinger process (see, e.g., Jacod and Shiryaev [6]), we have

$$h_t^n\Big(\frac{1}{2}\Big) = h_t^n\Big(\frac{1}{2}, P_{\alpha_n}^{\varepsilon_n,h}, P_{\alpha_n}^{\varepsilon_n}\Big) = \frac{1}{8}\int_0^t [h(s,\varepsilon_n X;\alpha_n)]^2 ds,$$

and since $h \in \mathcal{H}$, and hence is bounded, $h_t^n(\frac{1}{2})$ is bounded too, which provides (2.21). □

Proposition 2.2. *For each estimator $(\alpha_t^{\varepsilon,\psi})_{\varepsilon>0}$ with $\psi \in \Psi_0$ and each alternative $(P_\alpha^{\varepsilon,h})_{\varepsilon>0} \in (\mathbf{P}_\alpha^{\varepsilon,h})_{\varepsilon>0}$, the following relation holds true:*

$$\mathcal{L}\{\varepsilon^{-1}(\alpha_t^{\psi,\varepsilon} - \alpha) \mid P_\alpha^{\varepsilon,h}\} \xrightarrow{w} N([\gamma_{t0}^\psi(\alpha)]^{-1} b_t(\psi,h;\alpha), V_t(\psi;\alpha)),$$

where

$$b_t(\psi,h;\alpha) := \int_0^t \psi(s,Y^0(\alpha);\alpha)h(s,Y^0(\alpha);\alpha)\,ds.$$

Proof. Proposition 2.1 together with (2.16) provides that uniformly in α on each compact,

$$P_\alpha^{\varepsilon,h} - \lim_{\varepsilon\to 0} \varepsilon^{-1} r_{t\varepsilon}^{\psi}(\alpha) = 0,$$

and therefore we have to establish the limit distribution of random vector $[\gamma_{t0}^{\psi}(\alpha)]^{-1}\varepsilon L_t^{\psi,\varepsilon}$ under the measures $(P_\alpha^{\varepsilon,h})_{\varepsilon>0}$.

By virtue of the Girsanov Theorem, the process $L_t^{\psi,\varepsilon}(\alpha) = (L_s^{\psi,\varepsilon}(\alpha))_{0\le s\le t}$ is a semimartingale with canonical decomposition

$$L_s^{\psi,\varepsilon}(\alpha) = \widetilde{L}_s^{\psi,\varepsilon}(\alpha) + b_{\varepsilon,s}(\psi,h;\alpha), \quad 0 \le s \le t, \tag{2.22}$$

where $\widetilde{L}_t^{\psi,\varepsilon}(\alpha) = (\widetilde{L}_s^{\psi,\varepsilon}(\alpha))_{0\le s\le t}$ is a $P_\alpha^{\varepsilon,h}$-square integrable martingale, defined as follows:

$$\widetilde{L}_s^{\psi,\varepsilon}(X;\alpha) := \int_0^s \psi_\varepsilon(u,X;\alpha)(dX_u - (a_\varepsilon(u,X;\alpha) + \varepsilon h_\varepsilon(u,X;\alpha))du),$$

and

$$b_{\varepsilon,s}(\psi,h;\alpha) := \varepsilon \int_0^s \psi_\varepsilon(u,X;\alpha) h_\varepsilon(u,X;\alpha)\, du.$$

But $\langle \widetilde{L}^{\psi,\varepsilon}(\alpha)\rangle_t = \Gamma_{t\varepsilon}^{\psi}(\alpha)$, where $\Gamma_{t\varepsilon}^{\psi}(\alpha)$ is defined by (2.8). On the other hand, from Proposition 2.1 and (2.10) it follows that

$$P_\alpha^{\varepsilon,h} - \lim_{\varepsilon\to 0}\langle \varepsilon \widetilde{L}^{\psi,\varepsilon}(\alpha)\rangle_t = P_\alpha^{\varepsilon,h} - \lim_{\varepsilon\to 0} \varepsilon^2 \Gamma_{t\varepsilon}^{\psi}(\alpha) = P_\alpha^{\varepsilon} - \lim_{\varepsilon\to 0} \varepsilon^2 \Gamma_{t\varepsilon}^{\psi}(\alpha) = \Gamma_{t0}^{\psi}(\alpha)$$

uniformly in α on each compact, and hence

$$\mathcal{L}\{([\gamma_{t0}^{\psi}(\alpha)]^{-1}\varepsilon \widetilde{L}_t^{\psi,\varepsilon} \mid P_\alpha^{\varepsilon,h}\} \xrightarrow{w} N(0, V_t(\psi;\alpha)). \tag{2.23}$$

Finally, relation (2.23) together with (2.22) and relation

$$P_\theta^{\varepsilon,h} - \lim_{\varepsilon\to 0} \varepsilon b_{\varepsilon,t}(\psi,h;\alpha) = \int_0^t \psi(s,Y^0(\alpha);\alpha) h(s,Y^0(\alpha);\alpha) ds = b_t(\psi,h;\alpha)$$

provides the desirable result. □

2.3 Optimization criteria. Construction of optimal B-robust estimators

In this subsection, we state and solve an optimization problem, which results in construction of optimal B-robust estimator.

Initially, it should be stressed that the bias vector $\widetilde{b}_t(\psi, h; \alpha) := [\gamma_{t0}^{\psi}(\alpha)]^{-1}$ $b_t(\psi, h; \alpha)$ can be viewed as the influence functional of the estimator $(\alpha_t^{\psi,\varepsilon})_{\varepsilon>0}$ w.r.t. alternative $(P_\alpha^{\psi,h})_{\varepsilon>0}$.

Indeed, the expansion (2.15) together with (2.22) and (2.23) allows to conclude that

$$\mathcal{L}\{\varepsilon^{-1}(a_t^{\psi,\varepsilon} - \alpha - \varepsilon^2[\gamma_{t0}^{\psi}(\alpha)]^{-1}b_{\varepsilon t}(\psi, h; \alpha)) \mid P_\alpha^{\varepsilon,h}\} \xrightarrow{w} N(0, V_t(\psi; \alpha)),$$

and hence, the expression

$$\alpha + \varepsilon^2[\gamma_{t0}^{\psi}(\alpha)]^{-1}b_{\varepsilon t}(\psi, h; \alpha) - \alpha = \varepsilon^2[\gamma_{t0}^{\psi}(\varepsilon)]^{-1}b_{\varepsilon t}(\psi, h; \alpha)$$

plays the role of bias on the "fixed step ε" and it seems natural to interpret the limit

$$P_\alpha^{\varepsilon,h} - \lim_{\varepsilon\to 0} \frac{\alpha + \varepsilon^2[\gamma_{t0}^{\psi}(\alpha)]^{-1}b_{\varepsilon t}(\psi, h; \alpha) - \alpha}{\varepsilon} = [\gamma_{t0}^{\psi}(\varepsilon)]^{-1}b_{\varepsilon t}(\psi, h; \alpha)$$

as the influence functional.

For each estimator $(a_t^{\psi,\varepsilon})_{\varepsilon>0}$, $\psi \in \Psi_0$, defined the risk functional w.r.t. alternative $(P_\alpha^{\varepsilon,h})_{\varepsilon>0}$, $h \in \mathcal{H}$, as follows:

$$D_t(\psi, h; \alpha) = \lim_{a\to\infty} \lim_{\varepsilon\to 0} E_\alpha^{\varepsilon,h}((\varepsilon^{-2}|\alpha_t^{\psi,\varepsilon} - \alpha|^2) \wedge a),$$

where $x \wedge a = \min(x, a)$, $a > 0$, $E_\alpha^{\varepsilon,h}$ is an expectation w.r.t. measures $P_\alpha^{\varepsilon,h}$.

Using Proposition 2.2 it is not hard to verify that

$$D_t(\psi, h; \alpha) = |\widetilde{b}_t(\psi, h; \alpha)|^2 + \operatorname{tr} V_t(\psi; \alpha),$$

where $\operatorname{tr} A$ denotes the trace of matrix A.

Connect with each $\psi \in \Psi_0$ the function $\widetilde{\psi}$ as follows:

$$\widetilde{\psi}(t, x; \alpha) = [\gamma_{t0}^{\psi}(\alpha)]^{-1}\psi(t, x; \alpha).$$

Then $\widetilde{\psi} \in \Psi_0$ and

$$\gamma_{t0}^{\widetilde{\psi}}(\alpha) = Id,$$

where Id is a unit matrix,

$$V_t(\psi; \alpha) = V_t(\widetilde{\psi}; \alpha) = \Gamma_{t0}^{\widetilde{\psi}}(\alpha), \quad \widetilde{b}_t(\psi, h; \alpha) = \widetilde{b}_t(\widetilde{\psi}, h; \alpha) = b_t(\widetilde{\psi}, h; \alpha).$$

Therefore

$$D_t(\psi,h;\alpha) = D_t(\widetilde{\psi},h;\alpha) = |b_t(\widetilde{\psi},h;\alpha)|^2 + \operatorname{tr}\Gamma_{t0}^{\widetilde{\psi}}(\alpha). \tag{2.24}$$

Denote by $\mathcal{H}_r$ a set of functions $h \in \mathcal{H}$ such that for each $\alpha \in \mathcal{A}$,

$$\int_0^t |h(x,Y^0(\alpha);\alpha)|\,ds \le r,$$

where r, $r > 0$, is a constant.

Since, for each $r > 0$,

$$\sup_{h\in\mathcal{H}_r} |b_t(\widetilde{\psi},h;\alpha)| \le \text{const.}(r) \sup_{0\le s\le t} |\widetilde{\psi}_t(s,Y^0(\alpha);\alpha)|,$$

where constant depends on r, we call the function $\widetilde{\psi}$ an influence function of estimator $(\alpha_t^{\psi,\varepsilon})_{\varepsilon>0}$ and a quantity

$$\gamma_{t\psi}^*(\alpha) = \sup_{0\le s\le t} |\widetilde{\psi}(s,Y^0(\alpha);\alpha)|$$

is named as the (unstandardized) gross error sensitivity at point α of estimator $(\alpha_t^{\psi,\varepsilon})_{\varepsilon>0}$.

Define

$$\Psi_{0,c} = \Bigg\{\psi \in \Psi_0 : \int_0^t \psi(s,Y^0(\alpha);\alpha)[\dot{a}(s,Y^0(\alpha);\alpha)]'ds = Id, \tag{2.25}$$

$$\gamma_{t\psi}^*(\alpha) \le c\Bigg\}, \tag{2.26}$$

where $c \in [0,\infty)$ is a generic constant.

Taking into account the expression (2.24) for the risk functions, we come to the following optimization problem, known in robust estimation theory as Hampel's optimization problem: minimize the trace of the asymptotic covariance matrix of estimator $(\alpha_t^{\psi,\varepsilon})_{\varepsilon>0}$ over the class $\Psi_{0,c}$, that is,

$$\text{minimize} \quad \int_0^t \psi(s,Y^0(\alpha);\alpha)[\psi(s,Y^0(\alpha);\alpha)]'ds \tag{2.27}$$

under the side conditions (2.25) and (2.26).

Define the Huber function $h_c(z)$, $z \in R^m$, $c > 0$, as follows:

$$h_c(z) := z\min\left(1, \frac{c}{|z|}\right).$$

For arbitrary nondegenerate matrix A denote $\psi_c^A = h_c(Aa)$.

Theorem 2.1. *Assume that for given constant c there exists a nondegenerate $m \times m$ matrix $A_c^*(\alpha)$, which solves the equation (w.r.t. matrix A)*

$$\int_0^t \psi_c^A(s, Y^0(\alpha); \alpha)[\dot{a}(s, Y^0(\alpha)\alpha)]' ds = Id. \tag{2.28}$$

Then the function $\psi_c^{A_c^*(\alpha)} = h_c(A_c^*(\alpha)\dot{a})$ solves the optimization problem (2.27).

