.

Computational Business Administration

Edited by T. Toronjadze

GAU BUSINESS SCHOOL DEAN

DR. PROFESSOR

Tbilisi, Georgia, 2014

The book has been written by group of GAU Business School professors:

T. Toronjadze
T. Uzunashvili
T. Mosiashvili
G. Pertaia
L. Gachechiladze
I. Chelidze
B. Gogichashvili

Technical Editor: *M. Kvinikadze*

CONTENTS

PREFACE

The purpose of the current book is to introduce to students computational solutions of the cases in the business administration.

First part of the book contains general business models and includes linear programming, forecasting, and games theory. This part of the book is for all business administration students and accompanies Essentials of Business Administration book.

Second part of the book is dedicated to advance students with major in quantitative finance and deals with financial modeling problems.

BUSINESS MODELING

1. LINEAR PROGRAMMING – PRODUCTION MIX

This method determines the product mix that maximizes corporate profitability

Companies often need to determine the monthly (or weekly) production schedule that gives the quantity of each product that must be produced. In its simplest incarnation, the product mix problem involves how to determine the amount of each product that should be produced during a month to maximize profits. Product mix must often satisfy the following constraints:

- Product mix can't use more resources than are available.
- There is a limited demand for each product. We can't produce more of a product during a month than is demanded because the excess production is wasted

Let's now solve the following example of the product mix problem:

Let's say we work for a drug company that can produce six products at their plant. Production of each product requires labor and raw material. Row 4 in Figure 3-1 gives the hours of labor needed to produce a pound of each product, and row 5 gives the pounds of raw material needed to produce a pound of each product.

	A	B	C	D	E	F	G	H	I
1									
2			Pounds Made	150	160	170	180	190	200
3		Available	Product	1	2	3	4	5	6
4		4500	Labour	6	5	4	3	2.5	1.5
5		1600	Raw Material	3.2	2.6	1.5	0.8	0.7	0.3
6			Unit Price	$12.50	$11.00	$9.00	$7.00	$6.00	$3.00
7			Variable Cost	$6.50	$5.70	$3.60	$2.80	$2.20	$1.20
8			Demand	960	928	1041	977	1084	1055
9			Unit profit Cont.	$6.00	$5.30	$5.40	$4.20	$3.80	$1.80
10									
11									
12			Profit	4504					
13						Available			
14			Labor Used	3695	<=	4500			
15			Raw Material Use	1488	<=	1600			
16									

Figure 3-1.

For example, producing a pound of product 1 requires 6 hours of labor and 3.2 pounds of raw material. For each drug, the price per pound is given in row 6, the unit cost per pound is given in row 7, and the profit contribution per pound is given in row 9. For example, product 2 sells for $11.00 per pound, incurs a unit cost of $5.70 per pound, and contributes $5.30 profit per pound. This month's demand for each drug is given in row 8. For example, demand for product 3 is 1041 pounds. This month, 4500 hours of labor and 1600 pounds of raw material are available. How can this company maximize its monthly profit?

If we knew nothing about the Excel Solver, we would attack this problem by constructing a spreadsheet in which we track for each product mix the profit and resource usage associated with the product mix. Then we would use trial and error to vary the product mix to optimize profit without using more labor or raw material than is available and without producing more of any drug than there is demand. We use Solver in this process only at the trial-and-error stage. Essentially, Solver is an optimization engine that flawlessly performs the trial-and-error search.

A key to solving the product mix problem is efficiently computing the resource usage and profit associated with any given product mix. An important tool that we can use to make this computation is the SUMPRODUCT function. The SUMPRODUCT function multiplies corresponding values in cell ranges and returns the sum of those values. Each cell range used in a SUMPRODUCT evaluation must have the same dimensions, which implies that you can use SUMPRODUCT with two rows or two columns but not with a column and a row.

As an example of how we can use the SUMPRODUCT function in our product mix example, let's try to compute our resource usage. Our labor usage is given by

*(Labor used per pound of drug 1) * (Drug 1 pounds produced) + (Labor used per pound of drug 2) * (Drug 2 pounds produced) + ... (Labor used per pound of drug 6) * (Drug 6 pounds produced)*

We could compute labor usage in a tedious fashion as *D2*D4 + E2*E4 + F2*F4 + G2*G4 + H2*H4 + I2*I4*. Similarly, raw material usage could be computed as *D2*D5 + E2*E5 + F2*F5 + G2*G5 + H2*H5 + I2*I5*. Entering these formulas in a spreadsheet is time-consuming with six products. Imagine how long it would take if you were working with a company that produced, say, 50 products at their plant. A much easier way to compute labor and raw material usage is to copy from D14 to D15 the formula *SUMPRODUCT($D$2:$I$2,D4:I4)*. This formula computes *D2*D4 + E2*E4 + F2*F4 + G2*G4 + H2*H4 + I2*I4* (which is our labor usage) and is much easier to enter! Notice that I use the $ sign with the range D2:I2 so that when I copy the formula I still pull the product mix from row 2. The formula in cell D15 computes raw material usage.

In a similar fashion, our profit is given by

(Drug 1 profit per pound)(Drug 1 pounds produced) + (Drug 2 profit per pound)* (Drug 2 pounds produced) + ...(Drug 6 profit per pound)*(Drug 6 pounds produced).*

Profit is easily computed in cell D12 with the formula*SUMPRODUCT(D9:I9,$D$2:$I$2)*.

We now can identify the three parts of our product mix Solver model.

- **Target cell -** Our goal is to maximize profit (computed in cell D12).
- **Changing cells -** The number of pounds produced of each product (listed in the cell range D2:I2).
- **Constraints -** We have the following constraints:

- Do not use more labor and raw material than are available. That is, the values in cells D14:D15 (resources used) must be less than or equal to the values in cells F14:F15 (the available resources)
- Do not produce more of a drug than is in demand. That is, the values in the cells D2:I2 (pounds produced of each drug) must be less than or equal to the demand for each drug (listed in cells D8:I8).
- We can't produce a negative amount of any drug

Let's see how to input the target cell, changing cells, and constraints into Solver. Then, all we need to do is click the Solve button and Solver will find a profit-maximizing product mix!

To begin, select Tools, Solver. The Solver Parameters dialog box will appear, as shown in Figure 3-2.

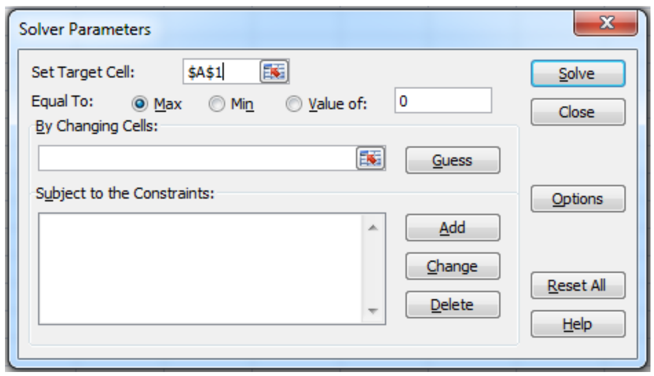


Figure 3-2. The Solver Parameters dialog box.

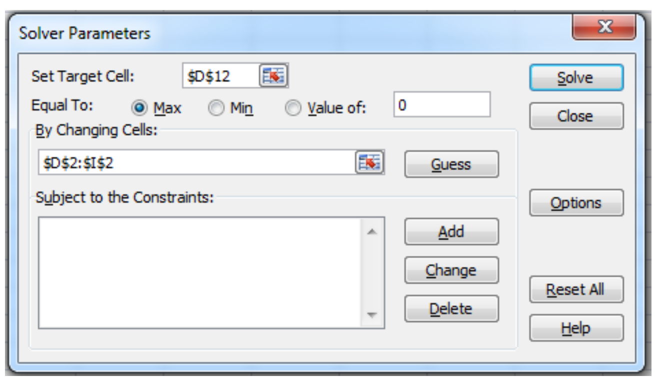


Figure 3-3. The Solver Parameters dialog box with the target cell and changing cells defined.

To input the target cell, click in the Set Target Cell box and then select our profit cell (cell D12). To input our changing cells, click in the By Changing Cells box and then point to the range D2:I2, which contains the pounds, produced of each drug. The dialog box should now look Figure 3-3.

We're now ready to add constraints to the model. Click the Add button. You'll see the Add Constraint dialog box, shown in Figure 3-4.

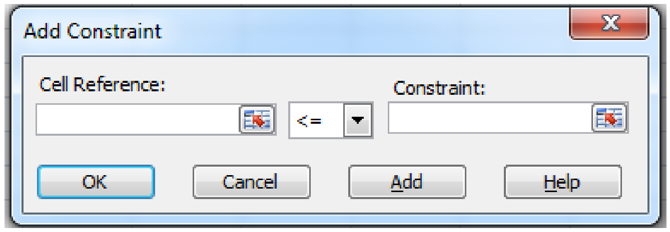


Figure 3-4. The Add Constraint dialog box.

To add the resource usage constraints, click in the box labeled Cell Reference and then select the range D14:D15. Select <= from the drop-down list in the middle of the dialog box. Click in the box labeled Constraint, and then select the cell range F14:F15. The Add Constraint dialog box should now look like Figure 3-5.

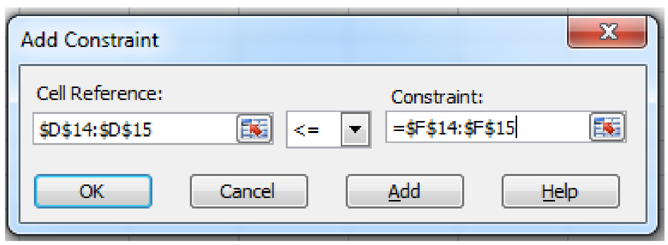


Figure 3-5. The Add Constraint dialog box with the resource usage constraints entered.

We have now ensured that when Solver tries different values for the changing cells, Solver will consider only combinations that satisfy both D14<=F14 (labor used is less than or equal to labor available) and D15<=F15 (raw material used is less than or equal to raw material available). Now click Add in the Add Constraint dialog box to enter the demand constraints. Simply fill in the Add Constraint dialog box as shown in Figure 3-6.

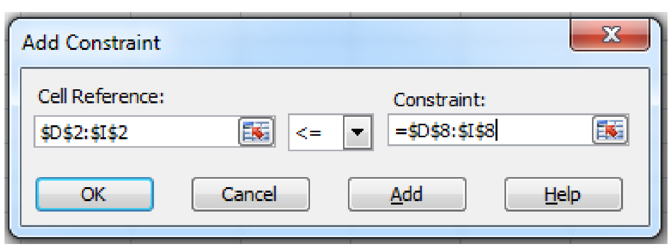


Figure 3-6. The Add Constraint dialog box with the demand constraints entered.

Adding these constraints ensures that when Solver tries different combinations for the changing cell values, Solver will consider only combinations that satisfy the following:

- D2<=D8 (the amount of drug 1 made is less than or equal to the demand for drug 1)
- E2<=E8 (the amount of drug 2 made is less than or equal to the demand for drug 2)
- F2<=F8 (the amount of drug 3 made is less than or equal to the demand for drug 3)
- G2<=G8 (the amount of drug 4 made is less than or equal to the demand for drug 4)
- H2<=H8 (the amount of drug 5 made is less than or equal to the demand for drug 5)
- I2<=I8 (the amount of drug 6 made is less than or equal to the demand for drug 6)

Click OK in the Add Constraint dialog box. The Solver window should look like Figure 3-7.

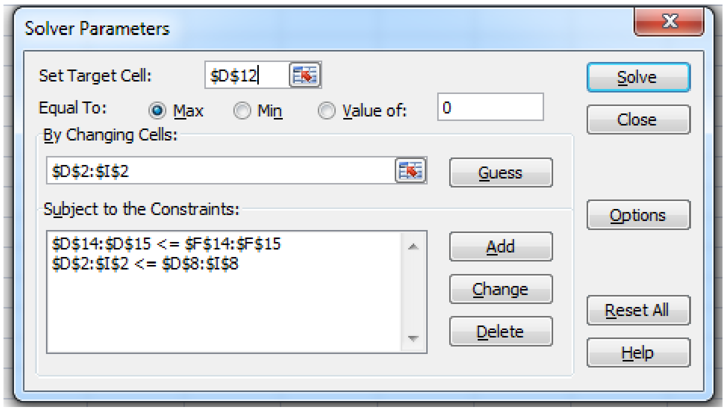


Figure 3-7. The final Solver window for the product mix problem.

We enter the constraint that all changing cells be nonnegative in the Solver Options dialog box. Click the Options button in the Solver Parameters dialog box. Select the options Assume Linear Model and Assume Non-Negative, as shown in Figure 3-8. Click OK.

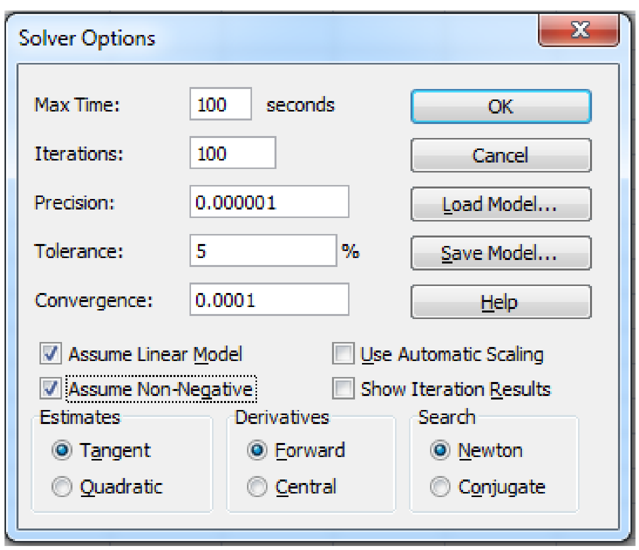


Figure 3-8. Solver options settings.

Selecting the Assume Non-Negative option ensures that Solver considers only combinations of changing cells in which each changing cell assumes a nonnegative value. We selected Assume Linear Model because the product mix problem is a special type of Solver problem called a linear model. Essentially, a Solver model is linear under the following conditions:

- The target cell is computed by adding together terms of the form *(changing cell)*(constant).*
- Each constraint satisfies the "linear model requirement." This means that each constraint is evaluated by adding together terms of the form *(changing cell)*(constant)* and comparing such sums to a constant.

Why is this Solver problem linear? Our target cell (profit) is computed as *(Drug 1 profit per pound) * (Drug 1 pounds produced) + (Drug 2 profit per pound) * (Drug 2 pounds produced) + …(Drug 6 profit per pound) * (Drug 6 pounds produced)*

This computation follows a pattern in which the target cell's value is derived by adding together terms of the form *(changing cell) * (constant)*.

Our labor constraint is evaluated by comparing the value derived from *(Labor used per pound of drug 1) * (Drug 1 pounds produced) + (Labor used per pound of drug 2) * (Drug 2 pounds produced) + ... (Labor used per pound of drug 6) * (Drug 6 pounds produced) to the labor available.*

Therefore, the labor constraint is evaluated by adding together terms of the form *(changing cell) * (constant)* and comparing such sums to a constant. Both the labor constraint and the raw material constraint satisfy the linear model requirement.

Our demand constraints take the form

(Drug 1 produced)<=(Drug 1 Demand)
(Drug 2 produced)<=(Drug 2 Demand)
:
(Drug 6 produced)<=(Drug 6 Demand)

Each demand constraint also satisfies the linear model requirement because each is evaluated by adding together terms of the form *(changing cell) * (constant)* and comparing such sums to a constant.

Having shown that our product mix model is a linear model, why should we care?

- If a Solver model is linear and we select Assume Linear Model, Solver is guaranteed to find the optimal solution to the Solver model. If a Solver model is not linear, Solver may or may not find the optimal solution.
- If a Solver model is linear and we select Assume Linear Model, Solver uses a very efficient algorithm (the simplex method) to find the model's optimal solution. If a Solver model is linear and we do not select Assume Linear Model, Solver uses a very inefficient algorithm (the GRG2 method) and might have difficulty finding the model's optimal solution.

After clicking OK in the Solver Options dialog box, we're returned to the main Solver dialog box, shown earlier in Figure 3-7. When we click Solve, Solver calculates an optimal solution (if one exists) for our product mix model. An optimal solution to the product mix model would be a set of changing cell values (pounds produced of each drug) that maximizes profit over the set of all feasible solutions. Again, a feasible solution is a set of changing cell values satisfying all constraints. The changing cell values shown in Figure 3-9 are a feasible solution because all production levels are nonnegative, no production levels exceed demand, and resource usage does not exceed available resources.

	A	B	C	D	E	F	G	H	I
1									
2			Pounds Made	150	160	170	180	190	200
3		Available	Product	1	2	3	4	5	6
4		4500	Labour	6	5	4	3	2.5	1.5
5		1600	Raw Material	3.2	2.6	1.5	0.8	0.7	0.3
6			Unit Price	$12.50	$11.00	$9.00	$7.00	$6.00	$3.00
7			Variable Cost	$6.50	$5.70	$3.60	$2.80	$2.20	$1.20
8			Demand	960	928	1041	977	1084	1055
9			Unit profit Cont.	$6.00	$5.30	$5.40	$4.20	$3.80	$1.80
10									
11									
12			Profit	4504					
13						Available			
14			Labor Used	3695	<=	4500			
15			Raw Material Use	1488	<=	1600			
16									

Figure 3-9. A feasible solution to the product mix problem fits within constraints.

The changing cell values shown in Figure 3-10 represent an infeasible solution for the following reasons:

- We produce more of drug 5 than is demanded.
- We use more labor than labor available.
- We use more raw material than raw material available

	A	B	C	D	E	F	G	H	I
1									
2			Pounds Made	300	0	0	0	1085	1000
3		Available	Product	1	2	3	4	5	6
4		4500	Labour	6	5	4	3	2.5	1.5
5		1600	Raw Material	3.2	2.6	1.5	0.8	0.7	0.3
6			Unit Price	$12.50	$11.00	$9.00	$7.00	$6.00	$3.00
7			Variable Cost	$6.50	$5.70	$3.60	$2.80	$2.20	$1.20
8			Demand	960	928	1041	977	1084	1055
9			Unit profit Cont.	$6.00	$5.30	$5.40	$4.20	$3.80	$1.80
10									
11									
12			Profit	$7 723.00					
13						Available			
14			Labor Used	6012.5	<=	4500			
15			Raw Material Use	2019.5	<=	1600			
16									

Figure 3-10. An infeasible solution to the product mix problem doesn't fit within the constraints we defined.

After clicking Solve, Solver quickly finds the optimal solution shown in Figure 3-11. You need to select Keep Solver Solution to preserve the optimal solution values in the spreadsheet.

	A	B	C	D	E	F	G	H	I
1									
2			Pounds Made	0	0	0	596.667	1084	0
3		Available	Product	1	2	3	4	5	6
4		4500	Labour	6	5	4	3	2.5	1.5
5		1600	Raw Material	3.2	2.6	1.5	0.8	0.7	0.3
6			Unit Price	$12.50	$11.00	$9.00	$7.00	$6.00	$3.00
7			Variable Cost	$6.50	$5.70	$3.60	$2.80	$2.20	$1.20
8			Demand	960	928	1041	977	1084	1055
9			Unit profit Cont.	$6.00	$5.30	$5.40	$4.20	$3.80	$1.80
10									
11									
12			Profit	$6 625.20					
13						Available			
14			Labor Used	4500	<=	4500			
15			Raw Material Use	1236.133	<=	1600			
16									

Figure 3-11. The optimal solution to the product mix problem.

Our drug company can maximize its monthly profit at a level of $6,625.20 by producing 596.67 pounds of drug 4, 1084 pounds of drug 5, and none of the other drugs! We can't determine if we can achieve the maximum profit of $6,625.20 in other ways. All we can be sure of is that with our limited resources and demand, there is no way to make more than $6,625.20 this month.

Does a Solver model always have a solution?

Suppose that demand for each product must be met. We then have to change our demand constraints from D2:I2<=D8:I8 to D2:I2>=D8:I8. To do this, open Solver, select the D2:I2<=D8:I8 constraint, and then click Change. The Change Constraint dialog box, shown in Figure 3-12, appears.

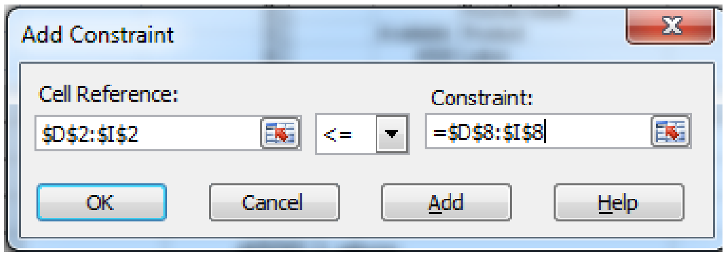


Figure 3-12. The Change Constraint dialog box.

Select >=, and then click OK. We've now ensured that Solver will consider only changing cell values that meet all demands. When you click Solve, you'll see the message "Solver could not find a feasible solution." This message means that with our limited resources, we can't meet demand for all products. We have not made a mistake in our model! Solver is simply telling us that if we want to meet demand for each product, we need to add more labor, more raw materials, or more of both.

Data Analyses and Business Modeling, Wayne L. Winston, H.B. Fenn and Company Ltd 2004, Chapter 25, pp. 197-208

2. LINEAR PROGRAMMING – SCHEDULING

This method efficiently schedules workforce to meet labor demands

Many organizations (banks, restaurants, postal services) know what their labor requirements will be at different times and need a method to efficiently schedule their workforce to meet their labor requirements. Excel Solver can be used to easily address a problem such as this. Here’s an example.

How can I efficiently schedule my workforce to meet labor demands? Bank 24 processes checks 7 days a week. The number of workers needed each day to process checks is given in row 14 of the file Bank24.xls, which is shown in Figure 4-1. For example, 13 workers are needed on Tuesday, 15 workers are needed on Wednesday, and so on. All bank employees work five consecutive days. What is the minimum number of employees Bank 24 can have and still meet its labor requirements?