Proof. (See, e.g., Hampel et al. [5].)

Let A be an arbitrary $m \times m$ matrix.

Since for each $\psi \in \Psi_0$, $\int \psi(\dot{a})' = Id$, $\int \dot{a}[\dot{a}]' = I^0(\alpha)$ (see (2.7)) and the trace is an additive functional, we have

$$\int (\psi - A\dot{a})(\psi - A\dot{a})' = \int \psi\psi' - A - A' + AI^0(\alpha)A'$$

(here and below we use simple evident notations for integrals).

Therefore instead of minimizing of tr $\int \psi\psi'$ we can minimize

$$\mathrm{tr} \int (\psi - A\dot{a})(\psi - A\dot{a})' = \int |\psi - A\dot{a}|^2,$$

and it is evident that a function $h_c(A\dot{a})$ minimizes the expression under integral sign, and hence the integral itself over all functions $\psi \in \Psi_0$, satisfying (2.26).

At the same time, the condition (2.25), generally speaking, can be violated. But, since a matrix A is arbitrary, we can choose $A = A_c^*(\alpha)$ from (2.28) which guarantees the validity of (2.25) with $\psi_c^* = \psi_c^{A_c^*(\alpha)}$. □

As we have seen, the resulting optimal influence function ψ_c^* is defined along the process $Y^0(\alpha) = (Y_s^0(\alpha))_{0 \le s \le t}$, which is a solution of equation (2.2).

But for constructing optimal estimator we need a function $\psi_c^*(s, x; \alpha)$, defined on whole space $[0, t] \times C_t \times \mathcal{A}$.

For this purpose, define $\psi_c^*(s, x; \alpha)$ as follows:

$$\psi_c^*(s, x; \alpha) = \psi_c^{A_c^*(\alpha)}(s, x; \alpha) = h_c(A_c^*(\alpha)\dot{a}(s, x; \alpha)), \tag{2.29}$$

and as usual $\psi_{c,\varepsilon}^* = \frac{1}{\varepsilon}\psi_c^*(s, \varepsilon x; \alpha)$, $0 \le s \le t$, $x \in C_t$, $\alpha \in \mathcal{A}$.

Definition 2.2. We say that $\psi_c^*(s, x; \alpha)$, $0 \le s \le t$, $x \in C_t$, $\alpha \in \mathcal{A}$, is an influence function of optimal B-robust estimator $(\alpha_t^{*,\varepsilon})_{\varepsilon>0} = (\alpha_t^{\psi_c^*,\varepsilon})_{\varepsilon>0}$ over the class of CULAN estimators $(\alpha_t^{\psi,\varepsilon})_{\varepsilon>0}$, $\psi \in \Psi_{0,c}$, if the matrix $A^*(\alpha)$ is differentiable in α.

From (2.9), (2.11), (2.28) and (2.29) it directly follows that

$$\gamma_{t0}^{\psi_c^*}(\alpha) = P_\alpha^\varepsilon - \lim_{\varepsilon\to 0} \varepsilon^2 \gamma_{t\varepsilon}^{\psi_c^*}(\alpha) = \int_0^t \psi_c^*(s, Y^0(\alpha);\alpha)(\dot{a}(s, Y^0(\alpha);\alpha))' ds = Id.$$

Besides, for each alternative $(P_\alpha^{\varepsilon,h})_{\varepsilon>0}$, $h \in \mathcal{H}$, according to Proposition 2.2, we have

$$\mathcal{L}\{\varepsilon^{-1}(\alpha_t^{*,\varepsilon} - \alpha) \mid P_\alpha^{\varepsilon,h}\} \xrightarrow{w} N(b_t(\psi_c^*, h;\alpha), V_t(\psi_c^*;\alpha)) \quad \text{as } \varepsilon \to 0,$$

where

$$b_t(\psi_c^*, h;\alpha) = \int_0^t \psi_c^*(s, Y^0(\alpha);\alpha) h(s, Y^0(\alpha);\alpha) ds,$$

and $V_t(\psi_c^*;\alpha) = \Gamma_{t0}^{\psi_c^*}(\alpha)$.

Hence, the risk functional for estimator $(\alpha_t^{*,\varepsilon})_{\varepsilon>0}$ is

$$D_t(\psi_c^*, h;\alpha) = |b_t(\psi_c^*, h;\alpha)|^2 + \operatorname{tr}\Gamma_{t0}^{\psi_c^*}, \quad h \in \mathcal{H},$$

and the (unstandardized) gross error sensitivity of $(\alpha_t^{*,\varepsilon})_{\varepsilon>0}$ is

$$\gamma_{\psi_c^*}(\alpha) = \sup_{0\le s\le t} |\psi_c^*(s, Y^0(\alpha);\alpha)| \le c.$$

Thus, we may conclude that $(\alpha_t^{*,\varepsilon})_{\varepsilon>0}$ is the optimal B-robust estimator over the class of estimators $(\alpha_t^{\psi,\varepsilon})_{\varepsilon>0}$, $\psi \in \Psi_{0,c}$, in the following sense: the trace of asymptotic covariance matrix of $(\alpha_t^{*,\varepsilon})_{\varepsilon>0}$ is minimal among all estimators $(\alpha_t^{\psi,\varepsilon})_{\varepsilon>0}$ with bounded by constant c gross error sensitivity, that is,

$$\Gamma_{t0}^{\psi_c^*}(\alpha) \le \Gamma_{t0}^{\psi}(\alpha) \quad \text{for all } \psi \in \Psi_{0,c}.$$

Note that for each estimator $(\alpha_t^{\psi,\varepsilon})_{\varepsilon>0}$ and alternatives $(P_\alpha^{\varepsilon,h})_{\varepsilon>0}$, $h \in \mathcal{H}$, the influence functional is bounded by const.$(r)\cdot c$. Indeed, we have for $\psi \in \Psi_{0,c}$,

$$\sup_{h\in\mathcal{H}_r} |b_t(\psi, h;\alpha)| \le \text{const.}(r)\cdot c := C(r;c),$$

and since from (2.24)

$$\inf_{\psi\in\Psi_{0,c}} \sup_{h\in\mathcal{H}_r} D_t(\psi, h;\alpha) \le C^2(r;c) + \operatorname{tr}\Gamma_{t0}^{\psi_c^*}(\alpha),$$

we can choose "optimal level" of truncation, minimizing the expression

$$C^2(r;c) + \operatorname{tr}\Gamma_{t0}^{\psi_c^*}(\alpha)$$

over all constants c, for which the equation (2.28) has a solution $A_c^*(\alpha)$. This can be done using the numerical methods.

For the problem of existence and uniqueness of solution of equation (2.28), we address to Rieder [7].

In the case of one-dimensional parameter α (i.e., $m = 1$), the optimal level c^* of truncation is given as a unique solution of the following equation (see Lazrieva and Toronjadze [1])

$$r^2c^2 = \int_0^t [\dot{a}(s, Y^0(\alpha); \alpha)]_{-c}^c \, \dot{a}(s, Y^0(\alpha); \alpha) ds - \int_0^t ([\dot{a}(s, Y^0(\alpha); \alpha)]_{-c}^c)^2 ds,$$

where $[x]_a^b = (x \wedge b) \vee a$ and the resulting function

$$\psi^*(s, x; \alpha) = [\dot{a}(s, x; \alpha)]_{-c}^c, \quad 0 \le s \le t, \quad x \in C_t,$$

is $(\Psi_0, \mathcal{H}_r)$ optimal in the following minimax sense:

$$\sup_{h \in \mathcal{H}_r} D_t(\psi^*, h; \alpha) = \inf_{\psi \in \Psi} \sup_{h \in \mathcal{H}_r} D_t(\psi, h; \alpha).$$

Appendix

Important feature of the stochastic volatility model is that volatility process Y is unobservable (latent) process. Clear that full knowledge of the model of the process Y is necessary and hence one needs to estimate the unknown parameter $\alpha = (\alpha_1, \dots, \alpha_m)$, $m \ge 1$.

A variety of estimation procedures are used, which involve either direct statistical analysis of the historical data or the use of implied volatilities extracted from prices of existing traded derivatives.

Consider the method based on historical data.

Fix the time variable t. From observations $Y_{t_0^{(n)}}, \dots, Y_{t_n^{(n)}}$, $0 = t_0^{(n)} < \dots < t_n^{(n)} = t$, $\max_j [t_{j+1}^{(n)} - t_j^{(n)}] \to 0$ as $n \to 0$, calculate the realization of yield process $R_t = \int_0^t \frac{dY_s}{Y_s}$, and then calculate the sum

$$S_n(t) = \sum_{j=0}^{n-1} \left| R_{t_{j+1}^{(n)}} - R_{t_j^{(n)}} \right|^2.$$

It is well known (see, e.g., Lipster and Shiryaev [2]) that

$$S_n(t) \xrightarrow{P} \int_0^t \sigma_s^2 \, ds \quad \text{as } n \to \infty.$$

Since $\sigma_t^2(\omega) = f(Y_t)$ is a continuous process, we get

$$\sigma_t^2(\omega) = \lim_{\Delta \downarrow 0} \frac{F(t+\Delta,\omega) - F(t,\omega)}{\Delta}\,,$$

where $F(t,\omega) = \int_0^t \sigma_s^2(\omega)\,ds$.

Hence, the realization $(y_t)_{0\le t\le T}$ of the process Y can be found by the formula $y_t = f^{-1}(\sigma_t^2)$, $0 \le t \le T$.

We can use the reconstructed sample path (y_t), $0 \le t \le T$, to estimate the unknown parameter α in the drift coefficient of diffusion process Y.

The second market price adjusted procedure of reconstruction the sample path of volatility process Y and parameter estimate was suggested by Renault and Touzi [8], where they used implied volatility data.

We present a quick review of this method, adapted to our model (1.1).

Suppose that the volatility risk premium $\lambda^\sigma \equiv 0$, meaning that the risk from the volatility process is non-compensated (or can be diversified away). Then the price $C_t(\sigma)$ of European call option can be calculated by Hull and White formula (see, e.g., Renault and Touzi [8]), and Black–Scholes (BS) implied volatility $\sigma^i(\sigma)$ can be found as an unique solution of the equation

$$C_t(\sigma) = C_t^{BS}(\sigma^i(\sigma)),$$

where $C^{BS}(\sigma)$ denotes the standard BS formula, written as a function of the volatility parameter σ.

Here (for further estimational purposes) only at-the-money options are used.

Under some technical assumptions (see Proposition 5.1 of Renault and Touzi [8]),

$$\frac{\partial \sigma_t^i(\sigma,\alpha)}{\partial \sigma_t} > 0 \tag{2.30}$$

(remember that the drift coefficient of process Y depends on unknown parameter α).

Fix current value of time parameter t, $0 \le t \le T$, and let $0 < T_1 < T_2 < \cdots < T_{k-1} < t < T_k$ be the maturity times of some traded at-the-money options.

Let $\sigma_{t_j^\varepsilon}^{i*}$ be the observations of an implied volatility at the time moments $0 = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_{[\frac{t}{\varepsilon}]} = t$, $\max\limits_j [t_{j+1}^\varepsilon - t_j^\varepsilon] \to 0$ as $\varepsilon \to 0$.

Then, using (2.30) and solving the equation

$$\sigma^i_{t^\varepsilon_j}(\sigma_{t^\varepsilon_j}, \alpha) = \sigma^{i^*}_{t^\varepsilon_j},$$

one can obtain the realization $\{\widetilde{\sigma}_{t^\varepsilon_j}\}$ of the volatility (σ_t), and thus, using the formula $y_{t^\varepsilon_j} = f^{-1}(\widetilde{\sigma}^2_{t^\varepsilon_j})$, the realization $\{y_{t^\varepsilon_j}\}$ of volatility process (Y_t), which can be viewed as the realization of nonlinear AR(1) process:

$$Y_{t^\varepsilon_{j+1}} - Y_{t^\varepsilon_j} = a(t^\varepsilon_j, Y_{t^\varepsilon_j}; \alpha)(t^\varepsilon_{j+1} - t^\varepsilon_j) + \varepsilon(w^\sigma_{t^\varepsilon_{j+1}} - w^\sigma_{t^\varepsilon_j}).$$

Using the data $\{y_{t^\varepsilon_j}\}$ one can construct the MLE $\widehat{\alpha}^\varepsilon_t$ of parameter α, see, e.g., Chitashvili et al. [9].

(Remember the scheme of construction of MLE. Rewrite the previous AR(1) process, using obvious simple notations, in form

$$Y_{j+1} - Y_j = a(t_j, Y_j; \alpha)\Delta + \varepsilon \Delta w^\sigma_j.$$

Then

$$\begin{aligned} \frac{\partial}{\partial y} P\{Y_{j+1} \le y \mid Y_j\} &= \frac{1}{\sqrt{2\pi\Delta}\varepsilon} \exp\left(- \frac{(y - Y_j - a(t_j, Y_j; \alpha)\Delta)^2}{2\varepsilon^2 \Delta} \right) \\ &=: \varphi_{j+1}(y, Y_j; \alpha), \end{aligned}$$

and the "likelihood" process $\ell_t = (\ell^{(1)}_t, \dots, \ell^{(m)}_t)$ is given by the relation

$$\ell^{(i)}_t = \sum_j \ell^{(i)}_{j+1}, \quad i = \overline{1, m},$$

where

$$\begin{aligned} \ell^{(i)}_{j+1}(y; \alpha) &= \frac{\partial}{\partial \alpha_i} \ln \varphi_{j+1}(y, Y_j; \alpha) \\ &= \frac{1}{\varepsilon^2 \Delta} \big(y - Y_j - a(t_j, Y_j; \alpha)\Delta\big) \dot{a}^{(i)}(t_j, Y_j; \alpha)\Delta. \end{aligned}$$

Hence MLE is a solution (under some conditions) of the system of equations

$$\frac{1}{\varepsilon^2 \Delta} \sum_j \big(y_{j+1} - y_j - a(t_j, y_j; \alpha)\Delta\big) \dot{a}^{(i)}(t_j, y_j; \alpha)\Delta = 0, \quad i = \overline{1, m},$$

where the reconstructed data $\{y_j\} = \{y_{t^\varepsilon_j}\}$ are substituted.)

Let us introduce the functionals

$$HW_\varepsilon^{-1} : \widehat{\alpha}_t^\varepsilon(p) \to \left(y_{t_j^\varepsilon}^{(p+1)},\ 0 \le j \le \left[\frac{t}{\varepsilon}\right]\right),$$
$$MLE_\varepsilon : \left(y_{t_j^\varepsilon}^{(p+1)},\ 0 \le j \le \left[\frac{t}{\varepsilon}\right]\right) \to \widehat{\alpha}_t^\varepsilon(p+1),$$

and

$$\phi_\varepsilon = MLE_\varepsilon \circ HW_\varepsilon^{-1}.$$

Starting with some constant initial value (or preliminary estimator obtained, e.g., from historical data), one can compute a sequence of estimators

$$\widehat{\alpha}_t^\varepsilon(p+1) = \phi_\varepsilon(\widehat{\alpha}_t^\varepsilon(p)), \quad p \ge 1.$$

If the operator ϕ_ε is a strong contraction in the neighborhood of the true value of the parameter, α^0, for a small enough ε, then one can define the estimator $\widehat{\alpha}_t^\varepsilon$ as the limit of the sequence $\{\widehat{\alpha}_t^\varepsilon(p)\}_{p\ge1}$ which is a strong consistent estimator of the parameter α.

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EDTECH SOLUTIONS FOR TRAINING ENGAGEMENT IN CLIP CORPORATE CULTURE

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Abstract

Nowadays organizations are facing challenges which cannot be resolved by the traditional managerial solutions. EdTech area has a relatively short history, it is a sphere in dynamic development process. Learning and Development practices in organizations tend to integrate and use the novelties from different training platforms. It stared with Excel, E-mails and Google Docs and today is present by complex solutions in Training Management Systems. The dynamism, uncertainty and high competition in business organizations make them adapt their learning systems and corporate cultures. From the comparative values organizational culture framework of Robert E. Quinn and Kim S. Cameron [1] companies are moving to 'clip' corporate culture and need of a different AI-based and scenario-based training system approaches.

Key words: learning and development, training management systems, EdTech, clip thinking, corporate culture, symbolic, simulacra, training engagement

Clip Corporate Culture and Corporate Simulation Environment

Alvin Toffler[2] futurism future already has happened. Human culture in general and organizational culture or corporate culture are based on the 'clip' thinking principles. These principles are easily recognizable in nowadays companies:

- Multi-focus in tasks and duties
- Less specialized jobs
- Wide variety of skills
- Multiple media (remote work processes)
- Fast pace of changes

[1] Cameron, Kim S., and Robert E. Quinn. 2011. Diagnosing and Changing Organizational Culture. 3rd ed. Chichester, England: Jossey Bass Wiley.
[2] Toffler, Alvin. Future Shock. New York: Random House, 1970.

- More service style by leaders
- Human resources development urgency (as reskilling, upskilling, etc.)
- Scaling of the learning programs (EdTech development)
- HR Branding and talent management (retaining talented employees with the help of positive organization's image)

Moreover, in organizations as reflecting overall culture development trends, there is a tendency to simulation of reality to become reality (simulacra and simulation of Jean Baudrillard[3]). In this system, the symbolic leadership and generally symbolic meanings play a greater role in managerial effectiveness. It is not enough to plan, organize, lead and control. It is a demand for managing intangible elements in communication, team work, relationships and meanings at work (purpose, mission, goals, vision, adjusted to employees needs, or at least in the average fit).

Corporate University for Scaling Learning and Development

In the light of the new meanings and directions, Corporate University takes a greater space in corporate Learning and Development programs (see Exhibit 1). Employee Learning usually considers newly hired employees (to train for the job and position and to teach standards) and current employees to keep them up-to-date (competences development; changes in conditions; changes of specific or general environment; company changes; talent pool/talent management). Moreover, the whole L&D System is set to promote high performance (Exhibit 2). The skills are divided into two categories: hard and soft skills (Exhibit 3), which means that the approach to Learning and Development and the selected solution should be useful for developing both clusters of skills. They have some specificity and need variable methods and activities to be used.

[3] Baudrillard, Jean. Simulacres Et Simulation: Simulacra and Simulation: The Body in Theory; University of Michigan: Ann Arbor, MI, 1994