	A	B	C	D	E	F	G	H	I	J
1										
2	**Total**									
3	**20**		**Working?**							
4										
5	Number Starting	Day Worker Starts	Monday	Tuesday	Wednesd	Thursday	Friday	Saturday	Sunday	
6	1	Monday	1	1	1	1	1	0	0	
7	3	Tuesday	0	1	1	1	1	1	0	
8	0	Wednesday	0	0	1	1	1	1	1	
9	4	Thursday	1	0	0	1	1	1	1	
10	1	Friday	1	1	0	0	1	1	1	
11	2	Saturday	1	1	1	0	0	1	1	
12	9	Sunday	1	1	1	1	0	0	1	
13		Number Working	17	0	3	3	3	3	3	
14			>=	>=	>=	>=	>=	>=	>=	
15			17	13	15	17	9	9	12	
16										

Figure 4-1. The data we’ll use to work through the bank workforce scheduling problem.

We begin by identifying the target cell, changing cells, and constraints for our Solver model.

- **Target cell -** Minimize total number of employees.
- **Changing cells -** Number of employees who start work (the first of five consecutive days) each day of the week. Each changing cell must be a nonnegative integer.
- **Constraints -** For each day of the week, the number of employees who are working must be greater than or equal to the number of employees required. _(Number of employees working)>=(Needed employees.)_

To set up our model, we need to track the number of employees working each day. Let us begin by entering trial values for the number of employees who start their five-day shift each day in the cell range A5:A11. For example, in A5, Let us enter 1, indicating that 1 employee begins work on Monday and works Monday through Friday. I entered each day's required workers in the range C14:I14.

To track the number of employees working each day, I entered in each cell in the range C5:I11 a 1 or a 0. The value 1 in a cell indicates that the employees who started working on the day designated in the cell's row are working on the day associated with the cell's column. For example, the 1 in cell G5 indicates that employees who started working on Monday are working on Friday; the 0 in cell H5 indicates that the employees who started working on Monday are not working on Saturday.

By copying from C12 to D12:I12 the formula*SUMPRODUCT($A$5:$A$11, C5:C11)*, I compute the number of employees working each day. For example, in cell C12 this formula evaluates to *A5+A8+A9+A10+A11*, which equals *(Number starting on Monday) + (Number starting on Thursday) + (Number starting on Friday) + (Number starting on Saturday) + (Number starting on Sunday)*. This total is indeed the number of people working on Monday.

After computing the total number of employees in cell A3 with the formula *SUM(A5:A11)*, I can enter our model in Solver as shown in Figure 4-2.

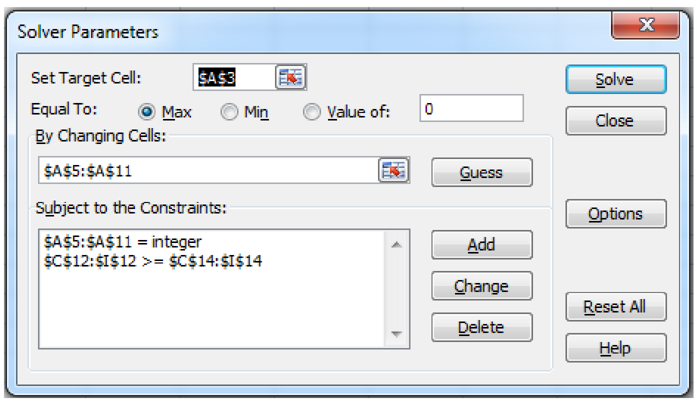


Figure 4-2. The Solver Parameters dialog box filled in to solve the bank workforce problem.

In the target cell (A3), we want to minimize the number of total employees. The constraint *C12:I12>=C14:I14* ensures that the number of employees working each day is at least as large as the number needed each day. The constraint *A5:A11 = integer* ensures that the number of employees beginning work each day is an integer. To add this constraint, I clicked Add in the Solver Parameters dialog box and filled in the Add Constraint dialog box as shown in Figure 4-3.

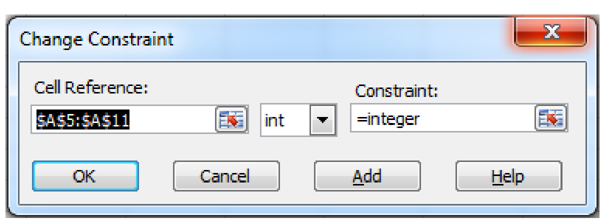


Figure 4-3. This constraint defines as an integer the number of workers who start each day.

I also selected the options Assume Linear Model and Assume Non-Negative for the changing cells by clicking Options in the Solver Parameters dialog box and then checking these options in the Solver Options dialog box. After clicking Solve, we find the optimal solution that's shown earlier in Figure 4-1.

A total of 20 employees are needed. One employee starts on Monday, 3 start on Tuesday, 0 start on Wednesday, 4 start on Thursday, 1 starts on Friday,2 starts on Saturday, and 9 starts on Sunday.

Note that this model is linear because the target cell is created by adding together changing cells and the constraint is created by comparing the result obtained by adding together the product of each changing cell times a constant (either 1 or 0) to the required number of workers.

Data Analyses and Business Modeling, Wayne L. Winston, H.B. Fenn and Company Ltd 2004, Chapter 27, pp. 215-220

3. LINEAR PROGRAMMING – TRANSPORTATION

This method determines the locations at which products shouldbe manufactured and from which they should be shipped to customers.

Many companies manufacture products at locations (often called supply points) and ship their products to customers (often called demand points). A natural question is what is the least expensive way to produce and ship products to customers and meet customer demand? This type of problem is called a transportation problem. A transportation problem can be set up as a linear Solver model with the following specifications:

- **Target cell -** Minimize total production and shipping cost.
- **Changing cells -** The amount produced at each supply point that is shipped to each demand point.
- **Constraints -** The amount shipped from each supply point can't exceed plant capacity. Each demand point must receive its required demand. Also, each changing cell must be nonnegative.

How can a drug company determine the locations at which they should produce drugs and from which they should ship drugs to customers?

You can follow along with this problem by looking at the file Transport.xls. Let's suppose a drug company produces a drug in Los Angeles, Atlanta, and New York. The Los Angeles plant can produce up to 10,000 pounds of the drug per month. Atlanta can produce up to 12,000 pounds of the drug per month, and New York can produce up to 14,000 pounds per month. Each month, the company must ship to the four regions of the United States—East, Midwest, South, and West—the number of pounds listed in cells B2:E2, as shown in Figure 5-1. For example, the West region must receive at least 13,000 pounds of the drug each month. The cost per pound of producing a drug at each plant and shipping the drug to each region of the country are given in cells B4:E6. For example, it costs $3.50 to produce a pound of the drug in Los Angeles and ship it to the Midwest region. What is the cheapest way to get each region the quantity of the drug they need?

To express our target cell, we need to track total shipping cost. After entering in the cell range B10:E12 trial values for our shipments from each supply point to each region, we can compute total shipping cost as the following:

<u>*(Amount sent from LA to East)* (Cost per pound of sending drug from LA to East) + (Amount sent from LA to Midwest)*(Cost per pound of sending drug from LA to Midwest) + (Amount sent from LA to South)*(Cost per pound of sending drug from LA to South) + (Amount sent from LA to West)*(Cost per pound of sending drug from LA to West) + ... + (Amount sent from New York City to West)*(Cost per pound of sending drug from New York City to West)*</u>

	A	B	C	D	E	F	G	H	I
1									
2	Demand	9000	6000	6000	13000				
3		EAST	MIDWEST	SOUTH	WEST	CAPACITY			
4	LA	$ 5.00	$ 3.50	$ 4.20	$ 2.20	10 000			
5	ATLANTA	$ 3.20	$ 2.60	$ 1.80	$ 4.80	12 000			
6	NEW YORK CITY	$ 2.50	$ 3.10	$ 3.30	$ 5.40	14 000			
7									
8	Shipments								
9		EAST	MIDWEST	SOUTH	WEST	Sent			
10	LA	0	0	0	10000	10000	<=	10000	
11	ATLANTA	0	3000	6000	3000	12000	<=	12000	
12	NEW YORK CITY	9000	3000	0	0	12000	<=	14000	
13	Received	9000	6000	6000	13000				
14		>=	>=	>=	>=				
15		9000	6000	6000	13000				
16									
17	Total Cost	$86 800.00							
18									

Figure 5-1. Data for a transportation problem that we'll set up to be resolved by Solver.

The SUMPRODUCT function can multiply corresponding elements in two separate rectangles (as long as the rectangles are the size) and add together the products. I've named the cell range B4:E6 as costs and the changing-cells range (B10:E12) as shipped. Therefore, our total shipping and production cost is computed in cell B18 with the formula SUMPRODUCT(costs, shipped).

To express our constraints, we first compute the total shipped from each supply point. By entering the formula *SUM (B10:E10)* in cell F10, we compute the total number of pounds shipped from Los Angeles as *(LA shipped to East) + (LA shipped to Midwest) + (LA shipped to South) + (LA shipped to West)*. Copying this formula to *F11:F12* computes the total shipped from Atlanta and New York City. Later I'll add constraints (called supply constraints) that ensure the amount shipped from each location does not exceed the plant's capacity.

Next I compute the total received by each demand point. I begin by entering in cell B13 the formula *SUM(B10:B12)*. This formula computes the total number of pounds received in the East as *(Pounds shipped from LA to East) + (Pounds shipped from Atlanta to East) + (Pounds shipped from New York City to East)*. By copying this formula from B13 to C13:E13, I compute the pounds of the drug received by the Midwest, South, and West regions. Later, I'll add constraints (called demand constraints) that ensure that each region receives the amount of the drug it requires.

We now open the Solver Parameters dialog box (click Tools, Solver) and fill it in as shown in Figure 5-2.

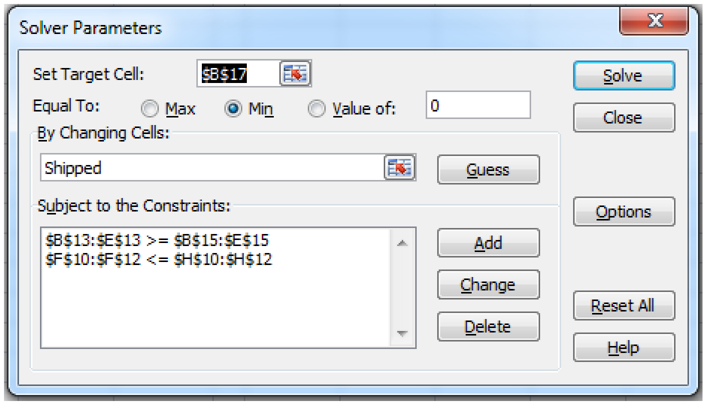


Figure 5-2. The Solver set up to answer our transportation example.

We want to minimize total shipping cost (computed in cell B18). Our changing cells are the number of pounds shipped from each plant to each region of the country. (These amounts are listed in the range named shipped, consisting of cells *B10:E12*.) The constraint *F10:F12<=H10:H12* (the supply constraint) ensures that the amount sent from each plant does not exceed itscapacity. The constraint *B13:E13>=B15:E15* (the demand constraint) ensures that each region receives at least the amount of the drug it needs.

Our model is a linear Solver model because our target cell is created by adding together terms of the form *(changing cell)*(constant)* and both our supply and demand constraints are created by comparing the sum of changing cells to a constant.

I now click Options in the Solver Parameters dialog box and check the Assume Linear Model and Assume Non-Negative options. After clicking Solve in the Solver Parameters dialog box, we're presented with the optimal solution shown earlier in Figure 5-1. The minimum cost of meeting customer demand is $86,800. This minimum cost can be achieved if the company uses the following production and shipping schedule:

- Ship 10,000 pounds from Los Angeles to the West region.
- Ship 3000 pounds from Atlanta to the West region and from Atlanta to the Midwest region. Ship 6000 pounds from Atlanta to the South region.
- Ship 9000 pounds from New York City to the East region and 3000 pounds from New York City to the Midwest region.

Data Analyses and Business Modeling, Wayne L. Winston, H.B. Fenn and Company Ltd 2004, Chapter 26, pp. 209-214

4. SIMULATION MODELS

Monte-Carlo simulations

Monte Carlo simulation is a method of analyzing the data in the cases when it is too complicated for conventional analytic method. Monte Carlo simulation allows us to do various complex calculations for example pricing exotic options or calculating risk associated with market fluctuations. At the core of Monte Carlo method is the idea of randomness. In order to carry out Monte Carlo simulation we need an efficient method for generating random numbers, thus we should use computers for it (doing Monte Carlo simulation without one is almost impossible!). Below will be given the examples of various random variables and methods of generating them.

Uniform random variable

Uniform random variable is the simplest type of random variable, in fact all other random variables are derived using uniform variables. Uniform random numbers have uniform distribution which assigns equal probabilities to each number on the given interval.

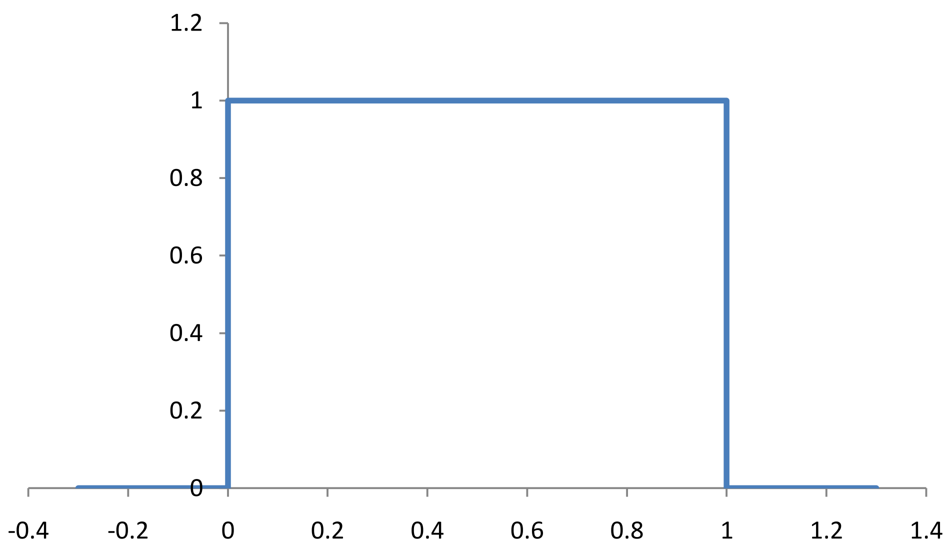


Figure 7-1.

Figure 7-1 shows the uniform distribution on interval [0,1]. As one can see from the graph this distribution gives equal probabilities to numbers in interval [0,1], for example probability that the random number will be in the interval [0.2,0.4] is the same as probability that number will be in interval [0.8,1] (and is equal to 20%)

We can generate uniform random variables in excel using formula “=rand()”

=RAND()
0.907809
0.825425
0.239661
0.529423
0.201589
0.179569
0.327986
0.780362
0.144663
0.659952
0.107574
0.656306
0.658285
0.08277

This formula give random variable in the interval [0,1]. But what if we need random variable in the interval [a,b]? in this case we can use simple formula:

$$rand[a,b] = a + (b - a) * rand[0,1] \quad (1)$$

For example if we need interval [5, 24]:

=5+(24-5)*RAND()
12.73544
20.89141
23.52896
15.55573
18.03485
16.0834
16.6205
19.457
15.15249
11.41532
8.956105
6.559959
8.864091
9.860163

The resulting random variable is uniformly distributed on [5,24].

There is another formula in excel that gives uniform random variable in the interval [a,b], this is “=randbetween(a,b)” however this formula gives only integer random variables for example it returns 5,7,12 but not 5.42 or 7.19, thus if continuous uniform random variable is necessary (1) formula should be used.

Normal random variable

Another widely used random variable is normal random variable. This kind of random numbers have normal distribution with given μ (miu) and σ (sigma), where μ indicates the center of the distribution, its mean, and σ is the spread around the mean. for example normal distribution wit μ =1 and σ=3 is given below:

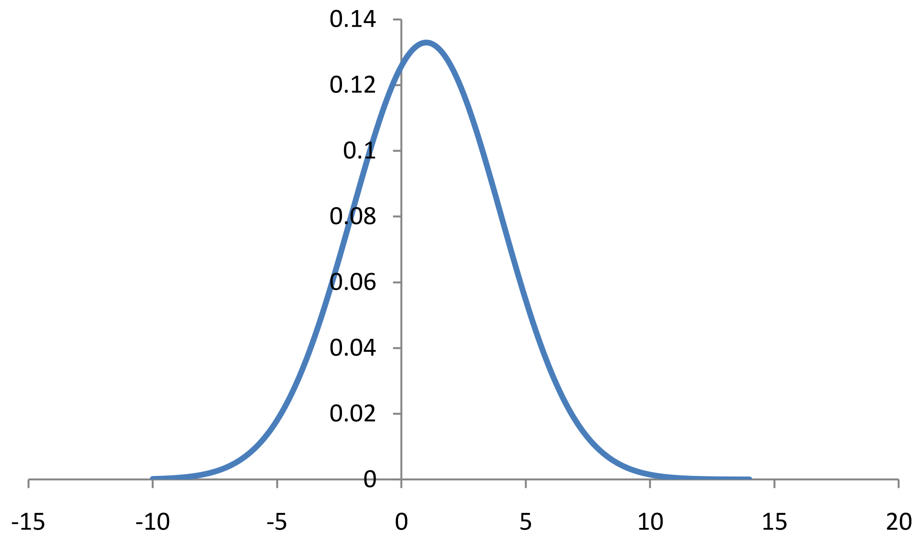
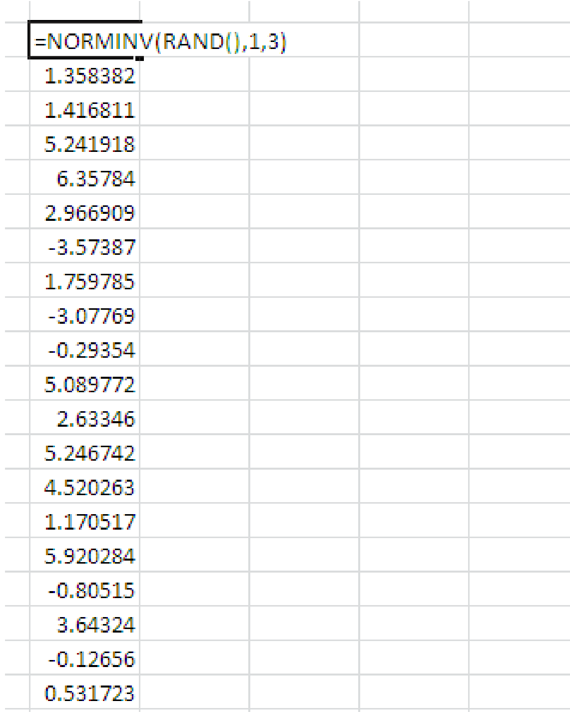


Figure 7-2.

Normal distribution (in contrast to uniform distribution) assigns larger probabilities to the values in the center, thus average values are most likely to occur.

Normal random variables can be generated using excel formula "=norminv(rand() , m , s)" where rand is uniform random variable and m,s are miu and sigma of the distribution.



=NORMINV(RAND(),1,3)
1.358382
1.416811
5.241918
6.35784
2.966909
-3.57387
1.759785
-3.07769
-0.29354
5.089772
2.63346
5.246742
4.520263
1.170517
5.920284
-0.80515
3.64324
-0.12656
0.531723

Microsoft Excel Data Analysis and Business Modeling, W.Winston, ch. 56

Simulating distributions given in tables

Suppose we have a distribution in the table form:

x	probability
4	0.1
5	0.2
6	0.3
7	0.25
8	0.15

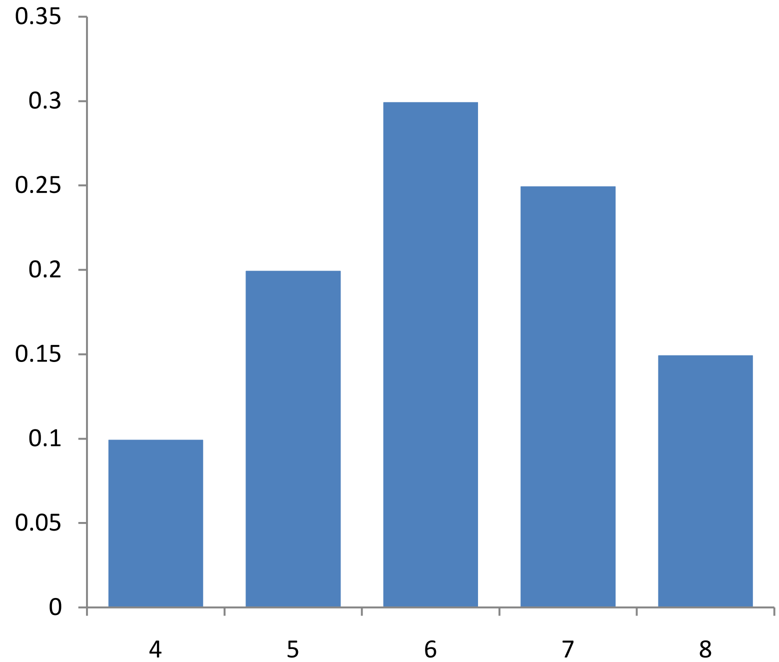


Figure 7-3.

To sample from this distribution we can use excels “**lookup”** function. Before we use this function we need to calculate the cumulative distribution. Cumulative distribution is just the sum of all previous probabilities (up to given x).

x	probability	cummulative
4	0.1	0.1
5	0.2	0.3
6	0.3	0.6
7	0.25	0.85
8	0.15	1

x	probability	cummulative
4	0.1	=C5
5	0.2	0.3
6	0.3	0.6
7	0.25	0.85
8	0.15	1

x	probability	cummulative
4	0.1	0.1
5	0.2	=C6+D5
6	0.3	0.6
7	0.25	0.85
8	0.15	1

Second step is to shift down cumulative probabilities by one step (this is required for excel formula)

x	probability	cummulative
4	0.1	0
5	0.2	0.1
6	0.3	0.3
7	0.25	0.6
8	0.15	0.85

In the first cell (instead of 0.1) we write 0 and delete the final cumulative probability which is always 1.

Next we generate uniform random numbers with "**rand()**" function and simulate the distribution with "**lookup**" function.

uniform rand	simulations
=RAND()	4
0.577541242	6
0.344743821	6
0.432587051	6
0.218155549	5
0.227361187	5
0.082764656	4
0.022450076	4
0.235582099	5

x	probability	cummulative
4	0.1	0
5	0.2	0.1
6	0.3	0.3
7	0.25	0.6
8	0.15	0.85

uniform rand	simulations
0.140817546	5
0.672046294	7
0.141878337	=LOOKUP(B16,$D$5:$D$9,$B$5:$B$9)
0.832339705	7
0.955879032	8
0.808137028	7
0.33363289	6

Microsoft Excel Data Analysis and Business Modeling, W.Winstonch. 58

5. FORECASTING MODELS

Forecasting

The purpose of this chapter is to introduce forecasting models. Forecasting is the process of estimating future values of some variable using historical data.

One factor linear regression

Linear regression is the simplest tool for forecasting the data. One factor linear regression tries to find the linear relationship between two variables. The formula for one factor linear regression is:

$$y = ax + b \tag{1}$$

a is the slope coefficient and b is the intercept with y axis. For example we have the following data:

x	y
3.05	3.17
3.24	4.61
3.54	3.53
3.75	3.44
3.91	3.92
4.05	4.98
4.55	4.86
4.71	4.65
4.91	5.02
5.68	5.34
5.7	5.54
5.73	5.99
5.9	5.9
6.37	6.61

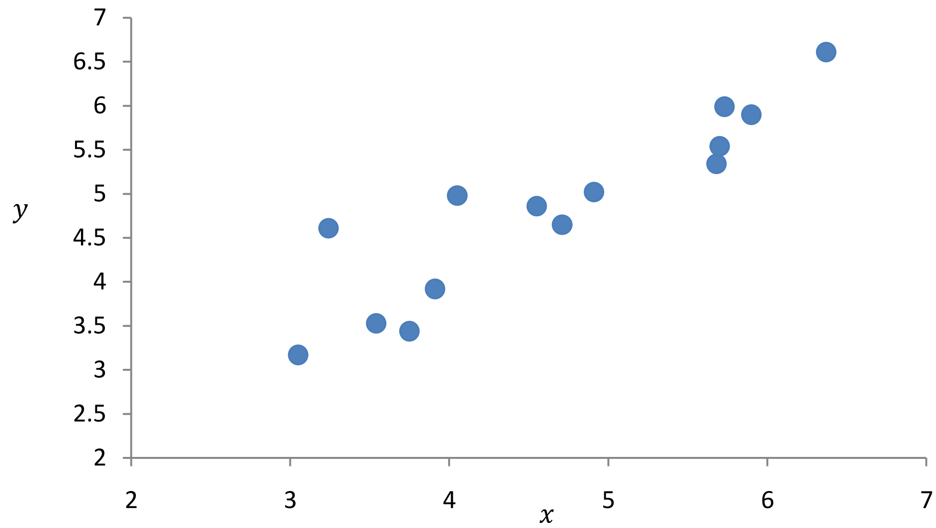


Figure 8-1.

We would like to perform linear regression analysis on this data. in order to do this we must “fit” our model to the data.

In order to fit the model we use following formulas:

$$a = \frac{cov(x, y)}{var(x)}$$
$$b = \bar{y} - a\bar{x}$$

$cov(x, y)$ is the covariance between $x$and y, $var(x)$ is the variance of x and $\bar{y}, \bar{x}$ are average of $y$and average of x.

Using above formulas for our example we get:

$$a = \frac{cov(x, y)}{var(x)} = \frac{0.9429}{1.1021} = 0.8556$$

And

$$b = \bar{y} - a\bar{x} = 4.8257 - 0.8556 * 4.6492 = 0.8477$$

Or

$$y = 0.8556 * x + 0.8477$$

Using this result we can forecast future values of y based on future values of x, for example if $x = 7$ than $y = 0.8556 * x + 0.8477 = 0.8556 * 7 + 0.8477 = 6.84$

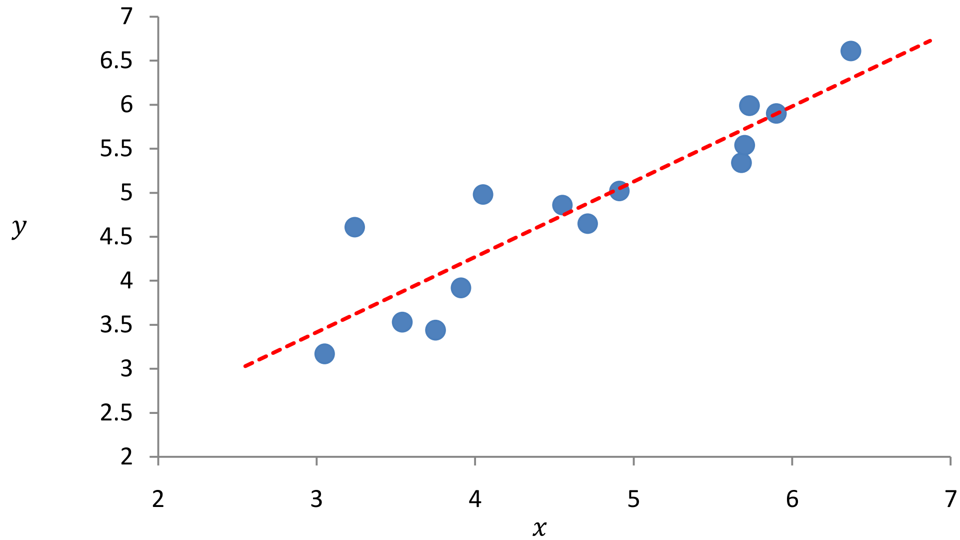


Figure 8-2.

Statics for Business and Economics, Sixth Edition,P. Newbold, W.Carlson B.Thorne ch. 12

Multifactor linear regression

In this section we will introduce multifactor linear regression. Multifactor linear regression can incorporate more than one variable in order to forecast y. The model for multifactor linear regression is:

$$y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b$$

Or

$$y = \sum_{i=1}^{n} a_i x_i \; + b$$

For example consider the case of 2 factor linear regression

X1	X2	Y
4.38	3.88	3.53
4.35	4.45	4.6
5.58	5.03	5.66
3.23	3.24	3.45
4.8	5.19	5.3
5.16	4.82	4.5
6.14	6.61	6.96

5.92	5.42	5.49
5.2	5.68	5.6
4.65	5.19	5
6.6	6.95	6.67
4.82	4.61	4.61
3.21	2.8	3.29
7.28	7.75	8.01
3.72	4.21	4.52
5.86	5.81	5.8
6.04	5.4	5.54
4.21	3.98	4.15
5.85	5.95	5.8
5.11	5.05	4.89

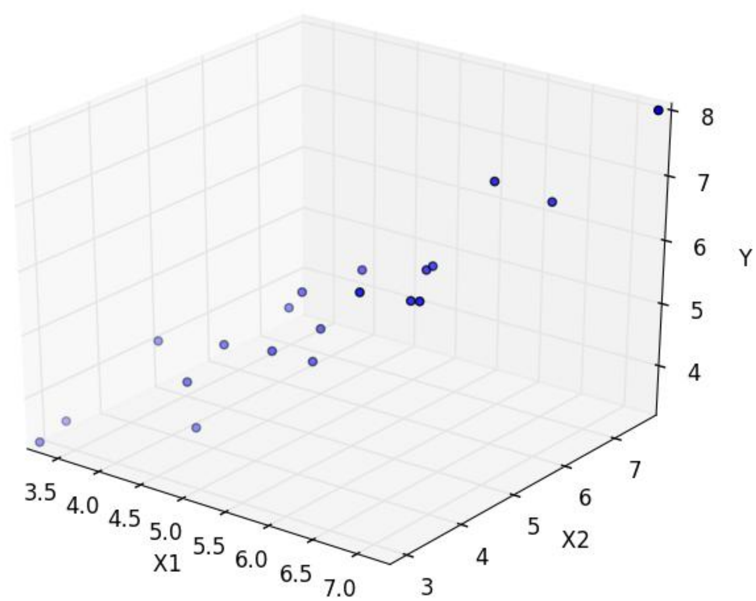


Figure 8-3.

In this case the points are in 3 dimensional space and the regression is no longer a line in space but a plane.

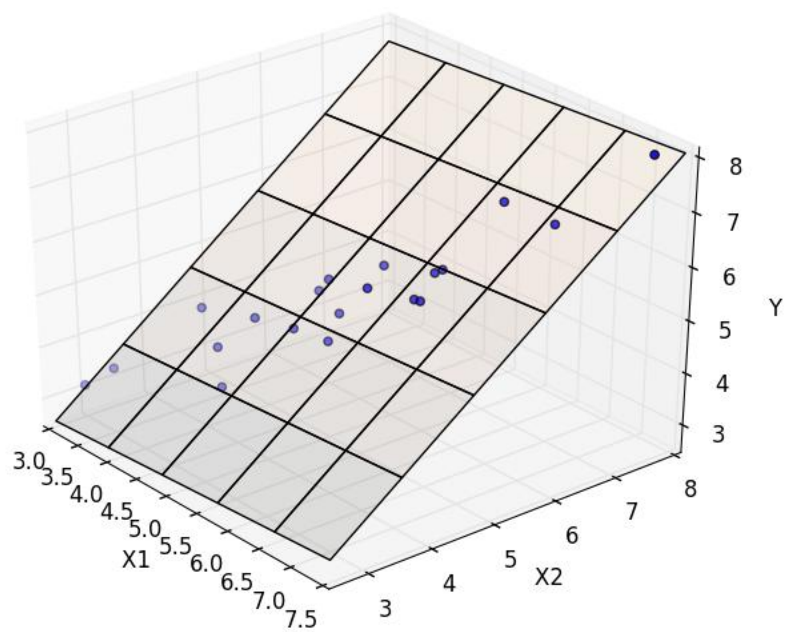
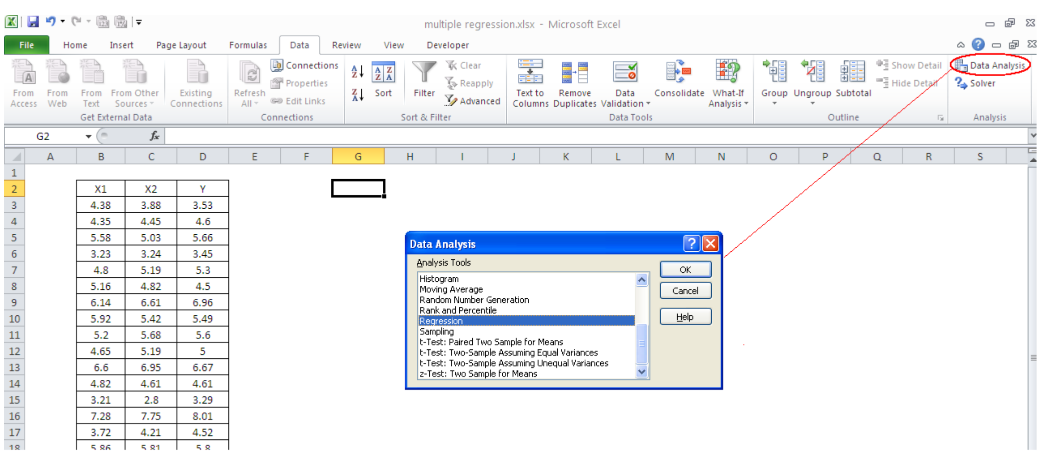


Figure 8-4

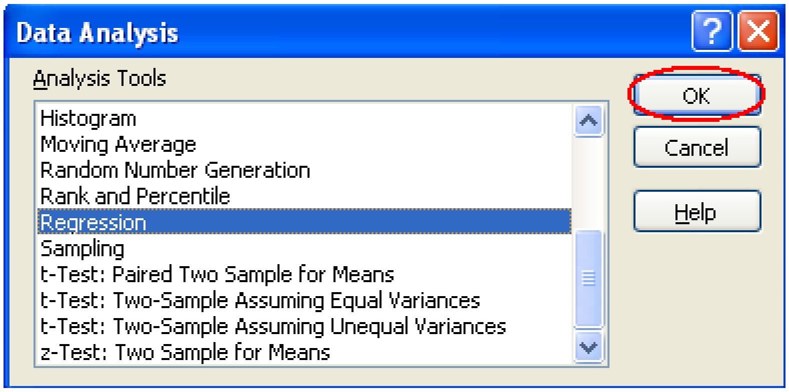
The equation of the plane on Figure 8-4 is:

$$y = a_1\, x_1 \; + a_2 x_2 \; + b = 0.0041 x_1 \; + 0.959 x_2 \; + 0.254$$

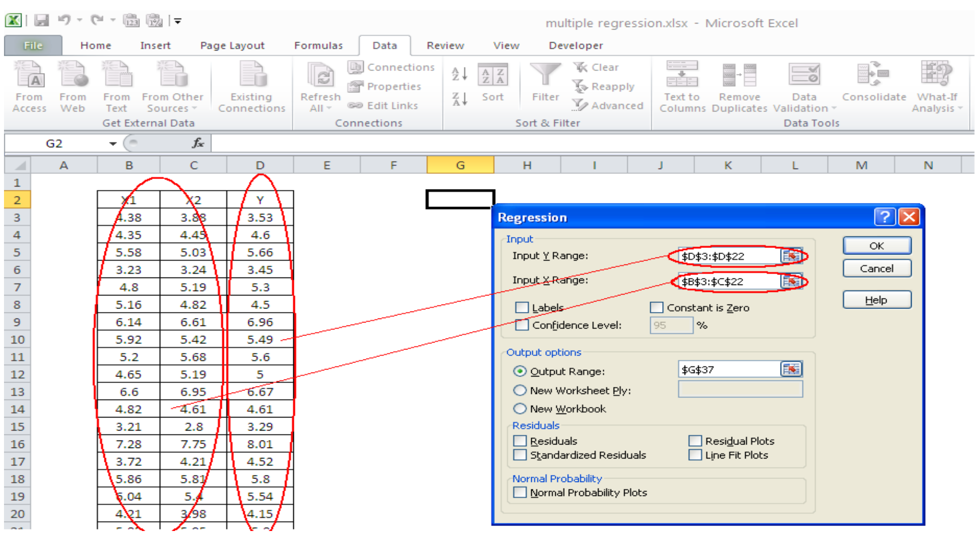
In order to get the above coefficients we can use Excels Data analysis package.



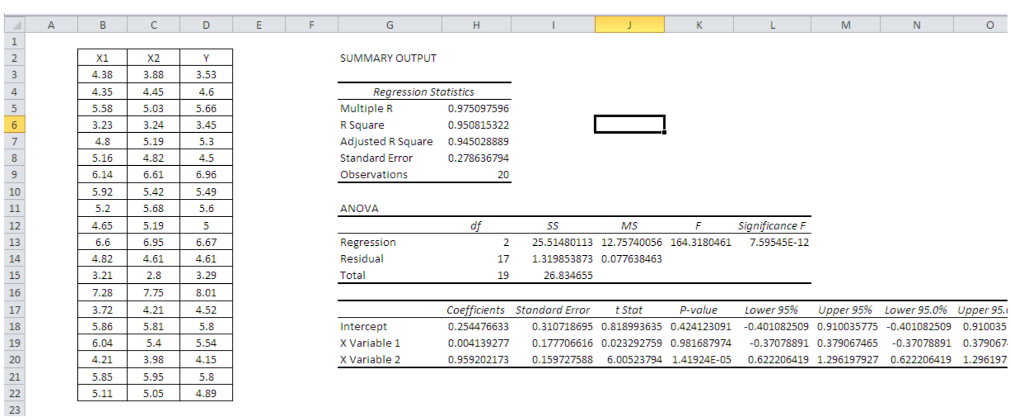
In the data analysis window we choose “Regression”



in the regression window we select the appropriate data columns and press ok



The result is:



X1	X2	Y
4.38	3.88	3.53
4.35	4.45	4.6
5.58	5.03	5.66
3.23	3.24	3.45
4.8	5.19	5.3
5.16	4.82	4.5
6.14	6.61	6.96
5.92	5.42	5.49
5.2	5.68	5.6
4.65	5.19	5
6.6	6.95	6.67
4.82	4.61	4.61
3.21	2.8	3.29
7.28	7.75	8.01
3.72	4.21	4.52
5.86	5.81	5.8
6.04	5.4	5.54
4.21	3.98	4.15
5.85	5.95	5.8
5.11	5.05	4.89

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.975097596
R Square	0.950815322
Adjusted R Square	0.945028889
Standard Error	0.278636794
Observations	20

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	2	25.51480113	12.75740056	164.3180461	7.59545E-12
Residual	17	1.319853873	0.077638463		
Total	19	26.834655			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95.0%*	*Upper 95.0*
Intercept	0.254476633	0.310718695	0.818993635	0.424123091	-0.401082509	0.910035775	-0.401082509	0.910035
X Variable 1	0.004139277	0.177706616	0.023292759	0.981687974	-0.37078891	0.379067465	-0.37078891	0.379067
X Variable 2	0.959202173	0.159727588	6.00523794	1.41924E-05	0.622206419	1.296197927	0.622206419	1.296197

The coefficients are given in “Coefficients” column

	Coefficients
Intercept	0.254476633
X Variable 1	0.004139277
X Variable 2	0.959202173

R square

In order to estimate how good out model explains the relationship amongst X and Y we use R square.

R Square is the number between 0 and 1, where 0 means that there is no linear relationship amongst variables and 1 means that there is exact linear relationship. If R square is 0.7 this means that our model explains 70% of relationship or X explains 70% of Y dynamics. R square is given in the output of excels Data analysis.

Regression Statistics	
Multiple R	0.975097596
R Square	0.950815322
Adjusted R Square	0.945028889
Standard Error	0.278636794
Observations	20

Statics for Business and Economics, Sixth Edition,P. Newbold, W.Carlson B.Thorne ch. 13

Moving average

Moving averages are simple yet useful tools for time series analysis. The main use of moving average is to remove the noise from the observations and highlight the trends and seasonality in the data. moving average simply averages previous n steps of time series thus it smoothest the data , greater the n the greater is the smoothing (however too large n s might distort the picture). The formula for moving average is:

$$y_i = \frac{y_{i-1} + y_{i-2} + y_{i-3} + \cdots + y_{i-n}}{n}$$

Consider example:

t	y
1	1.198387
1.5	0.459472
2	0.449052
2.5	-0.22116
3	-0.38461
3.5	0.03812
4	0.67639
4.5	0.846752
5	1.98064
5.5	2.589429
6	2.863289
6.5	3.248304
7	2.596059
7.5	2.004892
8	2.436832
8.5	1.295733
9	1.131351
9.5	1.766287
10	1.482776

Below is the scatter plot of the data.

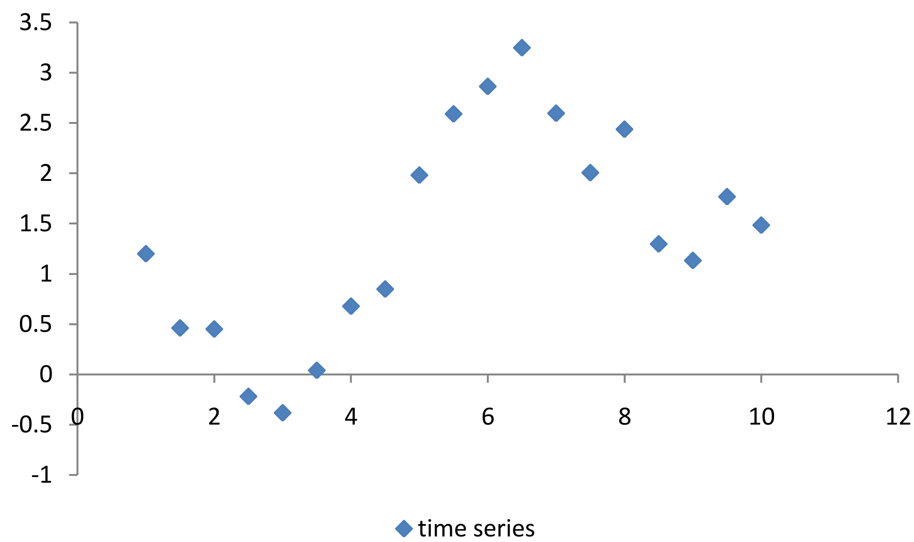


Figure 8-5.

After applying the moving average with n = 3 we see the following picture

t	y	Moving average
1	1.198387	
1.5	0.459472	
2	0.449052	
2.5	-0.22116	0.702304
3	-0.38461	0.229122
3.5	0.03812	-0.05224
4	0.67639	-0.18921
4.5	0.846752	0.109968
5	1.98064	0.520421
5.5	2.589429	1.167927
6	2.863289	1.805607
6.5	3.248304	2.477786
7	2.596059	2.90034
7.5	2.004892	2.902551
8	2.436832	2.616418
8.5	1.295733	2.345928
9	1.131351	1.912485
9.5	1.766287	1.621305
10	1.482776	1.39779

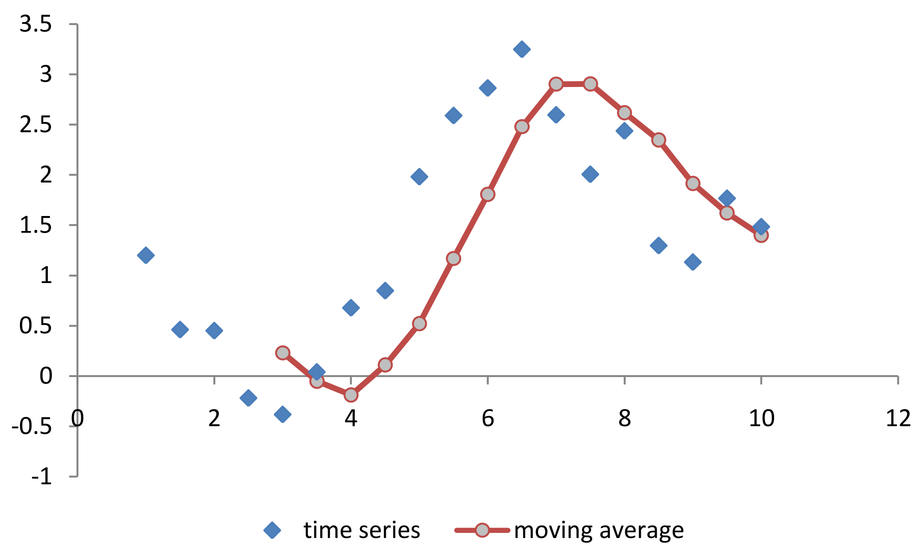


Figure 8-6.