Exhibit 1

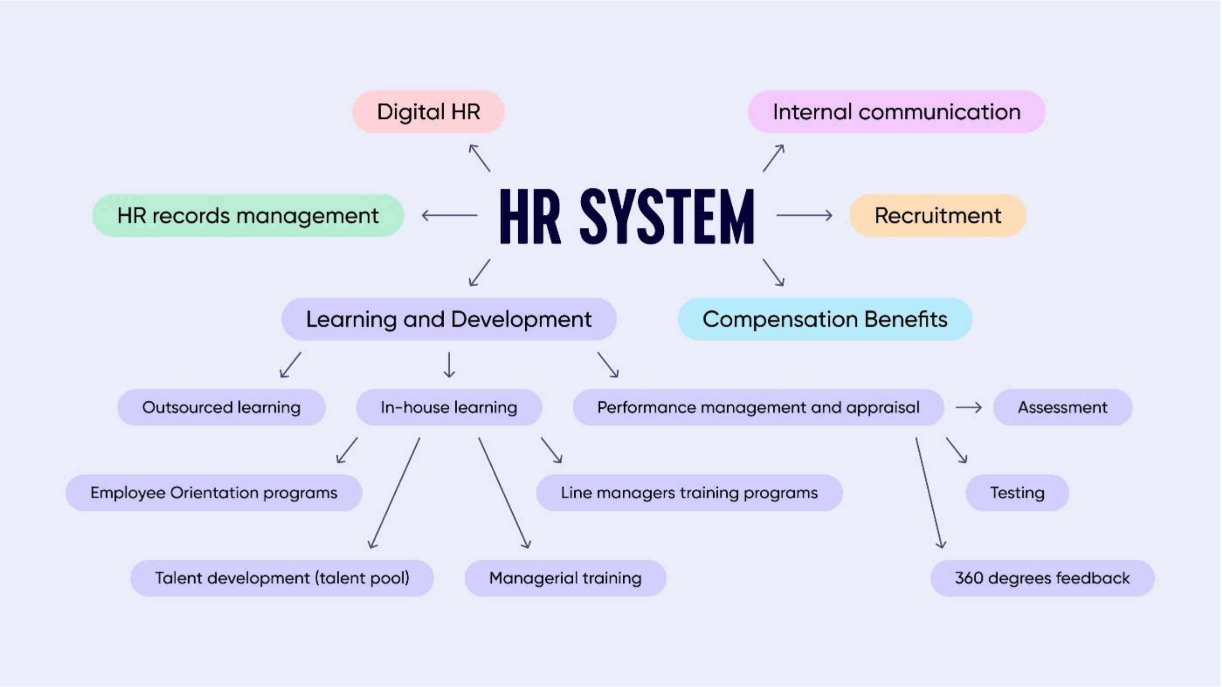


Exhibit 2

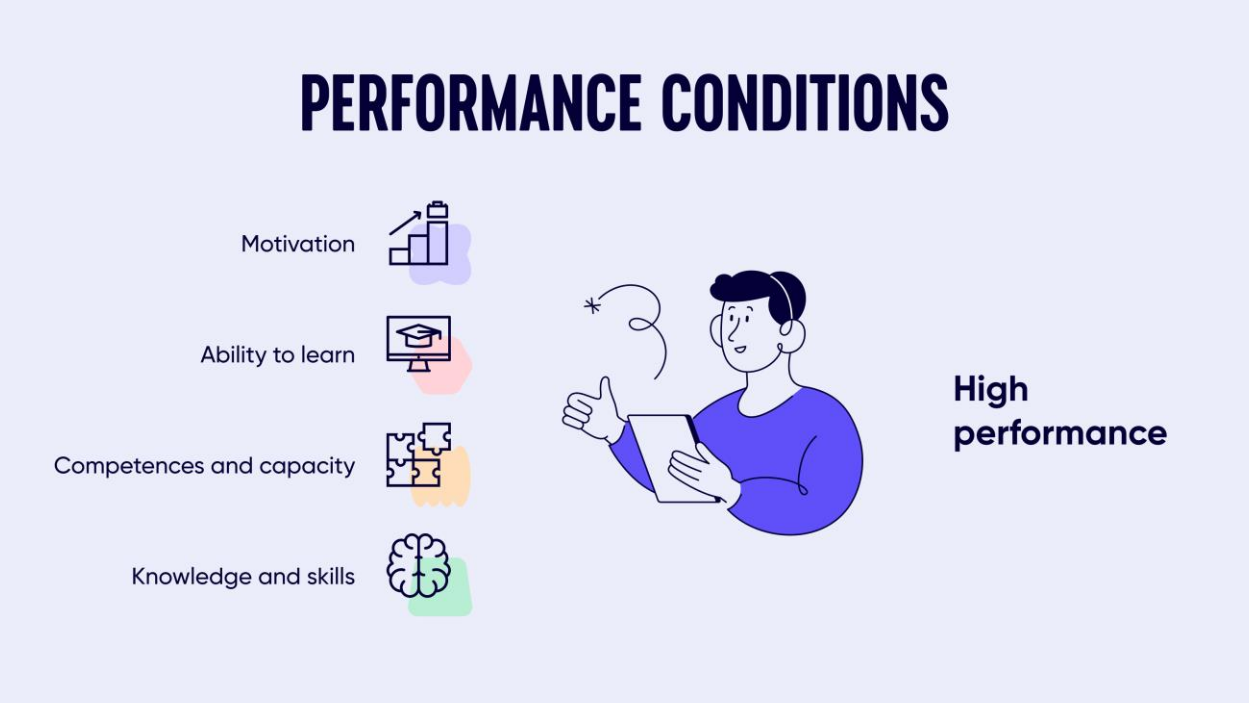
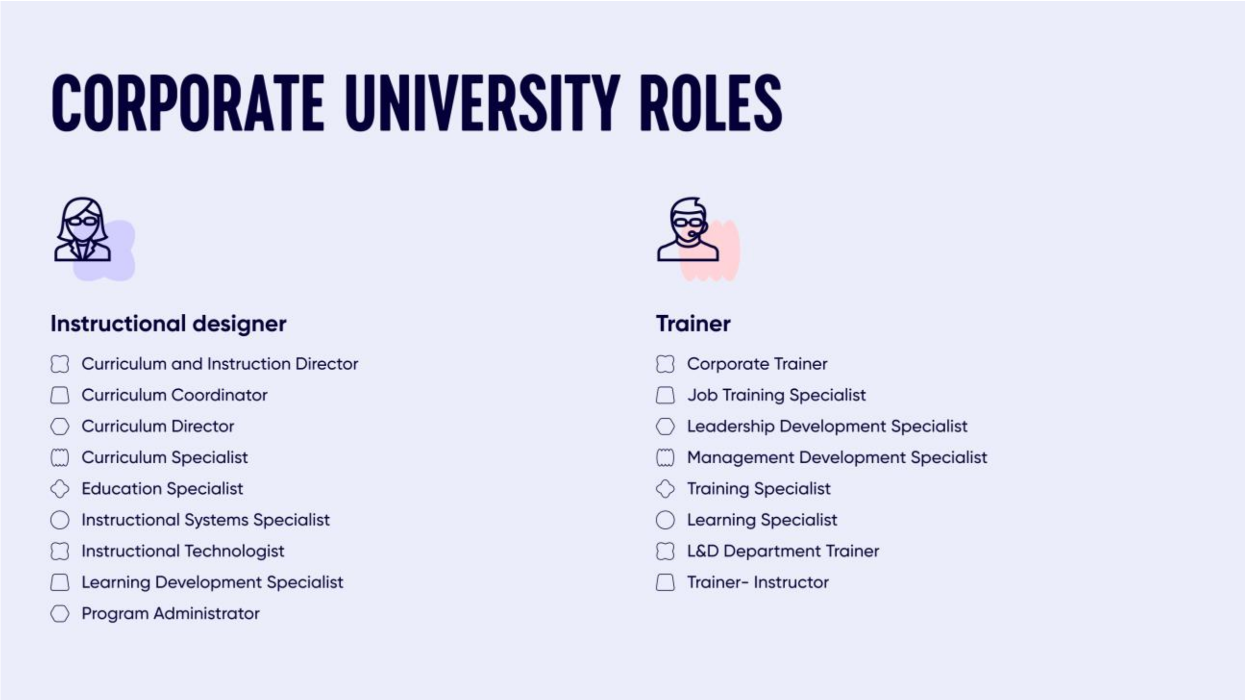


Exhibit 3

Corporate University resolved this demand and implements it in the complex strategic approach to corporate learning and development, as well as it is in a tight connection with organization's mission, business and strategic goals. For the scale-large learnings is the most effective solution for now. Corporate University reflects the clip culture, regardless to its strategic and deep connection, it still leans to introduce more scenario-based solution for the training programs. It means that skills are scaled, employees develop the skills in a less individualized application, but in a more unified standardized way for applying them to work. It matters much for service companies, call centers, hotel chains, software companies, banks, and other big corporations. In Corporate University there are different roles, and EdTech solutions (products) are looking for the ways how to serve each of them with the best tools (Exhibit 4). Instructional Designers are focused at designing the programs format (synchronous, asynchronous design; webinar, virtual training, facilitation format) and developing relevant activities to achieve learning goals and business goals. Trainers implement the training scenarios and activities developed by the Instructional Designers.

Exhibit 4

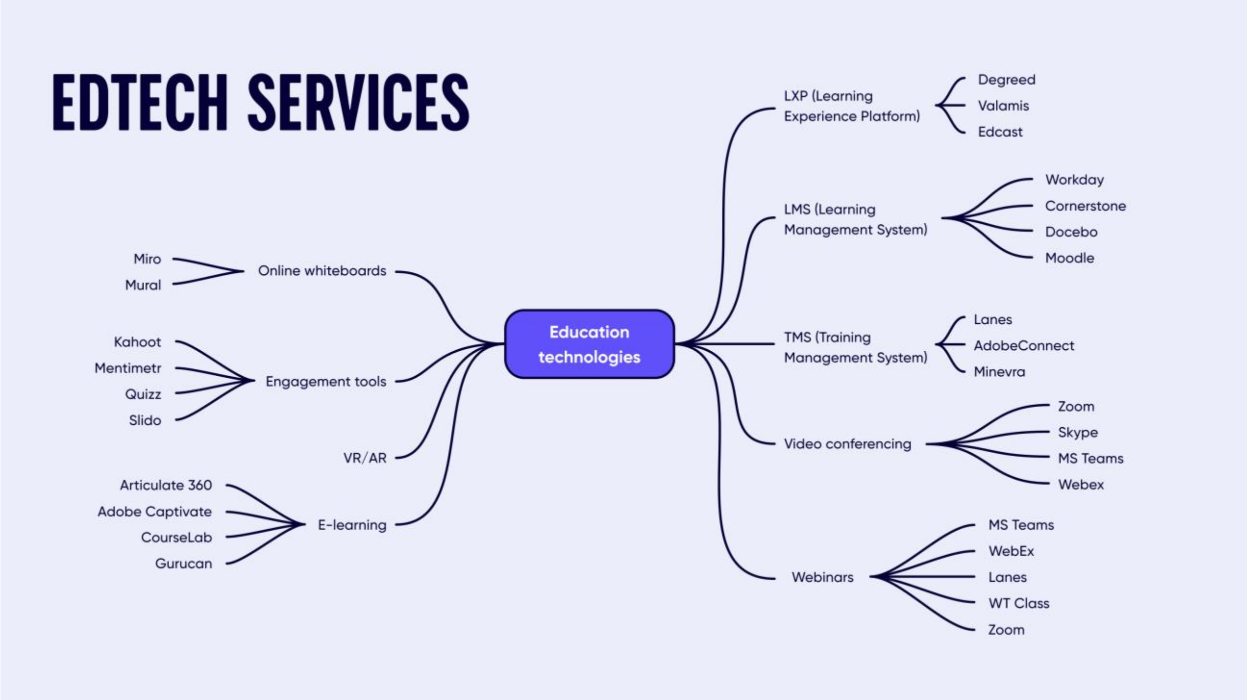


EdTech Market Today

EdTech is the practice of introducing information and communication technology tools into the classroom to create more engaging, inclusive and more individualized learning experience; hardware and software to enhance the quality of learning experience and get to high performance results. EdTech market has been permanently developing, as the demand is high. Nowadays it is present by the following services (Exhibit 5):

- LMS (Learning Management Systems) for asynchronous learning (e.g. Moodle)
- Online whiteboards (e.g. Miro and Mural)
- Engagement tools (for quizzes, polls, etc., as e.g. Mentimeter, Kahoot)
- E-learning (E-learning courses, as e.g. Gurucan, CourseLab)
- LXP (Learning Experience Platform)
- Video Conferencing and Webinars (as Zoom, Webex, MsTeams)
- VA and AR (Artificial Intelligence systems)
- TMS (Training Management Systems) for complex solutions (blended learning), (as Lanes, AdobeConnect)

Exhibit 5



The major two complex and effective systems for corporate learning (not freelance trainers, but for organizations) are LMS and TMS. As well, video conferencing (Zoom, MsTeams and Webex) are actively used for the corporate learning, but are less comfortable, and needs a lot of additional solutions and integrations (with engaging tolls for example).

LMS started with the elementary file drives, E-mails, Excel and Google docs, and nowadays is present as a complex asynchronous learning solution, or integrating video conferencing services for synchronous learning programs (Exhibit 6).

TMS in its turn goes beyond the limits and expands the solutions for all types of learning design – synchronous, asynchronous and blended. Developing all the needed instruments in one-stop complex classroom solution (Exhibit 7).

Exhibit 6

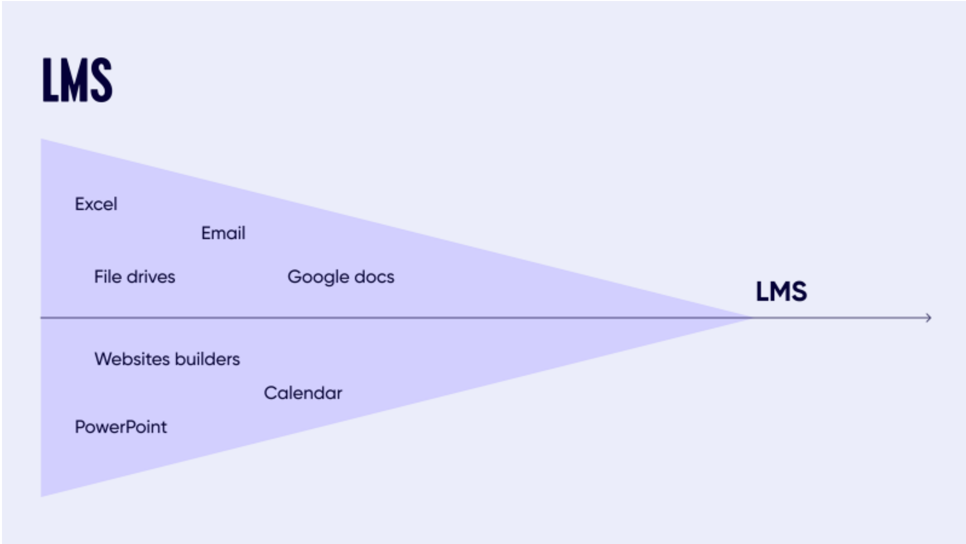
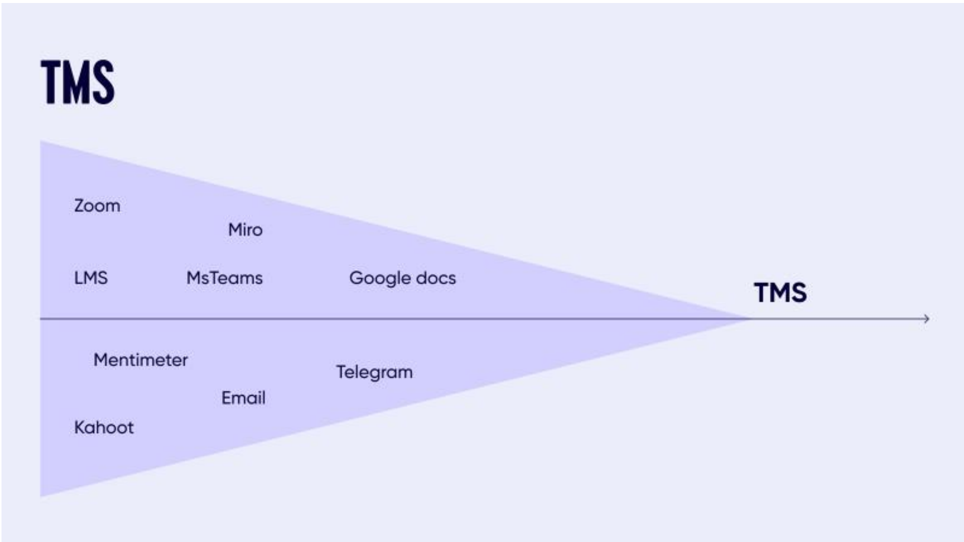


Exhibit 7



TMS Training Management Systems for Training Engagement in Clip Corporate Cultures

Training Management System is a centralized platform, where company can manage, create, schedule and deliver training courses. In virtual learning usually, engagement is the greatest challenge. Moreover, in a clip corporate culture where the attention is dispersed within various tasks and multiple skills, it is almost not possible to get 80-90% engagement in virtual training or asynchronous program.

Training Management Systems are developing the instruments and mechanics integrated in one platform to enhance the engagement and allow scenario training scaling. The main challenges for virtual learning and development are:

- Team work clarity and standards
- Scaling
- Engagement
- Measuring learning outcomes
- Optimizing the work of L&D Team
- Adapting the content for virtual learning
- Balancing workload
- Continuous improvement
- Feedback and assessment

The results with Training Management System can be impressive. For example, with Lanes platform [4] (some cases examples).

In Banking: Training sessions reduced to 3 hours, scalability enhanced, life-long learning culture and retention, new program design reduced to 1-2 days (compared to 2 weeks).

Healthcare company: cost training per employee reduced from $700 to $7, expertise developed company-wide easily, improved engagement and learning results.

Fast-food chain: 23% higher engagement rate, waiting period for leadership dramatically cut (from 6 months down to 3 days), expenditure on company-wide mandatory training has decreased by 39%.

Nowadays the leaders among TMS platforms worldwide are AdobeConnect[5] and Lanes[6] Exhibits 8-10).

The above mentioned results and the active development of TMS platforms show the tendency of engagement enhancement in virtual training programs for clip corporate cultures.

[4] https://www.lanes.ai/

[5] https://www.adobe.com/products/adobeconnect/learning.html

[6] https://www.lanes.ai/

Exhibit 8 AdobeConnect

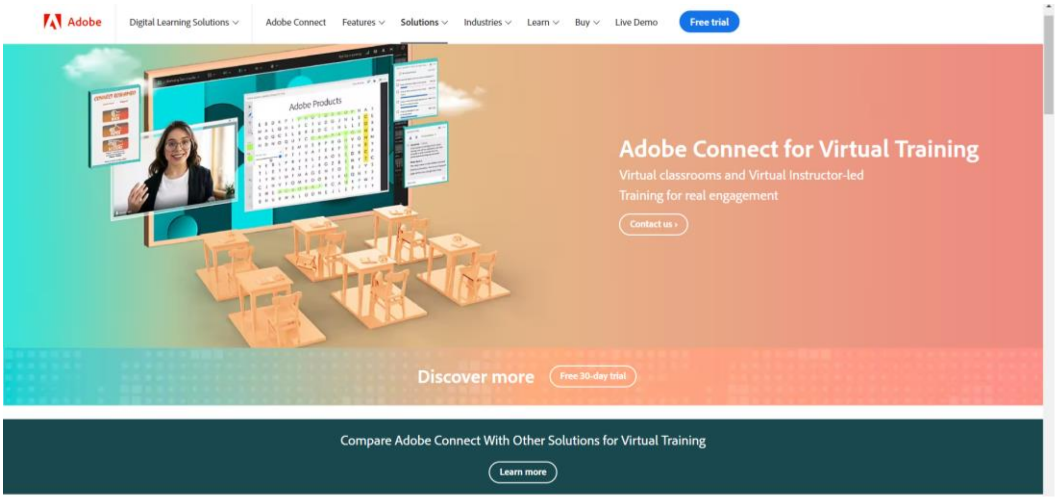


Exhibit 9 Lanes

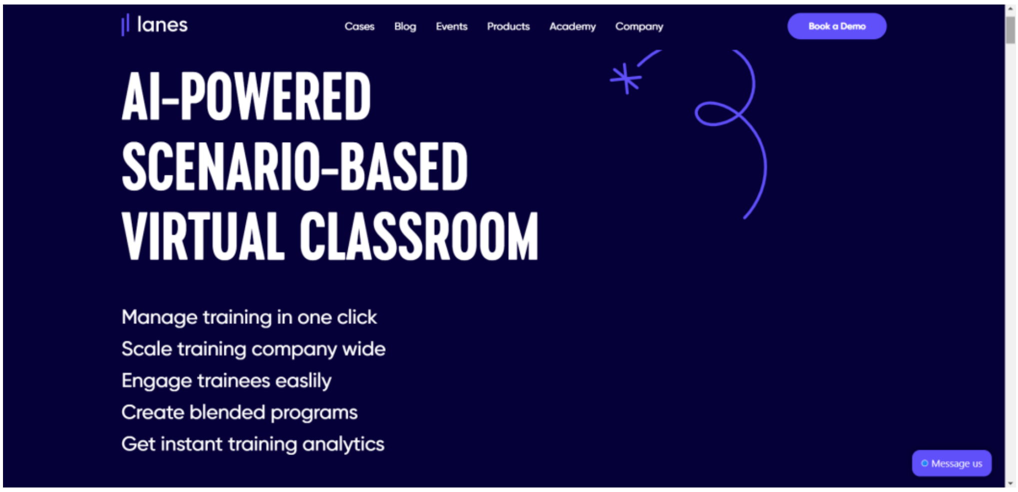
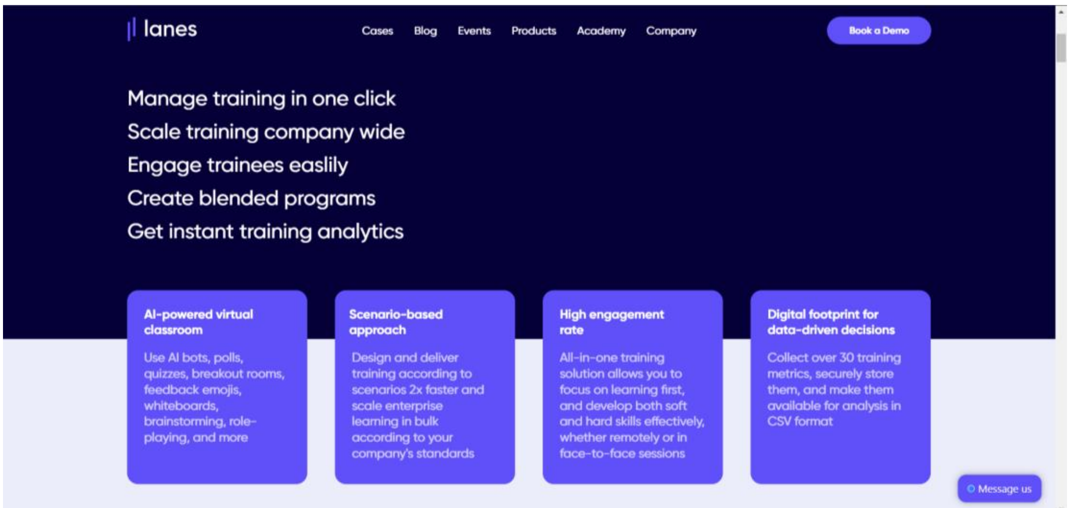


Exhibit 10 Lanes



ON ONE CONNECTION BETWEEN THE MOMENTS OF RANDOM VARIABLES

GEORGE GIORGOBIANI, VAKHTANG KVARATSKHELIA,
MARINE MENTESHASHVILI

ABSTRACT. A new elementary proof of one result of R. Fukuda is proposed. Some improvements are also presented.
Keywords— **probability space, random variable, mathematical expectation.**

1. INTRODUCTION

Let us start with formulation of an interesting result of S. Banach proved in 1933. This result is not used in this paper directly but it will help us to better discuss the problem.

Theorem 1. *(S. Banach, [1]) From any bounded orthonormal system* (φ_n) *a subsystem* (φ_{n_k}) *can be chosen so that the series* $\sum_{k=1}^{\infty} \alpha_k \varphi_{n_k}$ *converge in* $L_p([0,1])$ *for any* p, $1 \le p < \infty$, *whenever* $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$.

Another formulation of this result is as follows: there exists an isomorphism of the Hilbert space l_2 onto the closed linear manifold L in $L_p([0,1])$ spanned by the functions φ_{n_k}, $k = 1, 2, \ldots$.

Under this isomorphism the unit vectors $e_k = (0, \ldots, \overset{k}{1}, 0, \ldots)$, in l_2 correspond to the functions φ_{n_k}, i.e. the basic sequence φ_{n_k} is equivalent to the natural basis (e_k) in l_2. The existence of this isomorphism implies that there exists a constant $C \ge 1$ (the norm of the isomorphism), depending only on L and $p, 1 \le p < \infty$, such that for any $x \in L$ we have

$$\left(\int_0^1 |x(t)|^p \, dt \right)^{1/p} \le C \left(\int_0^1 |x(t)|^2 \, dt \right)^{1/2}.$$

Note that for $1 \le p \le 2$ the statement of the theorem is obvious (and $C = 1$ for this case).

Inspired by this result of S. Banach, in 1962 M.I. Kadec and A. Pelczynski [2] investigated more general version of the Banach theorem. In particular, for any sequence (x_n) in $L_p(0,1), p > 2$, they found the necessary and sufficient condition on (x_n) to contain a basic sequence (x_{n_k}) equivalent to the natural basis (e_k) in l_2.

2. Main Result

Investigating the Subgaussian random elements with values in Banach spaces and analyzing the results of [2], R. Fukuda [3] came to a result, which is improved in our Theorem 2 stated below.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\xi : \Omega \to \mathbb{R}^1$ be a real random variable and $\mathbb{E}$ be a mathematical expectation symbol.