From this chart we see that there is a downward trend . MA column was calculated as follows:

t	y	MA
1	1.198387	
1.5	0.459472	
2	0.449052	
2.5	-0.22116	=AVERAGE(C3:C5)
3	-0.38461	0.229122
3.5	0.03812	-0.05224
4	0.67639	-0.18921
4.5	0.846752	0.109968
5	1.98064	0.520421
5.5	2.589429	1.167927
6	2.863289	1.805607
6.5	3.248304	2.477786
7	2.596059	2.90034
7.5	2.004892	2.902551
8	2.436832	2.616418
8.5	1.295733	2.345928
9	1.131351	1.912485
9.5	1.766287	1.621305
10	1.482776	1.39779

t	y	MA
1	1.198387	
1.5	0.459472	
2	0.449052	
2.5	-0.22116	0.702304
3	-0.38461	0.229122
3.5	0.03812	-0.05224
4	0.67639	-0.18921
4.5	0.846752	0.109968
5	1.98064	=AVERAGE(G8:G10)
5.5	2.589429	1.167927
6	2.863289	1.805607
6.5	3.248304	2.477786
7	2.596059	2.90034
7.5	2.004892	2.902551
8	2.436832	2.616418
8.5	1.295733	2.345928
9	1.131351	1.912485
9.5	1.766287	1.621305
10	1.482776	1.39779

Thus we just let average with n values slide down by one step.

Statics for Business and Economics, Sixth Edition,P. Newbold, W.Carlson B.Thorne ch. 19.4

6. STRICTLY DETERMINED GAMES & MIXED STRATEGY GAMES

Game theory is relatively new branch of mathematics designed to help people in conflicting situations determine the best course of action out of several possible choices.

Consider two stores that sell HDTV's store R and store C. Each is trying to decide how to price a particular model. A market research firm supplies the following information.

$$\begin{array}{cc} & \text{Store C} \\ & \begin{array}{cc} \$499 & \$549 \end{array} \\ \text{Store R} \begin{array}{c} \$499 \\ \$549 \end{array} & \begin{bmatrix} 55\% & 70\% \\ 40\% & 66\% \end{bmatrix} \end{array} \tag{1}$$

The matrix entries indicate the percentage of the business that store R will receive. That is , if both stores price their HDTV as \$499, store R will receive 55% of all the business (store C will lose 55% and will get 45%.). if store R choses a price of \$499 an store C choses the price of \$549 , the store R will receive 70% of the business and store c will receive 30% of the business, and so on. Each store can choose its own price but cannot control the price of other. The object is for each store to determine a price that will ensure the maximum possible business in this competitive situation.

This marketing competition may be viewed as a game between store R and store C. A single play of this game requires store R to choose a row1 or row 2 in matrix (1) and simultaneously requires store c to choose column 1 or column2. It is common to designate the person choosing the row by R, for **row player** and C for **column player.** Each entry in matrix (1) is called the payoff for particular pair of moves by R and C. matrix (1) is called a **game matrix** or a **payoff matrix**. This game is **two-person zero sum-game** because there are only two players and one player's win is the others player's loss.

Strictly determined games

Actually, any m x n matrix may be considered a two-person zeros-sum matrix game. For example the 3x4 matrix:

$$\begin{bmatrix} 0 & 6 & -2 & -4 \\ 5 & 2 & 1 & 3 \\ -8 & -1 & 0 & 20 \end{bmatrix} \tag{2}$$

May be viewed as a matrix game where R has three moves and C has four moves. If R plays row 2 and C plays column 4 then R wins 3 units. If R plays row 3 and C plays column 1, then R "wins" – 8 units; that is, C wins 8 units. How should R and C play in matrix game (2)?

In order to answer this question we state a **fundamental principle of game theory**

Fundamental principle of game theory

1) A matrix game is played repeatedly.
2) Player R tries to maximize winnings.
3) Player C tries to minimize loses.

Thus Player R chooses row with the largest minimum payoff, while C chooses the column with smallest of the maximum payoff. In example (2) R chooses row 1 because it has the largest minimum value 1:

$$\begin{bmatrix} 0 & 6 & -2 & -4 \\ 5 & 2 & 1 & 3 \\ -8 & -1 & 0 & 20 \end{bmatrix}$$

While C chooses column 3 because it has the smallest maximum value 1

$$\begin{bmatrix} 0 & \boxed{6} & -2 & -4 \\ \boxed{5} & 2 & \boxed{1} & 3 \\ -8 & -1 & 0 & \boxed{20} \end{bmatrix}$$

Combining these two matrices:

$$\begin{bmatrix} 0 & \boxed{6} & -2 & -4 \\ \boxed{5} & 2 & \boxed{1} & 3 \\ -8 & -1 & 0 & \boxed{20} \end{bmatrix}$$

Thus choosing row 2 for R and column 3 for C is the optimal strategy for both and the payoff is 1. The optimal value is also called the **saddle value.**

Thus the procedure for locating the saddle value is:

Locating Saddle Value

1) circle thee minimum value in each row (it may occur in more than one place)
2) place square around the maximum value in each column (it may occur in more than one place)
3) any entry with both circle and square around is a saddle value.

If there are multiple saddle values, than they are equal. For example if the matrix has saddle values x and y, it can be shown that x = y.

Matrix game with a saddle value is called Strictly determined matrix game. It can happen that matrix has no saddle value in this case it is called the nonstrictly determined matrix game.

Finite Mathematics for Business, Economics, Life Sciences and Social Sciences (international edition) R. Barret, M.Zieger, K. Byleen,. ch. 10-1

Mixed-Strategy games

in this chapter we are going to consider at nonstrictly determined games and methods of choosing optimal strategy.

For example we have the following matrix:

$$\begin{bmatrix} \boxed{2} & -3 \\ -3 & \boxed{4} \end{bmatrix}$$

In this case there is no saddle point. What strategy should players choose? the answer is that player should play in some sort of mixed pattern. The best way to choose the mixed pattern is to choose randomly acording to some distribution.for example R might choose 1 row with probability 0.25 and row 2 with probability 0.75.

Strategies for R anc C

given the game matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

R's strategy is a probability row matrix :

P=$[p_1 \quad p_2]$ where $p_1, p_2 > 0$ and $p_1 + p_2 = 1$

C's strategy is a probability column matrix :

Q=$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ where $q_1, q_2 > 0$ and $q_1 + q_2 = 1$

In this case there is no saddle value, instead one uses **Expected Value of a Game.**

Expected value of the game is denoted as E(P,Q) and is calculated as:

$$E(P, Q) = PMQ$$

7. LINEAR PROGRAMMING - GAMES

Solution for 2x2 matrix game

Now we turn to the qestion of calculating optimal P and Q for the case of 2x2 matrix game. We wil consider general case of m x n matrix game in the next chapter where we will use linear programing to finde P and Q. in the case of simple 2x2 matrix the solution is:

Solution for 2x2 game matrix

for nonstrictly determined matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the optimal solution is:

$$P^* = [p_1^* \quad p_2^*] = \begin{bmatrix} \frac{d-c}{D} & \frac{a-b}{D} \end{bmatrix}$$

$$Q^* = \begin{bmatrix} q_1^* \\ q_2^* \end{bmatrix} = \begin{bmatrix} \frac{d-b}{D} \\ \frac{a-c}{D} \end{bmatrix}$$

$$E(P^*, Q^*) = \frac{ad - bc}{D}$$

where

$$D = a + d - b - c$$

Finite Mathematics for Business, Economics, Life Sciences and Social Sciences (international edition) R. Barret, M.Zieger, K. Byleen,. ch. 10-2

Solution of m x n matrix game using linear programing

In this chapter we are going to consider general (m x n) matrix game case and method for finding optimal strategies.

General matrix game

M is (m x n) nonstrictly determined matrix:

$$M = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \dots & \dots & \dots & \dots \\ r_{m1} & r_{m2} & \dots & r_{mn} \end{bmatrix}$$

find

$$P^* = [p_1^* \quad p_2^* \quad \dots \quad p_m^*]$$

And

$$Q^* = \begin{bmatrix} q_1^* \\ q_2^* \\ \dots \\ q_n^* \end{bmatrix}$$

Step 1) if at least one element in M is less or equal 0 ($r_{ij} \le 0$) than add some positive constant k (for example | min(M) | + 1) to each elemnet of M.

$$M_1 = \begin{bmatrix} r_{11} + k & r_{12} + k & \dots & r_{1n} + k \\ r_{21} + k & r_{22} + k & \dots & r_{2n} + k \\ \dots & \dots & \dots & \dots \\ r_{m1} + k & r_{m2} + k & \dots & r_{mn} + k \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Step 2) solve these linear programing problems:

(A) Minimize $y = x_1 + x_2 + \dots + x_m$

Subject to :

$M_1^T x \ge 1$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_m \end{bmatrix}$

$x_1, x_n, \dots x_m \ge 0$

(B) Maximize $\mathrm{y} = \mathrm{z}_1 + \mathrm{z}_2 + \dots + \mathrm{z}_\mathrm{n}$

Subject to :

$M_1 z \le 1$ where $z = \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_n \end{bmatrix}$

$z_1, z_n, \dots z_n \ge 0$

Step 3) $E_1(P^*, Q^*) = \frac{1}{y}$ note : $y = x_1 + x_2 + \dots + x_m = z_1 + z_2 + \dots + z_n$

$P^* = E_1(P^*, Q^*) * x^T \qquad Q^* = E_1(P^*, Q^*) * z^T$

And

$E(P^*, Q^*) = E_1(P^*, Q^*) - k$

Excel example

Consired the folowing matrix game:

$$M = \begin{bmatrix} 1 & -1 & 6 \\ -1 & 2 & -3 \end{bmatrix}$$

Finde P^*, Q^* and $E(P^*, Q^*)$

Step 1)the smallest element of M is -3 thus we choose k = 4 and:

$$M_1 = \begin{bmatrix} 5 & 3 & 10 \\ 3 & 6 & 1 \end{bmatrix}$$

Step 2)we have to solve these linear programming problems:

(A) Minimize $y = x_1 + x_2$

Subject to :

$$\begin{bmatrix} 5 & 5 \\ 3 & 6 \\ 10 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 1 \quad \text{or} \quad \begin{matrix} 5x_1 + 3x_2 \geq 1 \\ 3x_1 + 6x_2 \geq 1 \\ 10x_1 + x_2 \geq 1 \end{matrix}$$

$x_1, x_n, \ldots x_m \geq 0$

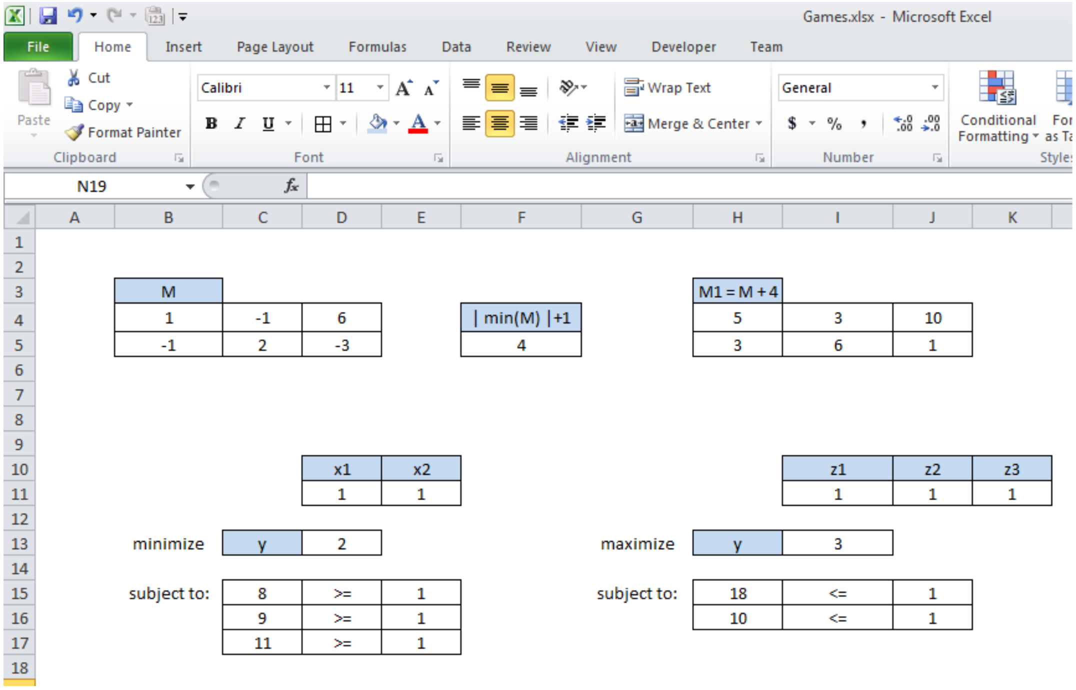
(B) Maximize $\mathrm{y} = \mathrm{z}_1 + \mathrm{z}_2 + \mathrm{z}_3$

Subject to :

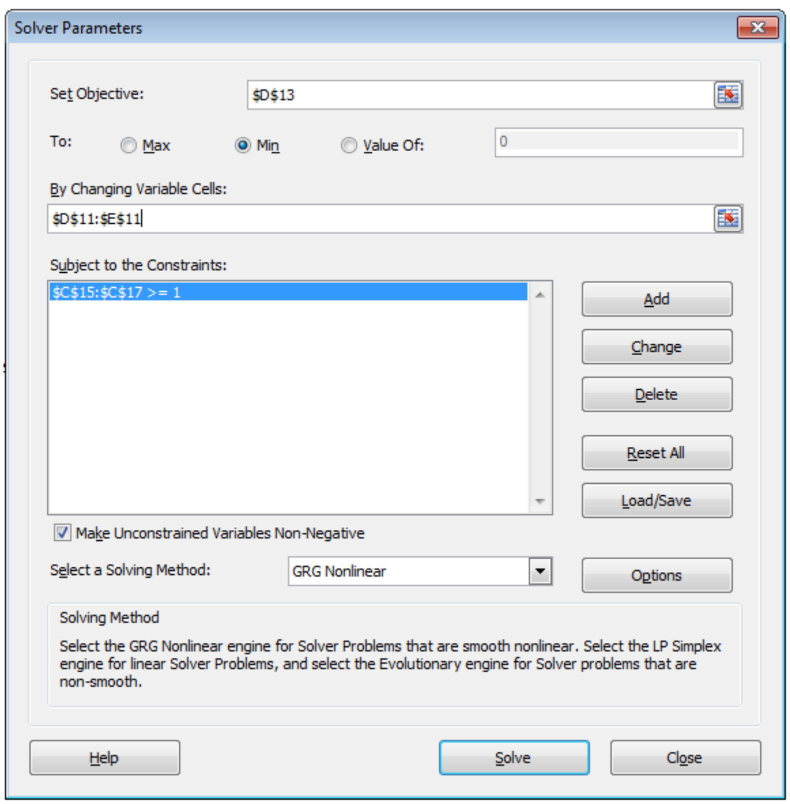
$$\begin{bmatrix} 5 & 3 & 10 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_{13} \end{bmatrix} \geq 1 \quad \text{or} \quad \begin{matrix} 5z_1 + 3z_2 + 10z_3 \leq 1 \\ 3z_1 + 6z_2 + z_3 \leq 1 \end{matrix}$$

$z_1, z_n, \ldots z_n \geq 0$

In order to solve, we write this problem in excel spreadsheet. Range D11:E11 is x vector (note that in this case it is a row vector not column vector, thus we must transpose it where necessary) before we start optimization we must write down some trial valuse for x, in this case we simply wrote 1, however any random number would work , D13 is the objective functio y (which is minimized in this case), it is simply the sum of x vector. Range C15:C17 is the constarint matrix,the formula for constarint matrix is MMULT(TRANSPOSE(H4:J5),TRANSPOSE(D11:E11)), where MMULT is matrix multiplication.



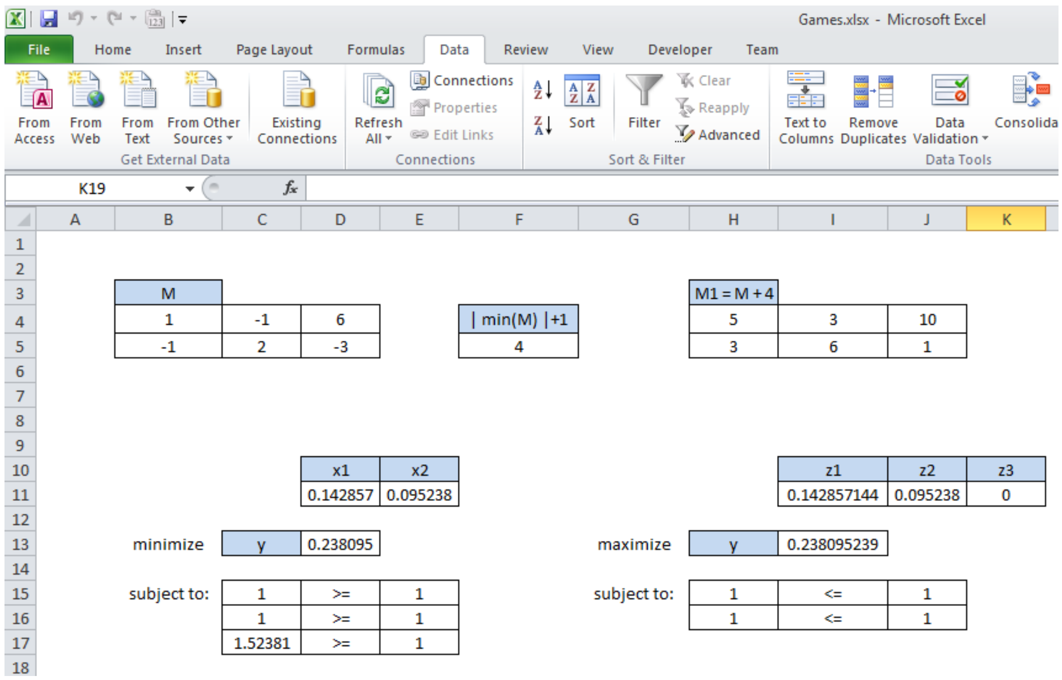
after prepearing the worksheet we start the solver and pass the following parameters:



Objective cell is D13 (y), the variable cells are D11:E11 (x vector) and we add constraint that every element of range C15:C17 should be greater than 1, finally we chose the minimization and click “Solve”.

We do the same calculation for the second problem the only difference is that every element of constraint matrix should be less than 1 and we want to maximize y not minimize.

The result of calculations is:



x = [0.1428, 0.0952] z = [0.1428, 0.0952, 0] and y = 0.2381. thus $E_1(P^*, Q^*) = \frac{1}{y} = \frac{1}{0.2381} = 4.2$.

finally using above given formulas :

$P^* = E_1(P^*, Q^*) * x^T = [0.6\ \ , 0.4]$ (note that in this case x is already transposed)

$Q^* = E_1(P^*, Q^*) * z^T = [0.6,\ 0.4,\ 0]$ (note that in this case z is already transposed)

Finite Mathematics for Business, Economics, Life Sciences and Social Sciences (international edition) R. Barret, M.Zieger, K. Byleen,. ch. 10-4

8. BASIC FINANCIAL CALCULATIONS

Introduction

We aims to give you some finance basics and their Excel implementation. If you have had a good introductory course in finance, most of the topics will probably be superfluous.

This problem covers the following:

- Present Value(PV)
- Future value (FV)
- Flat Payment Schedules (PMT)
- Discount Rate
- Net Present Value(NPV)
- Internal rate of return (IRR)
- Effective Annual Rate(EAR)
- Continuously compounded interest

Almost all financial problems center on finding the *value today* of a series of *cash receipts over time.*The cash receipts (or cash flows, as we will call them) may be certain or uncertain. We analyze the values of no risky cash flows—future receipts that we will receive with absolute certainty.

The basic concept to which we will return over and over is the concept of *opportunity cost.* Opportunity cost is the return that would be required of an investment to make it a viable alternative to other, similar, investments. As illustrated in this problem, when we calculate the net present value, we use the investment's opportunity cost as a discount rate. When we calculate the internal rate of return, we compare the calculated return to the investment's opportunity cost to judge its value.

In the financial literature you will find many synonyms for *opportunity cost,* among them *discount rate, cost of capital,* and *interest rate.* When it is applied to risky cash flows, we will sometimes call theopportunity cost the risk-adjusted discount rate (RADR) or the weighted average cost of capital (WACC).