Theorem 2. *Let* $p > q > 0$ *and for some* $C \geq 1$

$$\{\mathbb{E}|\xi|^p\}^{1/p} \leq C\,\{\mathbb{E}|\xi|^q\}^{1/q} < \infty. \tag{1}$$

Then for any $r, s,\ 0 < r, s \leq p$, *we have*

$$\{\mathbb{E}|\xi|^r\}^{1/r} \leq C^{\beta}\,\{\mathbb{E}|\xi|^s\}^{1/s},$$

where

$$\beta = \begin{cases} 0, & \text{if} \quad 0 < r \leq s \leq p, \\ 1, & \text{if} \quad q \leq s < r \leq p, \\ \frac{q(p-s)}{s(p-q)}, & \text{if} \quad 0 < s < q < r \leq p, \\ \frac{p(q-s)}{s(p-q)}, & \text{if} \quad 0 < s < r \leq q. \end{cases}$$

Proof. Since the expression $\{\mathbb{E}|\xi|^t\}^{1/t}$, as a function of t, $t > 0$, is nondecreasing, the statement of the theorem for the case $0 < r \leq s \leq p$ is evident. For the case $q \leq s < r \leq p$, the proof is also easy using the condition (1) of the theorem in addition. Therefore, we begin the proof with the case $0 < s < q < r \leq p$. Introduce the numbers $u = \frac{p(q-s)}{q(p-s)}$ and $v = \frac{s(p-q)}{q(p-s)}$. Clearly $0 < u, v < 1$ and $u + v = 1$. Using the Hölder inequality, we get the following inequality:

$$\mathbb{E}|\xi|^q = \mathbb{E}|\xi|^{uq}|\xi|^{vq} \leq \left\{\mathbb{E}|\xi|^{uqt}\right\}^{1/t}\left\{\mathbb{E}|\xi|^{vqt^\star}\right\}^{1/t^\star}, \tag{2}$$

where $1 < t, t^\star < \infty$ and $1/t + 1/t^\star = 1$. Choose now t by the condition $uqt = p$. It is clear that $t > 1$ and

$$t = \frac{p-s}{q-s}, \qquad t^\star = \frac{t}{t-1} = \frac{p-s}{p-q}.$$

For such a number t, the relation (2) leads to the following one

$$\mathbb{E}|\xi|^q \leq \{\mathbb{E}|\xi|^p\}^{\frac{q-s}{p-s}}\{\mathbb{E}|\xi|^s\}^{\frac{p-q}{p-s}} \leq$$
$$\leq C^{\frac{p(q-s)}{p-s}}\{\mathbb{E}|\xi|^q\}^{\frac{p(q-s)}{q(p-s)}}\{\mathbb{E}|\xi|^s\}^{\frac{p-q}{p-s}},$$

from which the following inequality, the key point for our proof, can easily be obtained

$$\{\mathbb{E}|\xi|^q\}^{1/q} \leq C^{\frac{p(q-s)}{s(p-q)}}\{\mathbb{E}|\xi|^s\}^{1/s}. \tag{3}$$

Using now the Hölder inequality, the assumption of the theorem and finally, the key inequality (3), we get:

$$\{\mathbb{E}|\xi|^r\}^{1/r} \leq \{\mathbb{E}|\xi|^p\}^{1/p} \leq C\,\{\mathbb{E}|\xi|^q\}^{1/q} \leq$$
$$\leq C^{\frac{q(p-s)}{s(p-q)}}\{\mathbb{E}|\xi|^s\}^{1/s}.$$

Now the case $0 < s < r \leq q$ is left, which can be reduced to the previous one.

$\square$

Note that applying Kahane's inequality, Fukuda in his paper [3] as a constant C^β for $r = p$ and $s = 1$ obtained the expression

$$C^{1+\frac{pq}{p-q}} \cdot q \cdot B^{-1}(1/q, \frac{p}{p-q} + 1), \tag{4}$$

where $B(\cdot,\cdot)$ is a beta function.

For the same values of the parameters $(r = p, s = 1)$, the constant obtained from Theorem 2 is equal to

$$C^\beta = \begin{cases} C, & if \quad 0 < q \leq 1, \\ C^{\frac{q(p-1)}{p-q}}, & if \quad 1 < q < p. \end{cases} \tag{5}$$

Using the computer program MAPLE we compared the values of (4) and (5) for different values of the parameters p and q, and it was found that the constant obtained by Theorem 2 is better, although it needs analytical confirmation.

Acknowledgment

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The Impact of Promotions on Consumer Behavior

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ABSTRACT

In a high velocity environment retail market is growing all over the world. Retailers use all types of promotional activities in order to be differentiated in the market. As a result of population and economic growth, retailers started to widen the application of various marketing in order to influence consumers. Peattie and Peattie (1994) stated that marketing activities are usually specific to a time period, place or customer group, which encourage a direct response from consumers or marketing intermediaries, through the offer of additional benefits. One of these activities is to use promotions (such as: discounts, buy one get one free, coupons, rebates, contests, cash-back offers and loyalty programs) and they directly influence individual to give quick decision and to finalize purchasing process. Since promotions are one of the most noticed marketing activities, promotions can greatly impact any company's market share and sustainability. It is therefore important to understand which promotions consumers prefer and also the effect of promotions on customers and their behavior. (Peattie, 1994)

According to the American Marketing Association, as noted by Kotler and Keller, marketing can be defined as "an organizational function and a set of processes for creating, communicating, and delivering value to consumers, also for managing consumer relationships in ways that benefit the organization and its stakeholders".

Product, price, place and promotion are the 4 Ps of marketing. All four of these elements are combined to make a successful marketing strategy. Promotion is used to communicate the company's message across to the consumer. The four main tools of promotion are advertising, sales promotion, public relation and direct marketing (Juneja, 2018).

- Advertising is defined as promotion for product, service and idea and any form of paid communication. Advertisement is not only used by companies. In many cases it is used by museum, charitable organizations and government. However, the treatment of advertising varies from organization to organization.

- The process of persuading a potential customer to buy the product is known as sales promotion. It is designed to be used as a short-term tactic for boosting sales – it is rarely suitable as a method of building long-term customer loyalty.

Types of sales promotion:

- When consumers are provided a sample of a product for free it is reffered as free samples. An organization may use this form of promotion as it can be effective in removing any monetary disincentive a consumer may have about purchasing and trailing a new product (Fripp, 2018)

- Premium offer - refer to a bonus offered to a consumer for buying one product. This bonus is given to the consumer for free or at a substantial discount.

- When companies offer consumers some form of bonus or reward for spending money at a specific store, they use - loyalty programs. An organization uses this form of promotion in order to create consumer loyalty and also to drive consumers to make repeat purchases.

- When organizations held competitions associated with their products, they use - contests. They use this form of promotion to offer additional incentive for consumers to purchases their product over similar competitive products.

- Vouchers that allow consumers to purchase products at a discounted price are coupons. This form of promotion is used by organizations to advertise a new product.

- There are discounts when products are temporarily offered at a lower price. This form of promotion is used by organizations in order to increase sales and attract new customers.

- Systems that allow customers to obtain a refund of some of the purchase price is known as cash-back. When cash-back are offered immediately at the time of purchase, this is an instant rebate. An organization uses this form of promotion to capture consumers' attention and offer an incentive for purchasing their product over similar competing products.

- Companies need to have a constant interaction with customers, employees and different stakeholders, without it they cannot survive. The public relation office makes this relation. Its major function is to handle press releases, support product publicity, create and maintain the corporate image, handle matters with lawmakers, guide management with respect to public issues (Juneja, 2018).

- Communication established through a direct channel without using any intermediaries is referred to as direct marketing. It can be used to deliver message or service. In recent yers, direct marketing has shown tremendous growth. In this growth story the internet has played major part. Direct marketing saves time, makes an experience personal and also pleasant. Direct marketing reduces cost for companies. Face to face selling, direct mail, catalog marketing, telemarketing, TV and kiosks are media for direct marketing (Juneja, 2018)

The consumer buying process consists of several stages which are as following:
The first step is problem recognition. During this step, the consumer realizes that she has an unfulfilled want or need.
The next step is to gather information relevant to what you need for solving the problem.

After gathering information, it is evaluated against a consumer's wants, needs, preferences, and financial resources, which are available for purchase.

At the purchase stage, the consumer will make a purchasing decision. The ultimate decision may be based on factors such as availability or price.

At the post-purchase evaluation stage, the consumer will decide whether the purchase actually satisfies her needs and wants (Kotler, 2003)

The research, which is used in this study, is descriptive in its nature. It can be explained by particular situation, telling some sort of things or some sort of noticeable facts. Research that explains the present situation instead of interpreting and making judgments is descriptive research (Creswell, 1994). The core purpose of descriptive research is to establish the accurateness of developed hypothesis that reflect the present position. This kind of research gives knowledge about the current scenario and concentrate on past or present for an instance in a community quality of life or customer attitude toward any marketing activity (Kumar, 2005).

The research was conducted in Carrefour, with a Carrefour consumer. Various products are the core business of the supermarket Carrefour. Their product offer is based on a number of unchanging principles, which are a broad selection, the lowest prices, the highest quality and compliance with manufacturing conditions – and promotes responsible consumption. The aim of offers are as follows:

- Satisfy the needs of a majority of customers
- Ensure product quality
- ensure adequate supply to avoid product shortage
- ensure the best prices every day
- Encourage sales of local products

The questionnaire focusing mainly on demographics and purchasing behavior. For the best results personal interviews were conducted at Carrefour.

The results of the consumer demographic data was analyzed using descriptive statistics and the presented values showed the frequency and the percentages of people that fall in each subcategory for each characteristic. Demographic questions enabled to have better understanding of the target group and their attitude.

The respondents were asked their opinion on several statements regarding the idea that promotions affect the buying behavior. The results of the impact of promotions on consumer behavior was analyzed using descriptive statistics and the values presented show the mean and standard deviation.

Hypotheses:

H_0: $\mu_1 - \mu_2 = 0$ cnsumers act as what they say, about buying behavior

H_1: $\mu_1 - \mu_2 \neq 0$ consumers do not act as what they say, about buying behavior

$$t = \frac{(\overline{x}_1 - \overline{x}_2) - (\mu_1 - \mu_2)_0}{\sqrt{s_p^2 / n_1 + s_p^2 / n_2}}$$

We reject H_0 if $z < -1.96$ or $z > 1.96$

We cannot reject H_0 hypothesis by 95% confidence level for buy-one-get-one-free, price-off promotions and brochures.

By far, the most successful promotion affecting consumer behavior, by increasing the likelihood of a purchase is the “buy-one-get-one-free” (BOGOF) offer. Therefore, if the “buy-one-get-one-free” offer is still profitable, by increasing the volume of sales, profits may increase via this promotion.

Brochures positioned in front of supermarkets are also very likely to cause sales. This is due to the fact that consumers like to check which products are on promotion at the time of their visit to the supermarket. Although consumers may have an intended shopping list while visiting the supermarket, the discovery of product under promotion always pushes towards spending money on discounted items.

In case of promotion, which enable consumers to save money, they are always encouraged to buy products on sales for saving reasons. In this manner, price-off promotions can help to increase volume sales of a product.

Demonstrations don't affect consumer behavior. If consumers do not have an initial interest in a product, they are not likely to pay enough attention to a demonstration.

Free trials offer a free item with the purchase of a different item. The free item in most of the cases does not represent the desired item, although the notion of having a gift for free triggers positive attitude, it doesn't increase volume sales of a product.

Research sub-question:

How do certain promotions, having impact on consumer behavior, relate to the attitude Georgian consumers display towards those promotions?

In response to this question related to the response of consumers to promotions and its relation to attitude, the following has been revealed. For example, research results revealed that the most attractive promotion if BOGOF offer which result into significantly boosted sales.

Brochures are fairly successful in affecting consumer behavior and are more effective in influencing the purchasing behavior of those with lower education levels.

Buy-one-get-one-free promotions are the most successful of the promotions and the success of buy-one-get-one-free promotions is not linked to any of the demographic characteristics.

Price-off promotions are also decently successful in influencing consumer behavior with females and younger consumers being those most affected by price-off promotions.

Demonstrations are not very influential with regards to purchasing behavior but are more likely to succeed in influencing the behavior of consumers with lower education levels.

Free trial promotions are less successful in influencing consumer behavior and are more successful in influencing the purchasing behavior of consumers with lower salaries.

Sale promotion plays a vital role for the dealers and retailers in the marketing programs. It generates large revenues and by using promotions sales can be increased.

The overall conclusion of this study is that I found positive consumer attitudes toward various promotions that influence their behavior, specifically the purchase process. The study confirmed that consumers behavior can be motivated through various kinds of elements, including promotion techniques such as BOGOF, price-off promotions and brochures. Furthermore, the structure offers new visions to understand that how different consumers respond to numerous promotion tools offered by marketers and their impacts on consumers buying behavior, which may be vital for marketers in order to use perfect promoting strategies and promotional tools to promote products.