Present Value (PV) and Net Present Value (NPV)

Both concepts, present value and net present value, are related to the value today of a set of future anticipated cash flows. As an example, suppose we are valuing an investment that promises \$100 per year at the end of this and the next four years. We suppose that there is no doubt that this series of five payments of \$100 each will actually be paid. If a bank would pay us an annual interest rate of 10 percent on a five-year deposit, then this 10 percent is the investment's opportunity cost, the alternative benchmark return to which we want to compare the investment. We may calculate the value of the investment by discounting its cash flows using this opportunity cost as a discount rate:

F19

	A	B	C	D	E	F	G	H	I
1									
2	Discount Rate	10%							
3	Present Value	$379.08	=NPV(B2,B7:B11)						
4									
5									
6	Year	Cash Flow							
7	1	$100.00							
8	2	$100.00							
9	3	$100.00							
10	4	$100.00							
11	5	$100.00							
12									
13									
14									

The present value (PV) of $379.08 is the value today of the investment. Suppose this investment was being sold for $400. Clearly it would not be worth its purchase price, since—given the alternative return (discount rate) of 10 percent—the investment is worth only $379.08. The net present value (NPV) is the applicable concept here. Denoting by r the discount rate applicable to the investment, the NPV is calculated as follows:

$$NPV = CF_0 + \sum_{t=1}^{N} \frac{CF_t}{(1+r)^t}$$

where CF_t is the investment's cash flow at time t and CF_0 is today's cash flow:

H9

	A	B	C	D	E	F
1						
2	Discount Rate	10%				
3	Net Present Value	-$20.92	=B7+NPV(B2,B7:B12)			
4						
5						
6	Year	Cash Flow				
7	0	-$400.00				
8	1	$100.00				
9	2	$100.00				
10	3	$100.00				
11	4	$100.00				
12	5	$100.00				
13						

A Note about Nomenclature

Excel's language about discounted cash flows differs somewhat from the standard finance nomenclature. Excel uses the letters NPV to denote the present value (not the net present value) of a series of cash flows.

To calculate the finance net present value of a series of cash flows using Excel, we have to calculate the present value of the future cash flows (using the Excel NPV function) and subtract from this present value the time-zero cash flow. (This is often the cost of the asset in question.)

Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter 1, pp. 6

The Internal Rate of Return (IRR) and Loan Tables

We continue with the same example. Suppose that we indeed paid \$400.00 for this series of cash flows. The internal rate of return (IRR) is defined as the compound rate of return r that makes the NPV equal to zero:

$$CF_0 + \sum_{t=1}^{N} \frac{CF_t}{(1+IRR)^t} = 0$$

Excel's function IRR will solve this problem; note that the IRR includes as arguments all of the cash flows of the investment, including the first (in this case negative) cash flow of −400:

C17 | *fx*

	A	B	C	D	E	F
1	IRR	7.931%	=IRR(B6:B11)			
2	NPV	-$20.92				
3						
4						
5	Year	Cash Flow				
6	0	-$400.00				
7	1	$100.00				
8	2	$100.00				
9	3	$100.00				
10	4	$100.00				
11	5	$100.00				
12						

Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter 1, pp. 8

Effective Annual Rate(EAR)

The term annual percentage rate (APR), also called nominal APR, and the term effective APR, also called EAR, describe the interest rate for a whole year (annualized), rather than just a monthly fee/rate, as applied on a loan, mortgage loan, credit card, etc.

Effective Annual Rate (EAR) or simply effective rate is the interest rate on a loan or financial product restated from the nominal interest rate as an interest rate with annual compound interest payable in arrears. It is used to compare the annual interest between loans with different compounding terms (daily, monthly, annually, or other). The effective interest rate differs in two important respects from the annual percentage rate (APR). The nominal APR is calculated as: the rate, for a payment period, multiplied by the number of payment periods in a year, while EAR incorporates reinvestment of periodic interest payments till the end of year

$$\text{APR} = \text{Per} - \text{period rate} \times \text{Periods per year}$$

Therefore, to obtain the EAR if there are n compounding periods in the year, we first recover the rate per period as APR/ m and then compound that rate for the number of periods in a year.

$$1 + \text{EAR} = (1 + \text{rate per period})^{\text{m}} = \left(1 + \frac{\text{APR}}{\text{m}}\right)^{\text{m}}$$

Rearranging,

$$\text{APR} = \left[(1 + EAR)^{(1/m)} - 1\right] \times \text{m}$$

The formula assumes that you can earn the APR each period. Therefore, after one year (when n periods have passed), your cumulative return would be (1 + APR/n) xn. Note that one needs to know the holding period when given an APR in order to convert it to an effective rate.

Excel deals with these formulas using function =EFFECT(APR; Number of periods) and =NOMINAL(EAR; Number of periods). See examples below.

Effective Annual Rate:

	A	B	C	D
1				
2	APR	10%		
3	Number of Periods	2	(Semianually)	
4				
5	EAR	10.25%	=EFFECT(B2;B3)	
6				

Annual Percentage Rate:

	A	B	C	D
1				
2	EAR	12%		
3	Number of Periods	2	(Semianually)	
4				
5	APR	11.66%	=NOMINAL(B2;B3)	
6				

The EAR diverges by greater amounts from the APR as n becomes larger (that is, as we compound cash flows more frequently). In the limit, we can envision continuous compounding when n becomes extremely large in equation above. With continuous compounding, the relationship between the APR and EAR becomes

$$1 + \mathrm{EAR} = e^{\mathrm{APR}}$$

Or equivalently

$$\mathrm{APR} = \ln(1 + \mathrm{EAR})$$

Mastering Financial Calculations, Robert Steiner, Prentis Hall, Chapter 1, p. 5

Flat Payment Schedules (PMT)

You take a loan for $10,000 at an interest rate of 7 percent per year. The bank wants you to make a series of payments that will pay off the loan and the interest over six years. We can use Excel's PMT function to determine how much should each annual fixed payment be:

B17 fx

	A	B	C	D	E
1					
2					
3	Loan Principal	10000			
4	Interest Rate	7%			
5	Loan Term	6	<--- Number of year over which loan is repaid made at end of each year		
6	Annual Payment	$2,097.96	=PMT(B4,B5,-B3)		
7					
8					

Notice that we have put "PV"—Excel's nomenclature for the initial loan principal—with a minus sign. Otherwise Excel returns a negative payment (a minor irritant).

Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter 1, pp.13

Future Values and Applications

Any reasonable investment or commitment of cash must provide for an increase in value over time. Given the amount of cash that you want to commit, you can find out how much that cash value will increase in the future once the expected rate of return is known. This calculation is called finding the future value of an investment

There is an easy formula to calculate future values:

$$FV = PV(1 + R)^N$$

Where

FV= Future value

PV=initial deposit (principal)

R=annual rate of interest

N=Number of years

We start with a triviality. Suppose you deposit $1,000 in an account, leaving it there for 10 years. Suppose the account draws annual interest of 10 percent. How much will you have at the end of 10 years? The answer, as shown in the following spreadsheet, is $2,593.74.

K15 *fx*

	A	B	C	D	E	F	G
1		SIMPLE FUTURE VALUE					
2							
3	Annual Interest Rate	10%					
4							
5	Year	Account	Interest	Total in			
6		balance	earned	account			
7		beg. Year	during year	end of year			
8	0	1000	100	1100	<==C8+B8		
9	1	1100.00	110.00	1210.00			
10	2	1210.00	121.00	1331.00			
11	3	1331.00	133.10	1464.10			
12	4	1464.10	146.41	1610.51			
13	5	1610.51	161.05	1771.56			
14	6	1771.56	177.16	1948.72			
15	7	1948.72	194.87	2143.59			
16	8	2143.59	214.36	2357.95			
17	9	2357.95	235.79	2593.74		=$B$3*B8	
18	10	2593.74					
19					=D8		
20							
21	A simple way	=1000*(1+10%)^10 =B8*(1+B3)^A18=2593.74					

As cell B21 shows, you don't need all these complicated calculations: The future value of $1,000 in 10 years at 10 percent per year is given by

$$\mathrm{FV} = 1000 * (1 + 10\%)^{10} = 2{,}593.74$$

Now consider the following, slightly more complicated, problem: Again, you intend to open a savings account. Your initial deposit of $1,000 this year will be followed by a similar deposit at the beginning of years 1,2, …, 9. If the account earns 10 percent per year, how much will you have in the account at the start of year 10?

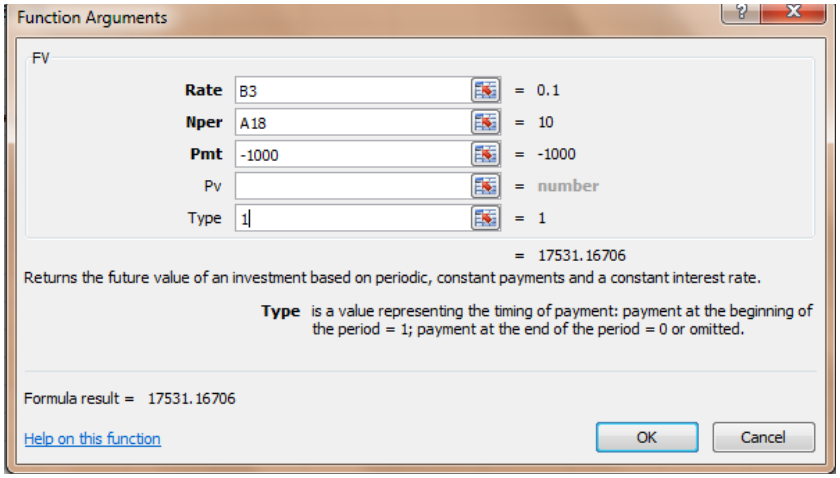
This problem is easily modeled in Excel:

	A	B	C	D	E	F	G	H
1			FUTURE VALUE WITH ANNUAL DEPOSIT					
2								
3	Annual Interest Rate	10%						
4								
5	Year	**Account**	**Deposit**	**Interest**	**Total in**			
6		**balance**	**beginning**	**earned**	**account**			
7		**beg. Year**	**of year**	**during year**	**end of year**			
8	0	0.00	1000.00	100.00	1100.00		=B8+C8+D8	
9	1	1100.00	1000.00	210.00	2310.00			
10	2	2310.00	1000.00	331.00	3641.00			
11	3	3641.00	1000.00	464.10	5105.10			
12	4	5105.10	1000.00	610.51	6715.61		=$B$3*(B8+C8)	
13	5	6715.61	1000.00	771.56	8487.17			
14	6	8487.17	1000.00	948.72	10435.89			
15	7	10435.89	1000.00	1143.59	12579.48			
16	8	12579.48	1000.00	1357.95	14937.42			
17	9	14937.42	1000.00	1593.74	17531.17			
18	10	17531.17						
19				=E8				
20								
21		Future Value		17531.17	=FV(B3,A18,-C8,,1)			

Thus the answer is that we will have $17,531.17 in the account at the beginning of year 10. This same answer can be represented as a formula that sums the future values of each deposit.

$$Total at beginning of year\ 10$$
$$= 1000 * (1 + 10\%)^{10} + 1000 * (1 + 10\%)^{9} + \cdots + 1000 * (1 + 10\%)^{1}$$
$$= \sum_{i=1}^{N} 1000 * (1 + 10\%)^{i}$$

An Excel Function Note from cell D21 that Excel has a function FV that gives this sum. The dialog box brought up by FV is the following:



We note three things about this function:

1. For positive deposits FV returns a negative number. There is an explanation for why this function isprogrammed in this way, but basically this outcome is an irritant. To avoid negative numbers, we have put the PMT in as $-1{,}000$.
2. The line PV in the dialog box refers to a situation where the account has some initial value other than 0 when the series of deposits is made. In this example, this line has been left blank, indicating that the initial account value is zero.
3. As noted in the picture, "Type" (either 1 or 0) refers to whether the deposit is made at the beginning or the end of each period.

Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter 1, pp. 14-16

Continuous Compounding

Suppose you deposit \$1,000 in a bank account that pays 5 percent per year. At the end of the year you will have 1,000 * (1.05) = \$1,050. Now suppose that the bank pays you 2.5 percent interest twice a year. After six months you'll have \$1,025, and after one year you will have

$$1000 * (1 + \frac{0.05}{2})^2 = 1050.625$$

By this logic, if you get paid interest m times per year, your accretion at the end of the year will be

$$\mathrm{FV} = \mathrm{P} * \mathrm{e}^{\mathrm{RN}}$$

Where

FV= Future Value

P=Initial Deposit (Principal)

R=Annual Rate of Interest

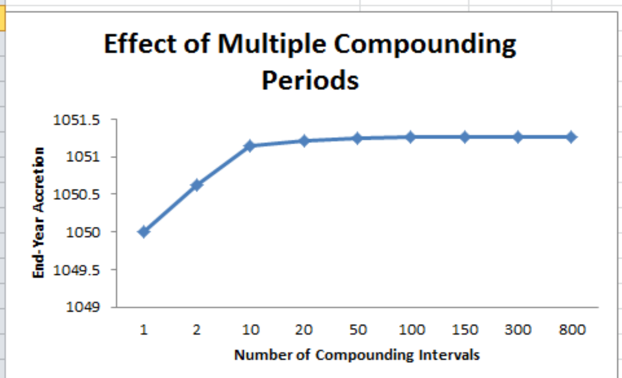
N=number of Years

As m increases, this amount gets larger, converging (rather quickly, as you will soon see) to $e^{0.5}$, which in Excel is written as the function Exp. When n is infinite, we refer to this process as continuous compounding. (By typing Exp(1) in a spreadsheet cell, you can see that e = 2.7182818285.…)

As you can see in the next display, \$1,000 continuously compounded for one year at 5 percent grows to \$1,000 $*e^{0.05}$= \$1,051.271 at the end of the year. Continuously compounded for N years, it will grow to \$1,000 $*e^{0.05N}$.

L10

	A	B	C	D	E	F
1	MULTIPLE COMPOUNDING PERIODS					
2						
3	Initial Deposit	1000				
4	Annual Interest Rate	5%		=B3*(1+B4/B5)^B5		
5	Namber of Compounding periods per year	2				
6	Interest per compounding period	2.50%		=B3*EXP(B4)		
7	Accretion In one year	1050.625			Number	End-year
8	Continius Compounding with EXP	1051.271096			of periods	Accretion
9					Per year	1050.625
10					1	1050
11					2	1050.625
12					10	1051.140132
13					20	1051.205503
14					50	1051.244832
15					100	1051.25796
16					150	1051.262338
17					300	1051.266717
18					800	1051.269454



Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter 1, pp. 18-19

9. CALCULATING THE COST OF CAPITAL

Introduction

The most widely used valuation method for firms is the discounted cash flow (DCF) method.We show how to use integrated accounting-based financial models for the firm to calculate the firm's free cash flows. Discounting these cash flows at an appropriately risk-adjusted discount rate will give us the value of the firm.we discuss how to calculate the firm's cost of capital, the discount rate applied to future cash flows. We consider two models for calculating the cost of equity, the discount rate applied to equity cash flows:

The Gordon model calculates the cost of equity based on the anticipated dividends of the firm

The capital asset pricing model (CAPM) calculates the cost of equity based on the correlation between the firm's equity returns and the returns of a large, diversified, market portfolio. As we will see, the CAPM can also be used to calculate the cost of the firm's debt.

We use all of these models to calculate the weighted average cost of capital (WACC), the appropriate discount rate for valuation of firm cash flows.

A Terminological Note As noted in the previous chapter, "cost of capital" is a synonym for the "appropriate discount rate" to be applied to a series of cash flows. In finance, "appropriate" is most often a synonym for "risk-adjusted." Hence, another name for the cost of capital is the "risk-adjusted discount rate" (RADR).

Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter 2, pp. 25

The Gordon Dividend Model

The Gordon dividendmodelderives the cost of equity from the following deceptively simple statement:

The value of a share is the present value of the future anticipated dividend stream from the share, where the future anticipated dividends are discounted at the appropriate risk-adjusted cost of equity.

Consider, for example, the case of a stock whose dividends are anticipated to grow at 10 percent per year. If next year's anticipated dividend is $3 per share, then the value of the stock today, P_0, is given by

$$P_0 = \frac{3}{1+R_E} + \frac{3*(1.10)}{(1+R_E)^2} + \frac{3*(1.10)^2}{(1+R_E)^3} + \frac{3*(1.10)^3}{(1+R_E)^4}$$

Where

R_E= Risk-adjusted cost of equity

The formula in cell B6 of the following spreadsheet discounts 67 years of dividends (not all for which are shown):

N10 f_x

	A	B	C	D	E	F
1	**Share Value and Anticipated Dividends**					
2						
3	Year 1 Anticipated Dividenc	3				
4	Grawth rate of Dividends	10%				
5	Cost of Equity	15%				
6	Share Value	56.95	=NPV(B5,C10:C22)			
7						
8						
9		Year	Anticipated Dividend			
10		1	3	=B3		
11		2	3.3	=C10*(1+$B$4)		
12		3	3.63	=C11*(1+$B$4)		
13		4	3.99			
14		5	4.39			
15		6	4.83			
16		7	5.31			
17		8	5.85			
18		9	6.34			
19		10	4.07			
20		11	7.78			
21		12	8.56			
22		13	9.42			

Notice that our "solution" is really only an approximation. We've simply taken the NPV for a very long series of dividends, whereas the actual problem in the equation relates to an infinite series of dividends. To do this infiniteseries calculation, we need to resort to some manipulation of the formula. We rewrite the formula using Div_1 to denote the next period anticipated dividend and using g to denote the anticipated growth rate of dividends sometimes called Growing perpetuity:

$$P_0 = \frac{Div_1}{1+R_E} + \frac{Div_1 * (1+g)}{(1+R_E)^2} + \frac{Div_1 * (1+g)^2}{(1+R_E)^3} + \frac{Div_1 * (1+g)^3}{(1+R_E)^4} + \cdots$$
$$= \sum_{i=1}^{\infty} \frac{Div_1 * (1+g)^{i-1}}{(1+g)^i} = \frac{Div_1}{R_E - g}$$

Provided $|g| < R_E$

Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter 2, pp. 26-27

Supernormal Growth" and the Gordon Model

Notice that if the condition $|g| < R_E$ is violated, the formula $P_0 = \frac{Div_1}{R_E - g}$ gives a negative answer. However, this does not mean that the value of the share is negative; rather it means that the basic condition has been violated. In finance examples, violations of $|g| < R_E$ usually occur for very fast-growing firms, in which—at least for short periods of time—we anticipate very high growth rates, so that $|g| < R_E$. In this case the original dividend discount formula shows that P_0 will have an infinite value. Since this result is clearly unreasonable (remember that we are valuing a security), it probably means either (1) that the long-term growth rate is less than the discount rate R_E, or (2) that the discount rate R_E is too low.

The following spreadsheet illustrates an initial, very high, growth rate that ultimately slows to a lower rate. We consider a firm whose current dividend is $8 per share. The firm's dividend is expected to grow at 35 percent for the next five years, after which the growth rate will slow down to 8 percent per year. The cost of equity, the discount rate for all of the dividends, is 18 percent:

	A	B	C	D	E	F	G	H	I	J
1	**The Gordon Dividend Model**									
2										
3	Current Dividend	$8.00				PV_0 During year 1-5	60.99	=NPV(B6,D8:D12)		
4	Growth Rate 1-5 years (g_1)	35%				PV_0 During year 6-∞	169.3450663	=(D13/(B6-B5))/(1+B6)^5		
5	Growth Rate 6-∞ years (g_2)	8%				Total PV_0	230.33			
6	Cost of Equity	18%								
7			year							
8			1	10.8	=B3*(1+B4)					
9			2	14.58	=D8*(1+B4)					
10			3	19.683	=D9*(1+B4)					
11			4	26.57205	=D10*(1+B4)					
12			5	35.87227	=D11*(1+B4)					
13			6	38.74205	=D12*(1+$B$5)					
14			7	41.84141						
15			8	45.18873						
16			9	48.80382						
17			10	52.70813						
18			11	56.92478						
19			12	61.47876						

To calculate the value of the firm's share, we first discount the dividends for years 1–5. Cell E4 shows that these five future dividends are worth $40. Now look at years 6–∞. Denote the long-term growth rate by g2 (in our example this is 8 percent). At time 0, the discounted dividend stream from years 6- ∞ looks like:

$$\frac{Div_5(1+g_2)}{(1+r_E)^6} + \frac{Div_5(1+g_2)^2}{(1+r_E)^7} + \frac{Div_5(1+g_2)^3}{(1+r_E)^8} = \sum_{t=6}^{\infty} \frac{Div_5 * (1+g_2)^{t-5}}{(1+r_E)^t}$$
$$= \frac{1}{(1+r_E)^5} \sum_{t=1}^{\infty} \frac{Div_5 * (1+g_2)^t}{(1+r_E)^t}$$

This last expression is basically the Gordon model discounted over five years.

$$\frac{1}{(1+r_E)^5}\sum_{t=1}^{\infty}\frac{Div_5*(1+g_2)^t}{(1+r_E)^t}=\frac{1}{(1+r_E)^5}\frac{Div_5*(1+g_2)^t}{r_E-\ g_2}$$

As shown in the spreadsheet, the value of the share is estimated at 230.33.

Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter 2, pp. 27-28

The Classic SML

The classic CAPM formula uses a security market line (SML) equation that ignores taxes.

$$Cost\ of\ Equity=\ r_f+\beta\left[E(r_M)-r_f\right]$$

Here r_f is the risk-free rate of return in the economy and$E(r_M)$ is the expected rate of return on the market. The choice of values for the SML parameters is often problematic. A common approach is to choose

nrf equal to the risk-free interest rate in the economy (for example, the yield on Treasury bills).
n$E(r_M)-r_f$ equal to the historic average of the "market risk premium," defined as the average return of a broadbased market portfolio minus the risk-free rate.

The following spreadsheet fragment illustrates this approach.

	A	B	C	D	E	F	G
1							
2							
3	CAPM cost of equity calculations						
4							
5	Abbots Beta		0.8065				
6							
7	Classical CAPM cost of equity						
8	Risk Premium	8.40%	=E(R_E)-R_f				
9	Risk -Free Rate	4.40%	=Treasury bill rate end 1998				
10	R_E the cost of equity	11.17%	=B9+C5*B8				
11							

Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter 2, pp. 32

Weighted Average Cost of Capital (WACC)

The preceding examples for the Gordon dividend model and the CAPM derive the cost of equity, the risk-adjusted discount rate that should be applied to the firm's equity payouts to shareholders. The discount rate that should be applied to the firm's free cash flows—the cash flows of the firm as a whole—is called the weighted average cost of capital (WACC). The WACC is a weighted average of the cost of equity and the cost of debt, where E is the market value of the firm's equity, D is the market value of the firm's debt, and TC is the corporate tax rate.

In the next spreadsheet we calculate Abbott's WACC for the case where the cost of debt is calculated by Method 1 but where we use both the Gordon model and the CAPM for the cost of equity

	A	B	C	D	E
1		**Calculating Abbortt's WACC**			
2	Calculation of end 1998 market value of equity				
3	Number of Share end 1998	1516063000			
4	Share Price end 1998	49			
5	Market value of Equity end 1998	74.28			
6					
7	Long-Term Debt	1339694000			
8	Short-Term Debt	1759076000			
9	Total Debt	3098770000			
10	Less Cash and Cash Equivalents	308230000			
11	Less Investment Activities	75087000			
12	Net Libilities (billion)	2.72			
13					
14	Cost of Debt	5.49%			
15					
16	Enterprise Value= Equity + Debt	77.00	=B5+B16		
17	Equity%	96.47%	=B5/B16		
18	Debt%	3.53%	=B12/B16		
19					
20					
21	Abbortts Tax rate	40%			
22	**Cost of Capital Calculations**				
23	Using Gordon Model				
24	Cost of Equity	13.40%			
25	WACC	13.04%	=(B17*B24+B18*B14*(1-B21))		

Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter 2, pp. 37

Exercises:

1. ABC Corp. has a stock price P_0 = 50. The firm has just paid a dividend of $3 per share, and knowledgable shareholders think that this dividend will grow by a rate of 5% per year. Use the Gordon dividend model to calculate the cost of equity of ABC.
2. Unheardof, Inc. has just paid a dividend of $5 per share. This dividend is anticipated to increase at a rate of 15% per year. If the cost of equity for Unheardof is 25%, what should be the market value of a share of the company?
3. Dismal.com is a producer of depressing Internet products. The company is not currently paying dividends, but its chief financial officer thinks that starting in 3 years it can pay a dividend of $15 per share, and that this dividend will grow by 20% per year. Assuming that the cost of equity of Dismal.com is 35%, value a share based on the discounted dividends.

10. FINANCIAL STATEMENT MODELING

Almost all financial-statement models are sales driven; this term means that as many as possible of the most important financial statement variables are assumed to be functions of the sales level of the firm. For example, accounts receivable may be taken to be a direct percentage of the sales of the firm. A slightly more complicated example might postulate that the fixed assets (or some other account) are a step function of the level of sales:

$$FixedAssets = \begin{cases} a if sales < A \\ b if A \leq sales < B \end{cases}$$

etc.

Order to solve a financial-planning model, we must distinguish between those financial-statement items that are functional relationships of sales and perhaps of other financial-statement items and those items that involve policydecisions. The asset side of the balance sheet is usually assumed to be dependent only on functional relationships. The current liabilities may also be taken to involve functional relationships only, leaving the mix between long-term debt and equity as a policy decision.

A simple example is the following. We wish to predict the financial statements for a firm whose current balance sheetand income statement are as follows:

	A	B	C
13			
14	**Year**	**0**	
15	***Income Statement***		
16	Sales	1000	
17	Cost of Goods Sold	-500	
18	Interest payments on debt	-32	
19	Interest earned on cash and marketable securitie	6	
20	Depreciation	-100	
21	Profit before tax	374	
22	Taxes	150	
23	Profit after tax	225	
24	Dividends	90	
25	Retained earnings	135	
26			
27	***Balance Sheet***		
28			
29	Cash and Marketable Securities	80	
30	Current Assets	150	
31	Fixed Assets		
32	At cost	1070	
33	Depreciation	-300	
34	Net fixed assets	770	
35	**Total Assets**	**1000**	
36			
37	Current Liabilities	80	
38	Debt	320	
39	Stock	450	
40	Accumulated Retained earnings	150	
41	**Total Liabilities and Equity**	**1000**	

The current (year 0) level of sales is 1,000. The firm expects its sales to grow at a rate of 10 percent per year. In addition, the firm anticipates the following financial-statement relations.

Current assets: Assumed to be 15 percent of end-of-year sales
Current liabilities: Assumed to be 8 percent of end-of-year sales
Net fixed assets: 77 percent of end-of year sales
Depreciation: 10 percent of the average value of assets on the books during the year
Fixed assets at cost: Sum of net fixed assets plus accumulated depreciation
Debt: The firm neither repays any existing debt nor borrows any more money over the five-year horizon of the pro-formas.

Projecting Next Year's Balance Sheet and Income Statement

We have already given the financial statement for year zero. We now project the financial statement for year one:

	A	B	C	D	E	F
1	**Setting up Financial Statement Model**					
2						
3	Sales Growth	10%				
4	Current assets/Sales	15%				
5	Current Liabilities/Sales	8%				
6	Net Fixed assets/Sales	77%				
7	Cost of Good Solds/Sales	50%				
8	Depreciation Rate	10%				
9	Interest rate of debt	10%				
10	Interest paid on cash and marketable securitie	8%				
11	Tax Rate	40%				
12	Dividend Payout Ratio	40%				
13						
14	**Year**	**0**	**1**			
15	**Income Statement**					
16	Sales	1000	1100	=B16*(1+$B$3)		
17	Cost of goods sold	-500	-550	=C16*(B7)		
18	Interest payments on debt	-32	-32	=-B9*(B37+C37)/2		
19	Interest earned on cash and marketable securi	8	9	=B10*(B28+C28)/2		
20	Depreciation	-100	-117	=-B10*(B32+C32)*2		
21	Profit befor tax	374	410	=SUM(C16:C20)		
22	Taxes	-150	-164	=C21*B11		
23	Profit after tax	225	246	=C22+C21		
24	Dividends	-90	-98	=C23*B12		
25	Retained earnings	135	148	=C24+C23		
26						
27	**Balance Sheet**					
28	Cash and Marketable securities	80	144	=C40-C29-C33		
29	Current Assets	150	165	=C16*B4		
30	Fixed Assets					
31	At cost	1070	1264	=C33-C32		
32	Depreciation	-300	-417	=B32-B8*(B31+C31)/2		
33	Net Fixed Assets	770	847	=C16*$B$6		
34	Total Assets	1000	1156	=C31+C32+C33		
35						
36	Current Liabilities	80	88	=C16*B10		
37	Debt	320	320	=B37		
38	Stock	450	450	=B38		
39	Accumulated Retained Earnings	150	298	=B39+C25		
40	**Total Lieabilities and Equity**	1000	1156	=sum(C36:C39)		
41						

Income Statement Equations

Sales = Initial sales * (1 + Sales growth)^year
Costs of goods sold = Sales * Costs of goods sold/Sales

The assumption is that the only expenses related to sales are costs of goods sold. Most companies also book an expense item called selling, general, and administrative expenses (SG&A). The changes you would have to make to accommodate this item are obvious.

Interest payments on debt = Interest rate on debt * Average debt over the year

This formula allows us to accommodate changes in the model for repayment of debt, as well as rollover of debt at different interest rates. Note that in the current version of the model, debt stays constant; but in other versions of the model to be discussed later debt will vary over time.

Interest earned on cash and marketable securities = Interest rate on cash * Average cash and marketable securities over the year

Depreciation = Depreciation rate * Average fixed assets at cost over the year

This calculation assumes that all new fixed assets are purchased during the year. We also assume that there is no disposal of fixed assets.

Profit before taxes = Sales − Costs of goods sold − Interest payments on debt + Interest earned on cash and marketable securities – Depreciation

Taxes = Tax rate * Profit before taxes
Profit after taxes = Profit before taxes – Taxes
Dividends = Dividend payout ratio * Profit after taxes

The firm is assumed to pay out a fixed percentage of its profits as dividends. An alternative would be to assume that the firm has a target for its dividends per share.

Retained earnings = Profit after taxes – Dividends

Balance Sheet Equations

Cash and marketable securities = Total liabilities − Current assets − Net fixed assets
Current assets = Current Assets/Sales * Sales
Net fixed assets = Net fixed assets/Sales * Sales
Accumulated depreciation = Previous year's accumulated depreciation + Depreciation rate * **Average fixed assets at cost over the year**.
Fixed assets at cost = Net fixed assets + Accumulated depreciation

Note that this model does not distinguish between plant property and equipment (PP&E) and other fixed assets such as land.

Current liabilities = Current liabilities/Sales * Sales

Debt is assumed to be unchanged. An alternative model, which we will explore later, assumes that debt is the balance-sheet plug.

Stock doesn't change (the company is assumed to issue no new stock).

Accumulated retained earnings = Previous year's accumulated retained earnings + Current year's additions to retained earnings

Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter3, pp. 49-53

Free Cash Flow (FCF): Measuring the Cash Produced by the Business

The most important calculation for valuation purposes is the free cash flow (FCF). FCF—the cash produced by a business without taking into account the way the business is financed—is the best measure of the cash produced by a business. The easiest way to define the free cash flow is as follows:

Defining the Free Cash Flow:

Profit after taxes: This is the basic measure of the profitability of the business, but it is an accounting measure that includes financing flows (such as interest), as well as noncash expenses such as depreciation. Profit after taxes does not account for either changes in the firm's working capital or purchases of new fixed assets, both of which can be important cash drains on the firm.

+ Depreciation: This noncash expense is added back to the profit after tax.

+ After-tax interest payments (net): FCF is an attempt to measure the cash produced by the business activity of the firm. To neutralize the effect of interest payments on the firm's profits, we

- Add back the after-tax cost of interest on debt (after-tax since interest payments are tax deductible).
- Subtract out the after-tax interest payments on cash and marketable securities.

− Increase in current assets: When the firm's sales increase, more investment is needed in inventories, accounts receivable, etc. This increase in current assets is not an expense for tax purposes (and is therefore ignored in the profit after taxes), but it is a cash drain on the company.

+ Increase in current liabilities: An increase in the sales often causes an increase in financing related to sales (such as accounts payable or taxes payable). This increase in current liabilities—when related to sales—provides cash to the firm. Since it is directly related to sales, we include this cash in the free cash flow calculations.

−**Increase in fixed assets at cost:** An increase in fixed assets (the long-term productive assets of the company) is a use of cash, which reduces the firm's free cash flow.

Here is the free cash flow for our firm:

	A	B	C	D	E	F	G	H
1								
2								
3	Year	0	1	2	3	4	5	
4	Free Cash Flow		170	188	201	214	229	
5								

Using the FCF to Value the Firm and Its Equity

The enterprise value of the firm is defined to be the value of the firm's debt, convertible securities, and equity. In financial theory, the enterprise value is the present value of the firm's future anticipated cash flows.

We can use the FCF projections and a cost of capital to determine the enterprise value of the firm. Suppose we have determined that the firm's weighted average cost of capital (WACC) is 20 percent. Then the enterprise value of the firm is the discounted value of the firm's projected FCFs plus its terminal value:

$$EnterpriseValue = \frac{FCF_1}{(1+WACC)^1} + \frac{FCF_2}{(1+WACC)^2} + \cdots + \frac{FCF_5}{(1+WACC)^5} + \frac{Year-5\ TerminalValue}{(1+WACC)^5}$$

In this formula, the year-5 terminal value is a proxy for the present value of all FCFs from year 6 onward

Here's an example that uses our projections:

	A	B	C	D	E	F	G	H	I
1	**Valuing the Firm**								
2	Weighted Average Cost of Capital	20%							
3	Growth Rate	10%							
4									
5	Year	0	1	2	3	4	5		
6	Future Cash Flow		178	188	201	214	233		
7	Terminal Value						$2,563.00	=G6*(1+B3)/(B2-B3)	
8	Adjust Cash flow		178	188	201	214	$2,796.00		
9	NPV= Enterprise Value	$1,622.06	=NPV(B2,C8:G8)						
10									
11									

Terminal Value

We have assumed that—after the year-5 projection horizon—the cash flows will grow at a rate equal to the sales growth of 10 percent. This assumption gives the terminal value as

$$Terminal\ Value at end of year\ 5 = \sum_{t=1}^{\infty} \frac{FCF_{t+5}}{(1+WACC)^t} = \sum_{t=1}^{\infty} \frac{FCF_5 * (1+growth)}{(1+WACC)^t}$$
$$= \frac{FCF_5 * (1+growth)}{WACC - Growth}$$

Financial Modeling,Second Edition,Simon Benninga, Massachusetts Inst. of Technology 2000, chapter3, pp. 54-56

FINANCIAL MODELING

11. INTRODUCTION TO PORTFOLIO MODELS

Modern portfolio theory, which has its origins in the work of Harry Markowitz, John Lintner, Jan Mossin, and William Sharpe, represents one of the great advances in finance.

In this chapter we review the basic mechanics of portfolio calculations. We start with a simple example of two assets, showing how to derive the return distributions from historical price data. We then discuss the general case of N assets; for this case it becomes convenient to use matrix notation and exploit Excel's matrix-handling capabilities.

It is useful before going on to review some basic notation: Each asset i (they may be shares, bonds, real estate, or whatever, although our numerical examples here will be confined to shares) is characterized by several statistics: $E(r_i)$, the expected return on asset i; $Var(r_i)$, the variance of asset i's return; and $Cov(r_i, r_j)$, the covariance of asset i's and asset j's returns. In our applications, it will often be convenient to write $Cov(r_i, r_j)$, as σ_{ij} and $Var(r_i)$ as σ_{ii} (instead of σ_i^2, as usual). Since the covariance of an asset's returns with itself, $Cov(r_i, r_i)$, is in fact the variance of the asset's returns, this notation is not only economical but also logical.

Suppose we have monthly price data for 12 months on two shares: one of Stock A and one of Stock B. The data look like this:

A	B	C
Month	Stock A	Stock B
0	25	45
1	24,12	44,85
2	23,37	46,88
3	24,75	45,25
4	26,62	50,87
5	26,5	53,25
6	28	53,25
7	28,88	62,75
8	29,75	65,5
9	31,38	66,87
10	36,25	78,5
11	37,13	78
12	36,88	68,23

These data give the closing price at the end of each month for each stock. The month-0 price is the initial price of the stock (i.e., the closing price at the end of the month proceeding month 1). We wish to calculate the relevant return statistics for each stock.

First we calculate the monthly return for each stock. This is the percentage return that would be earned by an investor who bought the stock at the end of a particular month $t-1$

and sold it at the end of the following month. For month t and Stock A, the monthly return r_{At} is defined as

$$r = \ln\frac{P_t}{P_{t-1}}$$

The return calculation is easily done in Excel. Setting up the proper formulas gives

	A	B	C	D	E	F	G	H
18		Calculating Returns						
19		Stock A			Stock B			
20	Month	Price	Return		Price	Return		
21	0	25			45			
22	1	24,12	-0,03583		44,85	-0,00334		
23	2	23,37	-0,03159		46,88	0,044268	<---LN(E21/E20)	
24	3	24,75	0,057372		45,25	-0,03539		
25	4	26,62	0,072837		50,87	0,117071		
26	5	26,5	-0,00452		53,25	0,045724		
27	6	28	0,05506		53,25	0		
28	7	28,88	0,030945		62,75	0,164161		
29	8	29,75	0,02968		65,5	0,042892		
30	9	31,38	0,053342		66,87	0,0207		
31	10	36,25	0,144269		78,5	0,160348		
32	11	37,13	0,023986		78	-0,00639		
33	12	36,88	-0,00676		68,23	-0,13382		

We now make a heroic assumption: We assume that the return data for the 12 months represent the distribution of the returns for the coming month. We thus assume that the past gives us some information about the way returns will behave in the future. This assumption allows us to assume that the average of the historic data represents the *expected monthly return* from each stock. It also allows us to assume that we may learn from the historic data what the variance of the future returns is. Using the **Average( ), Varp( ), and Stdevp( )** functions in Excel, we calculate the statistics for the return distribution:

	A	B	C	D	E	F	G	H	I
35	Monthly Mean		3,24%			3,47%	<---AVERAGE(F22:F33)		
36	Monthly Variance		0,23%			0,65%	<---VARP(F22:F33)		
37	Monthly Stand. Dev.		4,78%			8,03%	<---STDEVP(F22:F33)		
38									
39	Annual Mean		38,88%			41,62%	<---F35*12		
40	Annual Variance		2,75%			7,75%	<---F36*12		
41	Annual Stand. Dev.		16,57%			27,83%	<---SQRT(F40)		

Next we want to calculate the covariance of the returns. The covariance (and the correlation coefficient, which is derived from it) measures the degree to which the returns on the two assets move together. The definition is

$$Cov(r_A, r_B) = \frac{1}{M}\sum_t [r_{At} - E(r_A)] * [r_{Bt} - E(r_B)]$$

where M is the number of points in the distribution (in our case, M = 12).

	A	B	C	D	E	F	G	H	I	J
44		Covariance and Variance Calculation					=D48-$F$35			
45	Stock A			Stock B						
46	Returns	Returns-Mean		Returns	Returns-Mean		Product			
47										
48	-0,0358	-6,82%		-0,0033	-3,80%		0,0026			
49	-0,0316	-6,40%		0,0443	0,96%		-0,0006	<---B49*E49		
50	0,0574	2,50%		-0,0354	-7,01%		-0,0017			
51	0,0728	4,04%		0,1171	8,24%		0,0033			
52	-0,0045	-3,69%		0,0457	1,10%		-0,0004			
53	0,0551	2,27%		0,0000	-3,47%		-0,0008			
54	0,0309	-0,15%		0,1642	12,95%		-0,0002			
55	0,0297	-0,27%		0,0429	0,82%		-0,0000			
56	0,0533	2,09%		0,0207	-1,40%		-0,0003			
57	0,1443	11,19%		0,1603	12,57%		0,0141			
58	0,0240	-0,84%		-0,0064	-4,11%		0,0003			
59	-0,0068	-3,92%		-0,1338	-16,85%		0,0066			
60										
61					Covariance		0,0019	<---AVERAGE(G48:G59)		
62							0,0019	<---COVAR(D48:D59;A48:A59)		
63					Correlation		0,4959	<---G61/(C37*F37)		
64							0,4959	<---CORREL(D48:D59;A48:A59)		

This equation is easily set up in Excel: The column Product contains the multiple of the deviation from the mean in each month, that is, the terms **$[r_{At} - E(r_A)][r_{Bt} - E(r_B)]$**, for t = 1, …, 12. The covariance is **Average(Product) = 0.00191**. While it is worthwhile calculating the covariance this way at least once, there is a shorter way, which is also illustrated in the spreadsheets. Excel has an array function—**Covar(Array1,Array2)**—that calculates the covariance directly. To calculate the covariance using **Covar** there is no necessity to find the difference between the returns and the means. Simply use **Covar** directly on the columns, as illustrated in Cell G62 in the spreadsheet picture.

The covariance is a hard number to interpret, since its size depends on the units in which we measure the returns. (If we were to write the returns in percentages—i.e., 4 instead of 0.04—then the covariance would be 19.1, which is 10,000 times the number we just calculated) We can also calculate the correlation coefficient ρ_{AB}, which is defined

as $$\rho_{AB} = \frac{Cov(r_A, r_B)}{\sigma_A \sigma_B}$$

The correlation coefficient is unit-free; calculating it for our example gives $\rho_{AB} = 0.4958$. As we have illustrated, the correlation coefficient can be calculated directly in Excel using the function **Correl(Array1, Array2)**, where the arrays are the same column vectors used to calculate the covariance using the function **Covar**.

The correlation coefficient measures the degree of linear relation between the returns of Stock A and Stock B. The following facts can be proven about the correlation coefficient:

- The correlation coefficient is always between + 1 and − 1: $-1 \le \rho_{AB} \le 1$.
- If the correlation coefficient is +1, then the returns on the two assets are linearly related with a positive slope; that is, if $\rho_{AB} = 1$

- If the correlation coefficient is -1, then the returns on the two assets are linearly related with a positive slope; that is, if $\rho_{AB} = -1$
- If the return distributions are independent, then the correlation coefficient will be zero. (The opposite is not true: If the correlation coefficient is zero, this fact does not necessarily mean that the returns are independent.

Now suppose we form a portfolio composed half of Stock A and half of Stock B. What will be the mean and the variance of this portfolio? It is worth doing the brute-force calculations at least once in Excel.

It is easy to see that the mean portfolio return is exactly the average of the mean returns of the two assets:

$$\text{Expected Return} = E(r_P) = 0.5(r_A) + 0.5(r_B) = 0.5 * 3.24\% + 0.5 * 3.47\% = 3.35\%$$

In general the mean return of the portfolio is the weighted average return of the component stocks. If we denote by γ the proportion invested in Stock A, then

$$E(r_P) = \gamma(r_A) + (1-\gamma)(r_B)$$

However, the portfolio's variance is not the average of the two variances of the stocks! The formula for the variance is

$$\mathrm{Var}(r_P) = \gamma^2 \mathrm{Var}(r_A) + (1-\gamma)^2 \mathrm{Var}(r_B) + 2\gamma(1-\gamma)\mathrm{Cov}(r_A, r_B)$$

Or

$$\sigma_P^2 = \gamma^2\sigma_A^2 + (1-\gamma)^2\sigma_B^2 + 2\gamma(1-\gamma)\rho\sigma_A\sigma_B$$

A frequently performed exercise is to plot the means and standard deviations for various portfolio proportions γ. To do this we build a table using Excel's **Data|Table** command.

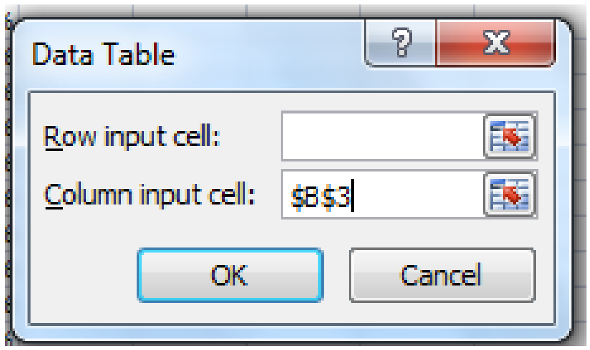
The first step is to set up the table's structure. In the next example, we put the formulas for the **Expected Return** and **Standard Deviation** on the top row, and we put the variable we wish to vary (in this case the proportion) in the first column. At this point the table looks like this:

	A	B	C	D	E	F
23	Proportion	Sigma	Mean			
24		5,60%	3,35%	<--- Table Header =E18		
25	0					
26	0,075					
27	0,15					
28	0,225					
29	0,3					
30	0,375					
31	0,45					
32	0,525					
33	0,6					
34	0,675					
35	0,75					
36	0,825					
37	0,9					
38	0,975					
39	1,05					
40	1,125					
41	1,2					
42	1,275					
43	1,35					
44	1,425					
45	1,5					
46	1,575					

The numbers directly under the labels "Sigma" and "Mean" refer to the corresponding formulas in the previous picture. Thus, if the cell E18 contains the calculation for the Expected Return, then the cell under the letters "Mean" contains the formula "=E18." Similarly, if the cell E20 contains the original calculation for the Variance, then the cell under "Sigma" in the table contains the formula "=E20."