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A stochastic model of predator-prey population dynamics

T. Kutalia and R. Tevzadze

Abstract

Abstract. We present results of an analysis for a randomized three-dimensional predator-prey model representing the dynamics of wolf-deer interactions.

1 Introduction

We study a predator – prey stochastic model (initiated by R. Chitashvili) in discrete and continuous time. For this we define the transition probabilities and Markov chain realization by the random difference schemes as well as the systems of stochastic differential equations. We show that in case of scaling, the solution of the system approaches the solution of the equation of ordinary differential equation. We present the graphs of these solutions for the specific parameter set and different initial conditions. Graphs illustrate the equilibrium points which the system approaches in infinite time.

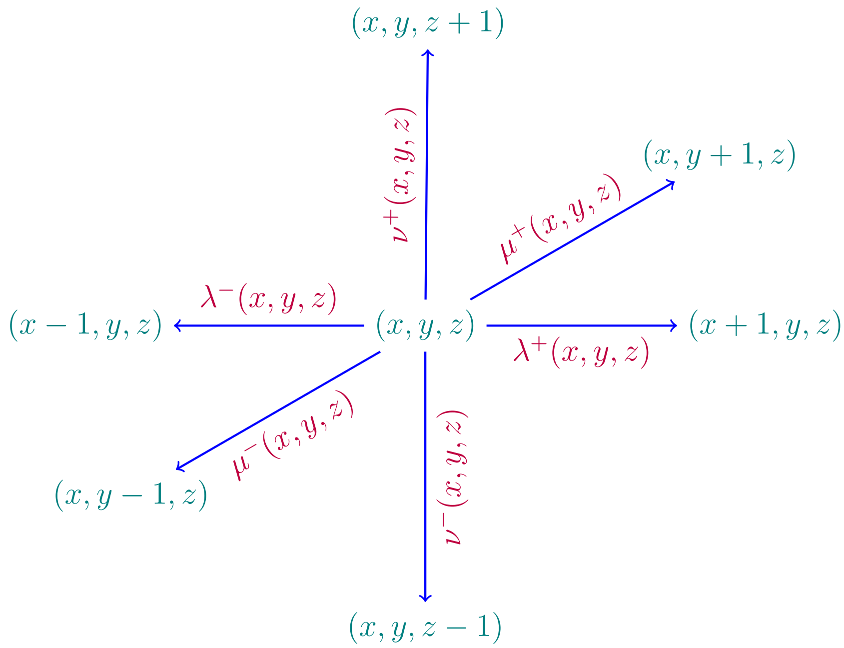
2 The discrete time Model

Let us consider two populations: deer and wolves and assume that deer divide into the group of strong and week ones. Denote by x, y, z the number of strong deer, week deer and wolves respectively.

Let $\lambda^{\pm}(x,y,z), \mu^{\pm}(x,y,z), \nu^{\pm}(x,y,z),\ (x,y,z) \in N^3$ be intensity of transitions

$$(x,y,z) \to (x \pm 1, y, z), (x,y,z) \to (x, y \pm 1, z), (x,y,z) \to (x,y,z \pm 1)$$

and $\lambda = \lambda^+ + \lambda^-, \mu = \mu^+ + \mu^-, \nu = \nu^+ + \nu^-$ respectively .



Let us introduce the following notations

$$p^{\pm}(x,y,z) = \frac{\lambda^{\pm}(x,y,z)}{(\lambda+\mu+\nu)(x,y,z)},$$
$$q^{\pm}(x,y,z) = \frac{\mu^{\pm}(x,y,z)}{(\lambda+\mu+\nu)(x,y,z)},$$
$$r^{\pm}(x,y,z) = \frac{\nu^{\pm}(x,y,z)}{(\lambda+\mu+\nu)(x,y,z)},$$
$$p = p^+ + p^-, q = q^+ + q^-, r = r^+ + r^-.$$

Then recurrent equations

$$\begin{aligned}
X_{n+1} &= X_n + I_{(\varepsilon_n < p^+(X_n,Y_n,Z_n))} - I_{(p^+(X_n,Y_n,Z_n)) \le \varepsilon_n < p(X_n,Y_n,Z_n)}, \qquad (1)\\
Y_{n+1} &= Y_n + I_{(p(X_n,Y_n,Z_n) \le \varepsilon_n < (p+q^+)(X_n,Y_n,Z_n))} - I_{(p+q^+)(X_n,Y_n,Z_n)) \le \varepsilon_n < (p+q)(X_n,Y_n,Z_n))},\\
Z_{n+1} &= Z_n + I_{((p+q)(X_n,Y_n,Z_n) \le \varepsilon_n < (p+q+r^+)(X_n,Y_n,Z_n))} - I_{((p+q+r^+)(X_n,Y_n,Z_n) \le \varepsilon_n < 1)},
\end{aligned}$$

2

where ε_n is i.i.d. with uniform distribution, define Markov chain with such transition probabilities. Indeed

$$\begin{aligned}
P(X_{n+1} = X_n \pm 1, Y_{n+1} = Y_n, Z_{n+1} = Z_n | X_n, Y_n, Z_n) = P(X_{n+1} = X_n \pm 1 | X_n, Y_n, Z_n) \\
= P(I_{(\varepsilon_n < p^+(X_n,Y_n,Z_n))} - I_{(p^+(X_n,Y_n,Z_n)) \le \varepsilon_n < p^+(X_n,Y_n,Z_n) + p^-(X_n,Y_n,Z_n))} = \pm 1) | X_n, Y_n, Z_n) \\
= P(\varepsilon_n < p^{\pm}(X_n, Y_n, Z_n) | X_n, Y_n, Z_n) = p^{\pm}(X_n, Y_n, Z_n)
\end{aligned}$$

and at cetera.

Assume that the share of strong deer increases proportionally to fraction of strong deer pairs into all deer pairs, i.e. by intensity $(x + y)\frac{x^2}{(x+y)^2}$. Similarly for week deers we get intensity $(x+y)(1-\frac{x^2}{(x+y)^2})$. The mortality of strong and week deer is defined as $d^s x + e^s xz,\ d^w y + e^w yz$ respectively. We define the rate of fecundity and mortality of wolfs as: $\alpha z + \beta'(x+y),\ \delta z + \gamma' \frac{z^2}{x+y}$. Therefore

$$\begin{aligned}
\lambda^+(x,y,z) = \frac{x^2}{x+y},\ \lambda^-(x,y,z) = (d^s x + e^s xz), \\
\mu^+(x,y,z) = \frac{(x+y)^2 - x^2}{x+y},\ \mu^-(x,y,z) = (d^w y + e^w yz), \\
\nu^+(x,y,z) = \alpha z + \beta'(x+y),\ \nu^-(x,y,z) = \delta z + \gamma' \frac{z^2}{x+y}.
\end{aligned}$$

We take $e^s = d^s = 0.2,\ e^w = d^w = 0.5,\ \alpha = \delta,\ \beta' = 1, \gamma' = 1$ and $(X_0, Y_0, Z_0) = (8, 8, 8)$. We graphically present the solution of (1) below

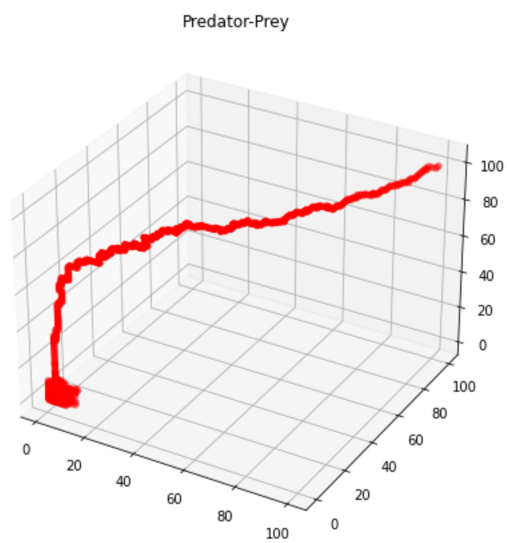


Figure 1: graph of random dynamics

3 The continuous time Model

Let $\mathcal{N}(duds)$ be the Poisson point process driven by Lebesgue measure $duds$ and $\tilde{\mathcal{N}}(duds) = \mathcal{N}(duds) - duds$. Then the Markov chain in continuous time is defined by the SDE

$$
\begin{aligned}
X_t &= X_0 + \int_0^t \int_0^\infty (I_{(u<\lambda^+(X_{s-},Y_{s-},Z_{s-}))} - I_{(\lambda^+(X_{s-},Y_{s-},Z_{s-}))u<\lambda(X_{s-},Y_{s-},Z_{s-})})\mathcal{N}(duds), \\
Y_t &= Y_0 + \int_0^t \int_0^\infty (I_{(\lambda(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu^+)(X_{s-},Y_{s-},Z_{s-}))} \\
&\quad -I_{(\lambda+\mu^+)(X_{s-},Y_{s-},Z_{s-}))\leq u<(\lambda+\mu)(X_{s-},Y_{s-},Z_{s-}))})\mathcal{N}(duds), \\
Z_t &= Z_0 + \int_0^t \int_0^\infty (I_{((\lambda+\mu)(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu+\nu^+)(X_{s-},Y_{s-},Z_{s-}))} \\
&\quad -I_{((\lambda+\mu+\nu^+)(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu+\nu)(X_{s-},Y_{s-},Z_{s-}))})\mathcal{N}(duds).
\end{aligned}
$$

Then

$$
\begin{aligned}
X_t &= X_0 + \int_0^t (\lambda^+(X_{s-}, Y_{s-}, Z_{s-})) - \lambda^-(X_{s-}, Y_{s-}, Z_{s-}))ds + L(t), \\
Y_t &= Y_0 + \int_0^t (\mu^+(X_{s-}, Y_{s-}, Z_{s-}) - \mu^+(X_{s-}, Y_{s-}, Z_{s-}))ds + M(t) \\
Z_t &= Z_0 + \int_0^t (\nu^+(X_{s-}, Y_{s-}, Z_{s-}) - \nu^+(X_{s-}, Y_{s-}, Z_{s-}))ds + N(t),
\end{aligned}
$$

where

$$
\begin{aligned}
L(t) &= \int_0^t \int_0^\infty (I_{(u<\lambda^+(X_{s-},Y_{s-},Z_{s-}))} - I_{(\lambda^+(X_{s-},Y_{s-},Z_{s-}))u<\lambda(X_{s-},Y_{s-},Z_{s-}))})\tilde{\mathcal{N}}(duds) \\
M(t) &= \int_0^t \int_0^\infty (I_{(\lambda(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu^+)(X_{s-},Y_{s-},Z_{s-}))} \\
&\quad -I_{(\lambda+\mu^+)(X_{s-},Y_{s-},Z_{s-}))\leq u<(\lambda+\mu)(X_{s-},Y_{s-},Z_{s-}))})\tilde{\mathcal{N}}(duds) \\
N(t) &= \int_0^t \int_0^\infty (I_{((\lambda+\mu)(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu+\nu^+)(X_{s-},Y_{s-},Z_{s-}))} \\
&- I_{((\lambda+\mu+\nu^+)(X_{s-},Y_{s-},Z_{s-})\leq u<(\lambda+\mu+\nu)(X_{s-},Y_{s-},Z_{s-}))})\tilde{\mathcal{N}}(duds).
\end{aligned}
$$

Obviously $\langle M, L\rangle = \langle M, N\rangle = \langle L, N\rangle = 0$ and

$$
\begin{aligned}
\langle L\rangle_t &= \int_0^t (\lambda^+(X_{s-}, Y_{s-}, Z_{s-})) + \lambda^-(X_{s-}, Y_{s-}, Z_{s-}))ds = \int_0^t \lambda(X_{s-}, Y_{s-}, Z_{s-})ds, \\
\langle M\rangle_t &= \int_0^t (\mu^+(X_{s-}, Y_{s-}, Z_{s-})) + \mu^-(X_{s-}, Y_{s-}, Z_{s-}))ds = \int_0^t \mu(X_{s-}, Y_{s-}, Z_{s-})ds, \\
\langle N\rangle_t &= \int_0^t (\nu^+(X_{s-}, Y_{s-}, Z_{s-})) + \nu^-(X_{s-}, Y_{s-}, Z_{s-}))ds = \int_0^t \nu(X_{s-}, Y_{s-}, Z_{s-})ds.
\end{aligned}
$$

Let $\lambda_K^\pm(i), \mu_K^\pm(i), \nu_K^\pm(i)$ denote $K\lambda^\pm(i/K), K\mu^\pm(i/K), K\nu_K^\pm(i/K), i = 0, 1, \ldots,$ and let (X^K, Y^K, Z^K) be the corresponding solution. Let $(\tilde{X}^K, \tilde{Y}^K, \tilde{Z}^K) = (X^K, Y^K, Z^K)/K$. The following proposition follows from results of [2] .