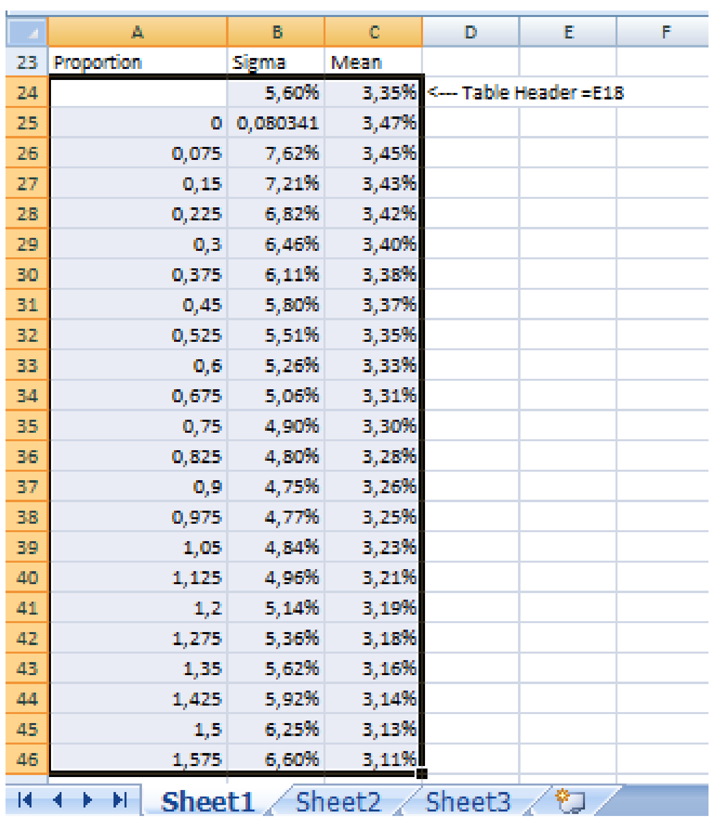
Now do the following:

- Highlight the table area (outlined in the dark border).
- Activate the command Data|Table. You will get a dialogue box that asks you to indicate a Row Input Cell and/or a Column Input Cell.



In this case, the variable we wish to change is in the left-hand column of our table, so we leave the Row Input Cell blank and indicate the cell B3 (which contains the Proportion in A) in the Column Input Cell box.

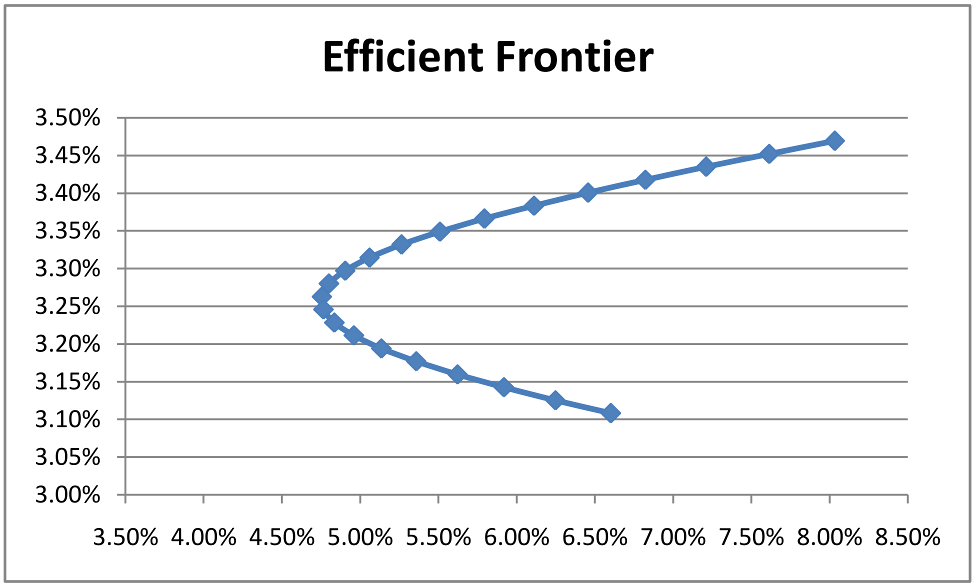
Here's the result:



	A	B	C	D	E	F
23	Proportion	Sigma	Mean			
24		5,60%	3,35%	<--- Table Header =E18		
25	0	0,080341	3,47%			
26	0,075	7,62%	3,45%			
27	0,15	7,21%	3,43%			
28	0,225	6,82%	3,42%			
29	0,3	6,46%	3,40%			
30	0,375	6,11%	3,38%			
31	0,45	5,80%	3,37%			
32	0,525	5,51%	3,35%			
33	0,6	5,26%	3,33%			
34	0,675	5,06%	3,31%			
35	0,75	4,90%	3,30%			
36	0,825	4,80%	3,28%			
37	0,9	4,75%	3,26%			
38	0,975	4,77%	3,25%			
39	1,05	4,84%	3,23%			
40	1,125	4,96%	3,21%			
41	1,2	5,14%	3,19%			
42	1,275	5,36%	3,18%			
43	1,35	5,62%	3,16%			
44	1,425	5,92%	3,14%			
45	1,5	6,25%	3,13%			
46	1,575	6,60%	3,11%			

Sheet1 Sheet2 Sheet3

The graph of the means and standard deviations looks like the following figure. To make the figure come out in this way, you have to use the **Graph Type XY** option.



Matrix notation greatly simplifies the writing of the portfolio problem. In the general case of N assets, suppose thatthe proportion of asset i in the portfolio is denoted by γi. It is often convenient to write the portfolio proportions as a column vector Γ:

$$\Gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \ldots \\ \gamma_N \end{bmatrix}$$

We may then write Γ^T as the transpose of Γ:

$$\Gamma^T = [\gamma_1 \gamma_2 \gamma_3 \ldots \gamma_N]$$

Now write E(r) as the column vector of asset returns, and $E(r)^T$ as the row vector of the asset returns:

$$E(r) = \begin{bmatrix} E(r_1) \\ E(r_2) \\ E(r_3) \\ \ldots \\ E(r_N) \end{bmatrix}$$

$$E(r)^T = [E(r_1)\ E(r_2) E(r_3) \ldots\ E(r_N)]$$

Then we may write the expected portfolio return in matrix notation as

$$E(r_P) = \sum_{i=1}^{N} \gamma_i E(r_i) = \Gamma^T E(r) = E(r)^T \Gamma$$

The portfolio's variance is given by

$$\mathrm{Var}(r_p) = \sum_{i=1}^{N}\sum_{j=1}^{N}\gamma_i\gamma_j\sigma_{ij}$$

The most economical representation of the portfolio variance uses matrix notation. It is also the easiest representation to implement for large portfolios in Excel. In this representation we call the matrix that has σ_{ij} in the i^{th} row and the j^{th} column the variance-covariance matrix

$$S = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \vdots & \sigma_{2N} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{3N} \\ & & \ldots & \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \sigma_{NN} \end{bmatrix}$$

Then the portfolio variance is given by $\mathbf{Var(r_p) = \Gamma^T S\, \Gamma}$.

Example: Suppose that there are four risky assets that have the following expected returns and variance-covariance matrix:

	A	B	C	D	E	F	G
1		**A Four-Asset Portfolio Problem**					
2							
3		Variance-covariance					
4		0,1	0,01	0,03	0,05		6%
5		0,01	0,3	0,06	-0,04		8%
6		0,03	0,06	0,4	0,02		10%
7		0,05	-0,04	0,02	0,5		15%

We wish to consider two portfolios of risky assets:

	A	B	C	D	E	F
9		Portfolio 1	0,2	0,3	0,4	0,1
10		Portfolio 2	0,2	0,1	0,1	0,6

For clarity of exposition, we first allocate space on the spreadsheet for the transposes of the two portfolios. We use the array function **Transpose** to insert these cells

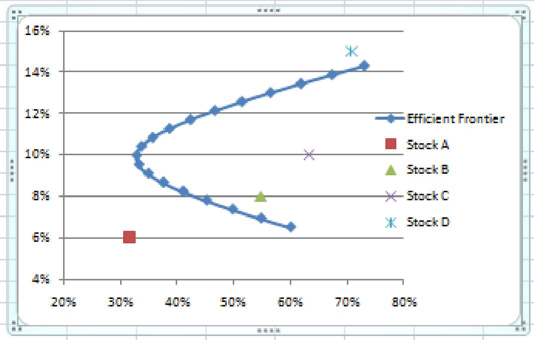
	A	B	C	D
18		Transpose		
19		Portfolio 1		Portfolio 2
20		0,2		0,2
21		0,3		0,1
22		0,4		0,1
23		0,1		0,6

We next calculate the means, variances, and covariance of the two portfolios. We use the Excel function **Mmult** for all the calculations:

	A	B	C	D	E	F	G	H	I	J	K
11											
12		Portfolio 1			Portfolio 2						
13		Mean	9,10%		Mean	12,00%	<---MMULT(C10:F10,G4:G7)				
14		Variance	12,16%		Variance	20,34%	<---MMULT(C10:F10,MMULT(B4:E7,D20:D23))				
15											
16		Covariance	0,0714	<---MMULT(C9:F9,MMULT(B4:E7,D20:D23))							
17		Correlation	0,454	<---C16/SQRT(C14*F14)							

We can now calculate the standard deviation and return of combinations of portfolios 1 and 2. Note that once we have calculated the means, variances, and the covariance of the returns of the two portfolios, the calculation of the mean and the variance of any portfolio is the same as for the two-asset case.

	A	B	C	D
25				
26	Calculating Returns of Combinations of Portfolio 1 and Portfolio 2			
27	Proportion	0,5		
28	Mean Return	0,1055		
29	Variance of Retunrs	0,11695		
30	St.Dev. Of Return	0,3419795		
31				
32				
33		Table of returns		
34		Propotion	Sigma	Mean
35			0,34198	0,1055
36		-0,8	0,72884	0,1432
37		-0,65	0,672294	0,13885
38		-0,5	0,617211	0,1345
39		-0,35	0,564021	0,13015
40		-0,2	0,513311	0,1258
41		-0,05	0,465892	0,12145
42		0,1	0,422874	0,1171
43		0,25	0,38573	0,11275
44		0,4	0,356303	0,1084
45		0,55	0,336624	0,10405
46		0,7	0,328448	0,0997
47		0,85	0,332625	0,09535
48		1	0,348712	0,091
49		1,15	0,375179	0,08665
50		1,3	0,410022	0,0823
51		1,45	0,451304	0,07795
52		1,6	0,497425	0,0736
53		1,75	0,547163	0,06925
54		1,9	0,599618	0,0649
55		Stock A	32%	6%
56		Stock B	55%	8%
57		Stock C	63%	10%



An *efficient portfolio* is the portfolio of risky assets that gives the lowest variance of return of all portfolios having the same expected return. Alternatively, we may say that an efficient portfolio has the highest expected return of all portfolios having the same variance.

Mathematically, we may define an efficient portfolio as follows: For a given return *m*, an efficient portfolio p is one that solves

$$\mathrm{Min} \sum_{\mathrm{i}=1}^{\mathrm{N}} \sum_{\mathrm{j}=1}^{\mathrm{N}} \gamma_i \gamma_j \sigma_{ij} = Var(\mathrm{r_p})$$

subject to

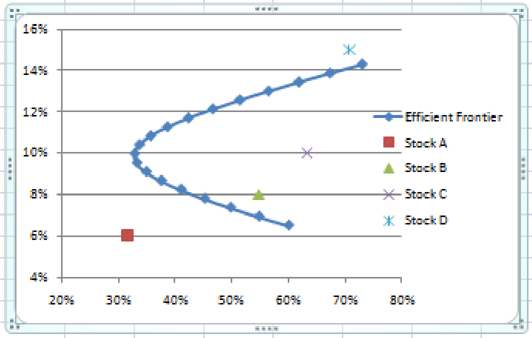
$$\sum_i x_i r_i = \mu = E(r_p)$$

$$\sum_i x_i = 1$$

The efficient frontier is the set of all efficient portfolios. The efficient frontier is the locus of all convex combinations of any two efficient portfolios.

To show that efficiency is a nontrivial concept, we show that the two portfolios whose combinations are graphed in four asset example are not efficient. This point is easy to see if we extend the data table to include numbers for the individual stocks:

	A	B	C	D
25				
26	Calculating Returns of Combinations of Portfolio 1 and Portfolio 2			
27	Proportion	0,5		
28	Mean Return	0,1055		
29	Variance of Retunrs	0,11695		
30	St.Dev. Of Return	0,3419795		
31				
32				
33		Table of returns		
34		Propotion	Sigma	Mean
35			0,34198	0,1055
36		-0,8	0,72884	0,1432
37		-0,65	0,672294	0,13885
38		-0,5	0,617211	0,1345
39		-0,35	0,564021	0,13015
40		-0,2	0,513311	0,1258
41		-0,05	0,465892	0,12145
42		0,1	0,422874	0,1171
43		0,25	0,38573	0,11275
44		0,4	0,356303	0,1084
45		0,55	0,336624	0,10405
46		0,7	0,328448	0,0997
47		0,85	0,332625	0,09535
48		1	0,348712	0,091
49		1,15	0,375179	0,08665
50		1,3	0,410022	0,0823
51		1,45	0,451304	0,07795
52		1,6	0,497425	0,0736
53		1,75	0,547163	0,06925
54		1,9	0,599618	0,0649
55		Stock A	32%	6%
56		Stock B	55%	8%
57		Stock C	63%	10%



Were the two portfolios efficient, then all of the individual stocks would fall on or inside the graph of the combinations.

Financial Modeling, 2nd ed., Simon Benninga, 2000; Chapter 7, pp.106-116
Investments, 8th ed., Zvi Bodie, Alex Kane, Alan J. Marcus, 2009; Chapter 7, pp. 197-204

12. VARIANCE-COVARIANCE MATRIX

In order to calculate efficient portfolios, we must be able to calculate the variance-covariance matrix from return data for stocks. In this chapter we discuss this problem, showing how to do the calculations in Excel. We illustrate several methods for calculating the variance-covariance matrix.

Throughout the chapter we shall use data for six stocks to illustrate our calculations.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1		Return Data for Variance-Covariance Calculations											
2													
3		AMR	BS	GE	HR	MO	UK	SP500					
4	1974	-0,3505	-0,1154	-0,4246	-0,2107	-0,0758	0,2331	-0,2647		AMR	American Airlines		
5	1975	0,7083	0,2472	0,3719	0,2227	0,0213	0,3569	0,372		BS	Bethlehem Steel		
6	1976	0,7329	0,3665	0,255	0,5815	0,1276	0,0781	0,2384		GE	General Electric		
7	1977	-0,2034	-0,4271	-0,049	-0,0938	0,0712	-0,2721	-0,0716		HR	International Harvester		
8	1978	0,1663	-0,0452	-0,0573	0,2751	0,1372	-0,1346	0,0656		MO	Philip Morris		
9	1979	-0,2659	0,0158	0,0896	0,0793	0,0215	0,2254	0,1844		UK	Union carbide		
10	1980	0,0124	0,4751	0,335	-0,1894	0,2002	0,3657	0,3242		SP500			
11	1981	-0,0264	-0,2042	-0,0275	-0,7427	0,0913	0,0479	-0,0491					
12	1982	1,0642	-0,1493	0,6968	-0,2615	0,2243	0,0456	0,2141					
13	1983	0,1942	0,368	0,311	1,8682	0,2066	0,254	0,2251					
14	Mean	0,2032	0,0531	0,1501	0,1529	0,1025	0,1200	0,1238	<---AVERAGE(H4:H13)				

This particular data set is calculated from annual price data for these six stocks. Thus, for example, AMR's return of −35.05 percent for 1974 is calculated as

$$r_{AMR,1974} = \frac{Price_{AMR,Dec\ 31,1974} - Price_{AMR,Dec\ 31,1973}}{Price_{AMR,Dec\ 31,1973}}$$

Suppose we have return data for N assets over M periods. We can write the return of asset i in period t as rit. Write the mean return of asset i as

$$\overline{r_i} = \frac{1}{N}\sum_{t=1}^{N} r_{it}\ , i = 1, \ldots$$

Then the covariance of the return of asset i and asset j is calculated as

$$\sigma_{ij} = \mathrm{Cov}(r_i r_j) = \frac{1}{M}\sum_{t=1}^{M}(r_{it} - \overline{r_i}) * (r_{jt} - \overline{r_j})\ , i, j = 1, \ldots$$

Our problem is to calculate these covariances efficiently.

By far the clearest method for calculating the variance-covariance matrix is an on-screen method involving the excess return matrix. Suppose that we have N risky assets and that for each asset we have return data for M periods. Then the excess return matrix will look like this:

$$A = MatrixofExcessReturns = \begin{bmatrix} r_{11} - \overline{r}_1 & \dots & r_{N1} - \overline{r}_N \\ r_{12} - \overline{r}_1 & \dots & r_{N1} - \overline{r}_N \\ \dots & \dots & \dots \\ r_{1M} - \overline{r}_M & \dots & r_{NM} - \overline{r}_N \end{bmatrix}$$

The transpose of this matrix is

$$A = MatrixofExcessReturns = \begin{bmatrix} r_{11} - \overline{r}_1 & r_{12} - \overline{r}_1 & r_{1M} - \overline{r}_M \\ \dots & \dots & \dots \\ r_{N1} - \overline{r}_N & r_{N1} - \overline{r}_N & r_{NM} - \overline{r}_N \end{bmatrix}$$

Multiplying AT times A and dividing through by the number of periods M gives the variance-covariance matrix:

$$S = [\sigma_{ij}] = \frac{A^T A}{M}$$

We illustrate this method with our numerical example. We first calculate the mean return for each asset (the last line of the following spreadsheet picture):

	A	B	C	D	E	F	G	H	I	J	K
1		Return Data for Variance-Covariance Calculations from Excess Returns									
2											
3		AMR	BS	GE	HR	MO	UK	SP500			
4	1974	-0,3505	-0,1154	-0,4246	-0,2107	-0,0758	0,2331	-0,2647			
5	1975	0,7083	0,2472	0,3719	0,2227	0,0213	0,3569	0,372			
6	1976	0,7329	0,3665	0,255	0,5815	0,1276	0,0781	0,2384			
7	1977	-0,2034	-0,4271	-0,049	-0,0938	0,0712	-0,2721	-0,0716			
8	1978	0,1663	-0,0452	-0,0573	0,2751	0,1372	-0,1346	0,0656			
9	1979	-0,2659	0,0158	0,0896	0,0793	0,0215	0,2254	0,1844			
10	1980	0,0124	0,4751	0,335	-0,1894	0,2002	0,3657	0,3242			
11	1981	-0,0264	-0,2042	-0,0275	-0,7427	0,0913	0,0479	-0,0491			
12	1982	1,0642	-0,1493	0,6968	-0,2615	0,2243	0,0456	0,2141			
13	1983	0,1942	0,368	0,311	1,8682	0,2066	0,254	0,2251			
14	Mean	0,2032	0,0531	0,1501	0,1529	0,1025	0,1200	0,1238	<---AVERAGE(H4:H13)		

The means were calculated by using the Excel function **Average( )** on each column of data. Next, we calculate the excess return matrix by subtracting each asset's mean return from each of the periodic returns:

	A	B	C	D	E	F	G	H	I
16			**Excess Return Matrix**						
17		AMR	BS	GE	HR	MO	UK		
18	1974	-0,5537	-0,1685	-0,5747	-0,3636	-0,1783	0,1131		
19	1975	0,5051	0,1941	0,2218	0,0698	-0,0812	0,2369		
20	1976	0,5297	0,3134	0,1049	0,4286	0,0251	-0,0419		
21	1977	-0,4066	-0,4802	-0,1991	-0,2467	-0,0313	-0,3921		
22	1978	-0,0369	-0,0983	-0,2074	0,1222	0,0347	-0,2546		
23	1979	-0,4691	-0,0373	-0,0605	-0,0736	-0,0810	0,1054		
24	1980	-0,1908	0,4220	0,1849	-0,3423	0,0977	0,2457		
25	1981	-0,2296	-0,2573	-0,1776	-0,8956	-0,0112	-0,0721		
26	1982	0,8610	-0,2024	0,5467	-0,4144	0,1218	-0,0744	<---G12-G$14	

The transpose of this matrix can be calculated by using the array function **Transpose( ):**

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
28														
29		Transpose Excess Return Matrix												
30		1974	1975	1976	1977	1978	1979	1980	1981	1982	1983			
31	AMR	-0,55371	0,50509	0,52969	-0,40661	-0,03691	-0,46911	-0,19081	-0,22961	0,86099	-0,00901			
32	BS	-0,16854	0,19406	0,31336	-0,48024	-0,09834	-0,03734	0,42196	-0,25734	-0,20244	0,31486			
33	GE	-0,57469	0,22181	0,10491	-0,19909	-0,20739	-0,06049	0,18491	-0,17759	0,54671	0,16091			
34	HR	-0,36357	0,06983	0,42863	-0,24667	0,12223	-0,07357	-0,34227	-0,89557	-0,41437	1,71533			
35	MO	-0,17834	-0,08124	0,02506	-0,03134	0,03466	-0,08104	0,09766	-0,01124	0,12176	0,10406	<---TRANSPOSE(B18:G27)		
36	UK	0,1131	0,2369	-0,0419	-0,3921	-0,2546	0,1054	0,2457	-0,0721	-0,0744	0,134			
37														
38			1. Mark the area B31:K36											
39			2. Type = TRANSPOSE(B18:G27)											
40			3. Instead of [Enter], press [Ctrl]+[Shift]+[Enter]											
41			4. Formula will appear as {TRANSPOSE(B18:G27)}											