Proposition 1. Let $\lambda_K^\pm(i), \mu_K^\pm(i), \nu_K^\pm(i)$ be Lipschitzz continuous and nonnegative on $\mathbb{R}_+^3$. Then $(\tilde{X}^K, \tilde{Y}^K, \tilde{Z}^K) \overset{K\to\infty}{\to} (x, y, z)$ in law in $D[0,T]^3$, where (x, y, z) is solution the of ODE

$$
\begin{aligned}
\dot{x} &= \lambda^+(x, y, z) - \lambda^-(x, y, z), \\
\dot{y} &= \mu^+(x, y, z) - \mu^-(x, y, z), \\
\dot{z} &= \nu^+(x, y, z) - \nu^-(x, y, z).
\end{aligned}
$$

In our model

$$\lambda^+ = \frac{X^2}{X+Y},\ \lambda^- = (d^s X + e^s XZ),$$
$$\mu^+ = \frac{(X+Y)^2 - X^2}{X+Y},\ \mu^- = (d^w Y + e^w YZ),$$
$$\nu^+ = \alpha Z + \beta'(X+Y),\ \nu^- = \delta Z + \gamma' \frac{Z^2}{X+Y}.$$

Therefore

$$dX_t = \left(\frac{X^2}{X+Y} - (d^s X + e^s XZ)\right) dt + dL(t)$$
$$dY_t = \left(\frac{(X+Y)^2 - X^2}{X+Y} - (d^w Y + e^w YZ)\right) dt + dM(t)$$
$$dZ_t = \left(\alpha Z + \beta'(X+Y) - \delta Z + \gamma' \frac{Z^2}{X+Y}\right) dt + dN(t).$$

In deterministic case one obtains

$$\dot{x} = \frac{x^2}{x+y} - (d^s x + e^s xz),$$
$$\dot{y} = \frac{(x+y)^2 - x^2}{x+y} - (d^w y + e^w yz), \qquad (2)$$
$$\dot{z} = \alpha z + \beta'(x+y) - \delta z - \gamma' \frac{z^2}{x+y}.$$

Such type of population model was studied in [1].

Remark. If $\tilde{y} = x + y$ then

$$\begin{aligned}
\dot{x} &= \frac{x^2}{\tilde{y}} - x(d^s + e^s z),\\
\dot{\tilde{y}} &= \tilde{y}(1 - d^w - e^w z) + x(d^w + e^w z - d^s - e^s z),\\
\dot{z} &= (\alpha - \delta) z + \beta' \tilde{y} - \gamma' \frac{z^2}{\tilde{y}}.
\end{aligned}$$

In particular case of parameters we have

$$\begin{aligned}
\dot{x} &= \frac{x^2}{x+y} - 0.2x(1+z), \\
\dot{y} &= \frac{(x+y)^2 - x^2}{x+y} - 0.5y(1+z), \qquad (3) \\
\dot{z} &= (x+y) - \frac{z^2}{x+y}.
\end{aligned}$$

or for $(x, \tilde{y}, z) = (x, x+y, z)$

$$\begin{aligned}
\dot{x} &= \frac{x^2}{\tilde{y}} - 0.2x(1+z), \\
\dot{\tilde{y}} &= \tilde{y} - (0.5\tilde{y} - 0.3x)(1+z), \\
\dot{z} &= \tilde{y} - \frac{z^2}{\tilde{y}}.
\end{aligned}$$

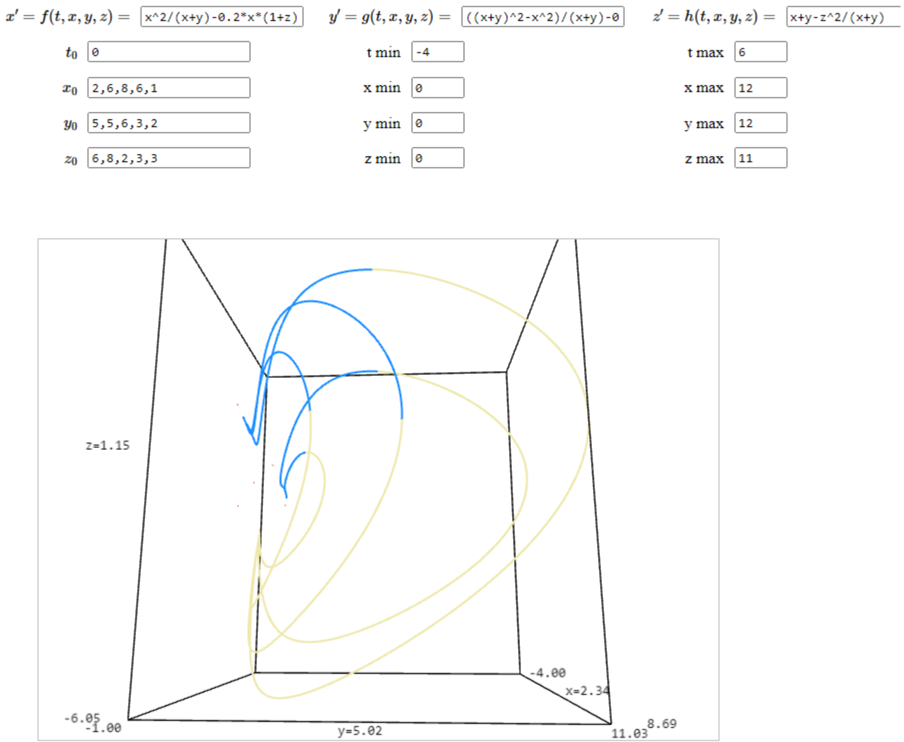


Figure 2: phase portrait of system (3)

The solution of

$$
\begin{aligned}
0 &= \frac{x^2}{x+y} - 0.2x(1+z), \\
0 &= \frac{(x+y)^2 - x^2}{x+y} - 0.5y(1+z), \\
0 &= (x+y) - \frac{z^2}{x+y}.
\end{aligned}
$$

defines the equilibrium points of the population system, which are marked in Figure 2 by red points.

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Healthcare Technologies and Big Data

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Abstract

An introduction of AI technologies in healthcare determined fundamental changes in healthcare. We are presenting major trends of adopted AI technologies worldwide in healthcare and current progress and challenges of AI adoption in Georgian healthcare systems.

Using technologies in Healthcare System is a part of the global healthcare development strategies. Capturing data and using them for analytic purposes gives possibilities to measure performance of the overall system.

Artificial Intelligence can serve as robust decision support system for healthcare workers, doctors, nurses, healthcare executives, payers, and governing bodies.

AI is based on the data which is processed by healthcare system players. Key players in Data production are doctors and nurses, who have the first access to the information generated during patient care.

AI can simplify the life of all health system workers by performing tasks which must be done by humans in a faster and cost-effective way.

Artificial Intelligence is the fastest growing business in the world. According to the CB INSIGHTS 2021 [1] was a year of unprecedented AI private market activity. AI funding increased by 108% in 2021 and Healthcare account for nearly a fifth of total funding.

AI is reinventing modern healthcare through machines that can predict, comprehend, learn, and act.
AI platforms are based on big data sources and during analyzing medical data and predicting or coming up with diagnosis, AI can be the way more trusted that humans, who have limited capabilities, according to the physiological and anatomic specification of the human brain.
AI is using data generated by healthcare players, including doctors, nurses, healthcare managers combined in HER systems. Also, field scientists, educational and research materials etc.

AI is using Machine Learning (ML) and Deep Learning (DL) Algorithms and statistical models to perform its tasks. Most important in ML and DL is Data captured in the systems using Electronic Health Records EHRs, scientific materials and medical literature.
However, the benefits of AI are very significant, still the field needs to adopt more regulatory rules for maintaining confidentiality of the medical data, raises more ethical concerns and system protection issues. Several countries are in the process of maintaining internal regulations on using AI platforms. E.g., Singapore adopted Artificial Intelligence in Healthcare Guidelines, which consists with all ruled for using AI systems in the country, to minimize the possibilities of the harm on patient care and population health.
Annually more than 400 000 inpatients are suffering with preventable harm, with 100 000 thousand of death cases, which can be avoided using AI technologies [2].

There are several ways how AI can save lives of patients [3]:

1. One of the most common problems during patient care can be delays, which can mean difference between life and death. AI can detect an issue and notify care teams to provide faster treatment decisions and save lives.

2. Using AI can help pathologists to come up with more precise diagnosis.
3. AI algorithms are used to diagnose and treat illness.
4. AI's deep learning platforms are used to analyze radiology images, blood tests, EKGs, genomics, patient medical history and support doctors in decision making process.
5. AI are used for screening purposes to detect cancer at the earliest stage and support diagnostic and treatment process by algorithms.
6. AI is used to identify harmful bacteria in patient blood and laboratory settings.
7. AI is used in Gastroenterology to identify early signs of GI diseases.
8. AI is used in Imaging technologies like Ultrasound, CT, MRI, and X-Ray for diagnostic purposes. Using big data in their applications, systems can come up with more precise diagnosis, compared with doctors.
9. AI is used in research for clinical trials for new treatment tools, drug development etc.
10. Combining AI and physics, AI can support drug development process and it can be done within days, rather than weeks.

Big organizations started to use AI applications during their everyday operations.
Here are several examples of how AI helped healthcare organizations perform on a higher quality level with better outcomes in patient care.
Cleveland Clinic in United States started partnership with IBM [4] to build an infrastructure that supports research in areas such as genomics, chemical and drug discovery, and population health.
John Hopkins Medicine [5] started partnership with GE Healthcare to adopt a system which supports patient flow process and as a result hospital has assigned patients to emergency departments to beds 38% faster.
AI system Babylon supports doctors [6] with deep learning tools, using symptom-checkers and up-to-date medical information.

AI can be used in healthcare management field, which generates huge amounts of data and usability of these data can be achieved by AI technologies. AI can analyze a data which can be useful for public and population health purposes, for supporting drug development process in pharmacy field, in analyzing payers' information and helping them to predict healthcare costs and make systems more cost-effective, can support governing regulatory agencies in population health management and planning purposes.

Systems used in Georgian Healthcare:

1. Mydoc.ge https://mydoc.ge – already it is used, and they have some statistics.
2. https://www.heaps.ai/index_in.html – we are starting to use this system in American Hospital Tbilisi – this is used for post-discharge follow-up and chronic disease management.

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