We can now calculate our variance-covariance matrix by multiplying AT times A. Again we use the array function **MMult(A_Transpose, A)/N:**

	A	B	C	D	E	F	G	H
43								
44								
45		**Product of Transpose Excess Returns Matrix times Excess Returns/10**						
46		AMR	BS	GE	HR	MO	UK	
47	AMR	0,20596	0,03752	0,10776	0,04926	0,02083	0,00586	
48	BS	0,03752	0,07903	0,03548	0,10283	0,00890	0,04027	
49	GE	0,10776	0,03548	0,08673	0,04428	0,01944	0,01468	
50	HR	0,04926	0,10283	0,04428	0,44352	0,01925	0,02565	
51	MO	0,02083	0,00890	0,01944	0,01925	0,00831	-0,00159	
52	UK	0,00586	0,04027	0,01468	0,02565	-0,00159	0,03894	
53								
54		1. Mark the Whole Area						
55		2. Type =Mmult(B31:K36,B18:G27)/10						
56		3. Instead of [Enter], press [Ctrl]+[Shift]+[Enter]						
57		4. Formula will appear as {=MMULT(B31:K36,B18:G27)/10}						

The single-index model (SIM) is an attempt to simplify some of the computational complexities of calculating the variance-covariance matrix. The model's basic assumption is that the returns of each asset can be linearly regressed on some market index:

$$\hat{R}_i = \alpha_i + \beta_i \hat{R}_x + \hat{\varepsilon}_i$$

Given this assumption, it is easy to establish the following two facts:

- $E(R_i) = \alpha_i + \beta_i E(R_x)$. This fact is trivial.
- $\sigma_{ij} = \beta_i \beta_j \sigma^2_x$. This fact requires a little more work. Writing the definition of σ_{ij} and expanding gives

$$\begin{aligned}
\sigma_{ij} &= E\{[\hat{R}_i - E(\hat{R}_i)][\hat{R}_j - E(\hat{R}_j)]\} \\
&= E\{[\alpha_i + \beta_i \hat{R}_x - [\alpha_i + \beta_i E(\hat{R}_x)]][\alpha_j + \beta_j \hat{R}_x - [\alpha_j + \beta_j E(\hat{R}_x)]]\} \\
&= E\{\beta_i [\hat{R}_x - E(\hat{R}_x)] \beta_j [\hat{R}_x - E(\hat{R}_x)]\} \\
&= \beta_i \beta_j [\hat{R}_x - E(\hat{R}_x)][\hat{R}_x - E(\hat{R}_x)] = \beta_i \beta_j \sigma_x^2
\end{aligned}$$

The SIM can lead to great simplifications in the calculation of the variance-covariance matrix. We illustrate with our six-portfolio example, adding a seventh column for the returns on the S&P 500. Regressing the returns of each asset on the Standard & Poor's 500 portfolio, we get the following table of β_s:

	A	B	C	D	E	F	G	H
3		AMR	BS	GE	HR	MO	UK	SP500
4	1974	-0,3505	-0,1154	-0,4246	-0,2107	-0,0758	0,2331	-0,2647
5	1975	0,7083	0,2472	0,3719	0,2227	0,0213	0,3569	0,372
6	1976	0,7329	0,3665	0,255	0,5815	0,1276	0,0781	0,2384
7	1977	-0,2034	-0,4271	-0,049	-0,0938	0,0712	-0,2721	-0,0716
8	1978	0,1663	-0,0452	-0,0573	0,2751	0,1372	-0,1346	0,0656
9	1979	-0,2659	0,0158	0,0896	0,0793	0,0215	0,2254	0,1844
10	1980	0,0124	0,4751	0,335	-0,1894	0,2002	0,3657	0,3242
11	1981	-0,0264	-0,2042	-0,0275	-0,7427	0,0913	0,0479	-0,0491
12	1982	1,0642	-0,1493	0,6968	-0,2615	0,2243	0,0456	0,2141
13	1983	0,1942	0,368	0,311	1,8682	0,2066	0,254	0,2251
14								
15	Beta	1,48214	1,08388	1,31081	1,29925	0,26223	0,49095	
16								
17								
18		=SLOPE(B4:B13,$H$4:$H$13)						
19		=COVAR(B4:B13,H4:H13)/VARP(H4:H13)						
20								

The circled formulas show two ways of calculating the betas of the shares (this topic will be discussed further in the next chapter).

To calculate the variance-covariance matrix, we have to calculate a matrix with entries $\beta_i \beta_j \sigma^2_{SP}$. This calculation is easily accomplished by putting the βs of the six assets on the borders of our variance-covariance matrix:

	B	C	D	E	F	G	H	I
10								
11	Var(SP500)		0,035938					
12								
13			**AMR**	**BS**	**GE**	**HR**	**MO**	**UK**
14			**1,4821**	**1,0839**	**1,3108**	**1,2993**	**0,2622**	**0,4910**
15	**AMR**	**1,4821**	0,0789	0,0577	0,0698	0,0692	0,0140	0,0262
16	**BS**	**1,0839**	0,0577	0,0422	0,0511	0,0506	0,0102	0,0191
17	**GE**	**1,3108**	0,0698	0,0511	0,0617	0,0612	0,0124	0,0231
18	**HR**	**1,2993**	0,0692	0,0506	0,0612	0,0607	0,0122	0,0229
19	**MO**	**0,2622**	0,0140	0,0102	0,0124	0,0122	0,0025	0,0046
20	**UK**	**0,4910**	0,0262	0,0191	0,0231	0,0229	0,0046	0,0087
21								
22				=D$14*$C15*$D$11				

To create the entries of the matrix, we use the mixed cell-reference feature of the spreadsheet. Thus, for example, the upper right-hand cell of the variance-covariance matrix (which contains AMR's SIM variance of 0.0790) has the formula = D$14 * $C15 * $D$11. The cell D11 contains the index variance (in this case = 0.0359) and D$14 and $C15 refer to the borders of the matrix, which contain the β_s of the assets. When this formula is copied to the whole matrix, we create the variance-covariance matrix according to the single-index model

It is clear that the variance-covariance matrix as estimated by the SIM differs from the exact variance-covariance matrix computed from the returns. In particular, as long as β_s are positive (nearly always the case), the SIM's variance-covariance matrix will have no negative entries; it cannot thus accommodate the negative covariance between two assets. In our example, the difference between the two matrices is significant.

	A	B	C	D	E	F	G	H	I
12									
13				**AMR**	**BS**	**GE**	**HR**	**MO**	**UK**
14				**1,4821**	**1,0839**	**1,3108**	**1,2993**	**0,2622**	**0,4910**
15		**AMR**	**1,4821**	0,0789	0,0577	0,0698	0,0692	0,0140	0,0262
16		**BS**	**1,0839**	0,0577	0,0422	0,0511	0,0506	0,0102	0,0191
17		**GE**	**1,3108**	0,0698	0,0511	0,0617	0,0612	0,0124	0,0231
18		**HR**	**1,2993**	0,0692	0,0506	0,0612	0,0607	0,0122	0,0229
19		**MO**	**0,2622**	0,0140	0,0102	0,0124	0,0122	0,0025	0,0046
20		**UK**	**0,4910**	0,0262	0,0191	0,0231	0,0229	0,0046	0,0087
21									
22					=D$14*$C15*$D$11				
23									
24									
25			**AMR**	**BS**	**GE**	**HR**	**MO**	**UK**	
26		**AMR**	0,205955	0,037518	0,107755	0,049264	0,020831	0,005863	
27		**BS**	0,037518	0,079035	0,035476	0,102835	0,008903	0,040267	
28		**GE**	0,107755	0,035476	0,086729	0,044284	0,019442	0,014677	
29		**HR**	0,049264	0,102835	0,044284	0,443524	0,019252	0,025647	
30		**MO**	0,020831	0,008903	0,019442	0,019252	0,00831	-0,00159	
31		**UK**	0,005863	0,040267	0,014677	0,025647	-0,00159	0,03894	
32									
33									
34			**AMR**	**BS**	**GE**	**HR**	**MO**	**UK**	
35		**AMR**	0,1270	-0,0202	0,0379	-0,0199	0,0069	-0,0203	
36		**BS**	-0,0202	0,0368	-0,0156	0,0522	-0,0013	0,0211	
37		**GE**	0,0379	-0,0156	0,0250	-0,0169	0,0071	-0,0085	
38		**HR**	-0,0199	0,0522	-0,0169	0,3829	0,0070	0,0027	
39		**MO**	0,0069	-0,0013	0,0071	0,0070	0,0058	-0,0062	
40		**UK**	-0,0203	0,0211	-0,0085	0,0027	-0,0062	0,0303	

Financial Modeling, 2nd ed., Simon Benninga, 2000; Chapter 8, pp.122-130
Investments, 8th ed., Zvi Bodie, Alex Kane, Alan J. Marcus, 2009; Chapter 7; 8 pp. 245-280

13. EFFICIENT PORTFOLIOS WITH SHORT SALES

The structure of the chapter is as follows: We begin with some preliminary definitions and notation. We then state the major results (proofs are given in the appendix to the chapter). In succeeding sections we implement these results, showing you

- How to calculate efficient portfolios.
- How to calculate the efficient frontier.

Throughout this chapter we use the following notation: There are N risky assets, each of which has expected return $E(r_i)$. The variable R is the column vector of expected returns of these assets:

$$R = \begin{bmatrix} E(r_1) = \overline{r_1} \\ E(r_2) = \overline{r_2} \\ \vdots \\ E(r_N) = \overline{r_N} \end{bmatrix}$$

and S is the N × N variance-covariance matrix:

$$S = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \vdots & \sigma_{2N} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{3N} \\ & & \cdots & \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \sigma_{NN} \end{bmatrix}$$

A portfolio of risky assets (when our intention is clear, we shall just use the word portfolio) is a column vector *x* whose coordinates sum to 1:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_N \end{bmatrix} \sum_{i=1}^{N} x_i = 1$$

Each coordinate x_i represents the proportion of the portfolio invested in risky asset i.

The expected portfolio return E(rx) of a portfolio x is given by the product of x and R:

$$E(r_x) = x^T R = \sum_{i=1}^{N} x_i E(r_i)$$

The *variance of portfolio x'sreturn,* $\sigma^2_x \equiv \sigma_{xx}$is given by the product

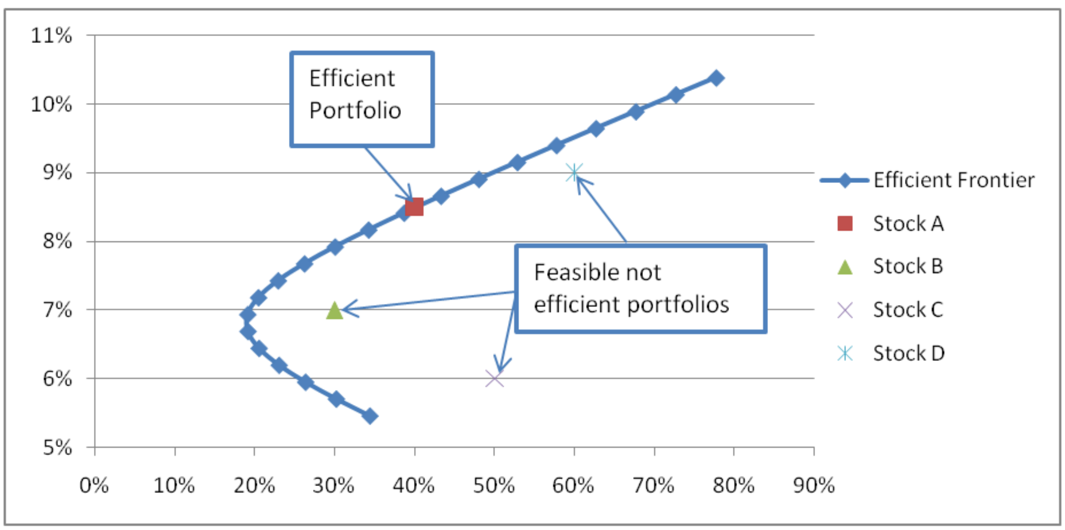
$$x^T S x = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}$$

The covariance between the return of two portfolios x and y, Cov(r_x, r_y), is defined by the product

$$\sigma_{xy} = x^T S y = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i y_j \sigma_{ij}$$

Note that $\sigma_{xy} = \sigma_{yx}$.

The following graph illustrates four concepts. A feasible portfolio is any portfolio whose proportions sum to one. The feasible set is the set of portfolio means and standard deviations generated by the feasible portfolios; this feasible set is the area inside and to the right of the curved line. A feasible portfolio is on the envelope of the feasible set if for a given mean return it has minimum variance. Finally, a portfolio x is an efficient portfolio if it maximizes the return given the portfolio variance (or standard deviation). That is, x is efficient if there is no other portfolio y such that $E(R_y) > E(R_x)$ and $\sigma_y \leq \sigma_x$. The set of all efficient portfolios is called the efficient frontier; this frontier is the heavier line in the graph.



To construct *Efficient Frontier*, we must find any two efficient portfolios. Any two efficient portfolios are enough to establish the whole frontier.Given any two envelope portfolios $x = \{x_1, \ldots, x_N\}$ and $y = \{y_1, \ldots, y_N\}$, all *efficient portfolios* are convexcombinations of x and y.

In this section we calculate the efficient frontier using Excel. We consider a world with four risky assets having the following expected returns and variance-covariance matrix:

	A	B	C	D	E	F	G
4							
5	**Variance-Covariance matrix**					**Mean Returns**	
6	0,4	0,03	0,02	0		0,06	
7	0,03	0,2	0,001	-0,06		0,05	
8	0,02	0,001	0,3	0,03		0,07	
9	0	-0,06	0,03	0,1		0,08	

We separate our calculations into two parts: First we calculate two efficient portfolios of the feasible set. Then we calculate the efficient frontier.

We must solve the system R − c = Sz for z, where we use two different values for c.

The c's we solve for are somewhat arbitrary, but to make life easy, we first solve this system for c = 0. This procedure gives the following results:

	A	B	C
12		Z	X
13		0,1019	0,0540
14		0,5657	0,2998
15		0,1141	0,0605
16		1,1052	0,5857
17		1,88689	
18			

The formulas in the cells are as follows:

- For z: =**MMult(MInverse(A6:D9), F6:F9)**. The range A6:D9 contains the variance-covariance matrix, and the cells F6:F9 contain the mean returns of the assets.
- For x: Each cell contains the associated value of z divided by the sum of all the z's. Thus, for example, cell C13 contains the formula =**B13/B14**

We now solve this system for some other constant c. This solution involves a few extra definitions, as the following picture from the spreadsheet shows:

	A	B	C	D	E	F	G	H
4								
5		Variance-Covariance matrix				Mean Returns	Mean Returns - Constant	
6	0,4	0,03	0,02	0		0,06	-0,005	
7	0,03	0,2	0,001	-0,06		0,05	-0,015	
8	0,02	0,001	0,3	0,03		0,07	0,005	
9	0	-0,06	0,03	0,1		0,08	0,015	
10								
11				Constant	0,065			
12		Z	X		Z	Y		
13		0,1019	0,0540		-0,0101	-0,1163		
14		0,5657	0,2998		-0,0353	-0,4067		
15		0,1141	0,0605		0,0047	0,0544		
16		1,1052	0,5857		0,1274	1,4687		

Each cell of the column vector labeled Mean minus constant contains the mean return of the given asset minus the value of the constant c (in this case c = 0.065). The second set of z's and its associated envelope portfolio y is givenby

	D	E	F	G
10				
11	Constant	0,065		
12		Z	Y	
13		-0,0101	-0,1163	
14		-0,0353	-0,4067	
15		0,0047	0,0544	
16		0,1274	1,4687	
17		0,086754		
18				

This vector z is calculated in a manner similar to that of the fir vector, except that the array function in the cells is **MMult(MInverse(A6:D9), G6:G9).**

To complete the basic calculations, we compute the means, standard deviations, and covariance of returns for the portfolios *x and y*.

	A	B	C	D	E	F
18						
19	Transpose X					
20	0,0540	0,2998	0,0605	0,5857		
21						
22	Transpose Y					
23	-0,1163	-0,4067	0,0544	1,4687		
24						
25	Mean X	0,0693		Mean Y	0,0940	
26	Var X	0,0367		Var Y	0,3341	
27	Sigma X	0,1917		Sigma Y	0,5780	
28						
29	Cov XY	0,049809				
30	Corr XY	0,449585				

The transpose vectors of x and of y are inserted using the array function Transpose. Now we calculate the **mean, variance,** and **covariance** as follows:

Mean(x) uses the formula **MMult(transpose_x, means)**.
Var(x) uses the formula **MMult(MMult(transpose_x, var_cov), x)**.
Sigma(x) uses the formula **Sqrt(var_x)**.
Cov(x, y) uses the formula **MMult(MMult(transpose_x, var_cov), y)**.
Corr(x, y) uses the formula**cov(x, y)/(sigma_x * sigma_y)**.

The following spreadsheet illustrates everything that has been done in this subsection.

	A	B	C	D	E	F	G	H
5		Variance-Covariance matrix				Mean Returns	Mean Returns - Constan	
6	0,4	0,03	0,02	0		0,06	-0,005	
7	0,03	0,2	0,001	-0,06		0,05	-0,015	
8	0,02	0,001	0,3	0,03		0,07	0,005	
9	0	-0,06	0,03	0,1		0,08	0,015	
10								
11				Constant	0,065			
12		Z	X		Z	Y		
13		0,1019	0,0540		-0,0101	-0,1163		
14		0,5657	0,2998		-0,0353	-0,4067		
15		0,1141	0,0605		0,0047	0,0544		
16		1,1052	0,5857		0,1274	1,4687		
17		1,88689			0,08675			
18								
19	Transpose X							
20	0,0540	0,2998	0,0605	0,5857				
21								
22	Transpose Y							
23	-0,1163	-0,4067	0,0544	1,4687				
24								
25	Mean X	0,0693		Mean Y	0,0940			
26	Var X	0,0367		Var Y	0,3341			
27	Sigma X	0,1917		Sigma Y	0,5780			
28								
29	Cov XY	0,04981						

Convex combinations of the two portfolios calculated before allow us to calculate the whole frontier of the feasible set. Suppose we let *p* be a portfolio that has proportion α invested in portfolio *x* and proportion (1−α) invested in *y*. Then—as discussed before—the mean and standard deviation of *p*'s return are

$$E(r_P) = \alpha(r_x) + (1-\alpha)(r_y)$$

$$\sigma_p^2 = \alpha^2\sigma_x^2 + (1-\alpha)^2\sigma_y^2 + 2\alpha(1-\alpha)\sigma_{xy}$$

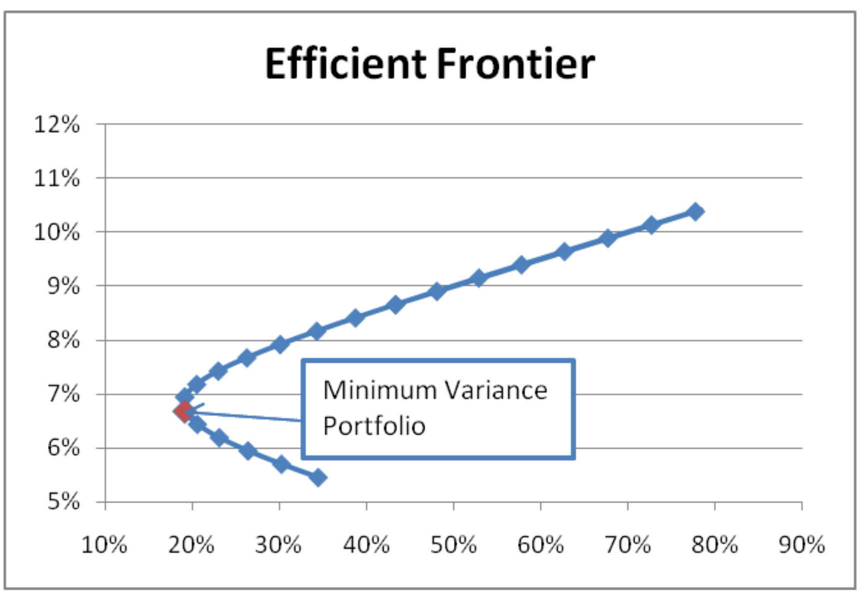
Here's a sample calculation for our two portfolios:

	A	B
31		
32	**Single Portfolio Calculation**	
33	Proportion of X	0,3
34	p's Mean Return	0,086586
35	p's Sigma	0,433516
36		

We can turn this calculation into a **data table** to get the following table:

	A	B	C	D	E	F	G
38			Data Table for Efficient				
39			Sigma	Return			
40			0,43352	0,08659	<-- Data Table Header		
41		-0,4	0,77777	0,10385			
42		-0,3	0,72738	0,10138			
43		-0,2	0,67725	0,09892			
44		-0,1	0,62743	0,09645			
45		0	0,57801	0,09398			
46		0,1	0,52911	0,09152			
47		0,2	0,48087	0,08905			
48		0,3	0,43352	0,08659			
49		0,4	0,38738	0,08412			
50		0,5	0,34295	0,08165			
51		0,6	0,30098	0,07919			
52		0,7	0,26266	0,07672			
53		0,8	0,22982	0,07425			
54		0,9	0,20510	0,07179			
55		1	0,19167	0,06932	<-- Minimum Variance Portfolio		
56		1,1	0,19193	0,06685			
57		1,2	0,20581	0,06439			
58		1,3	0,23088	0,06192			
59		1,4	0,26396	0,05946			
60		1,5	0,30244	0,05699			

The data table itself has been outlined in black. The five data points in the fourth column give the expected return of the portfolio in the cell to the left; these data points are graphed as a separate data series in the following figure.



Suppose a risk-free asset exists, and suppose that this asset has expected return r_f. Let M be the efficient portfolio which is constructed using weights solved for this equation:

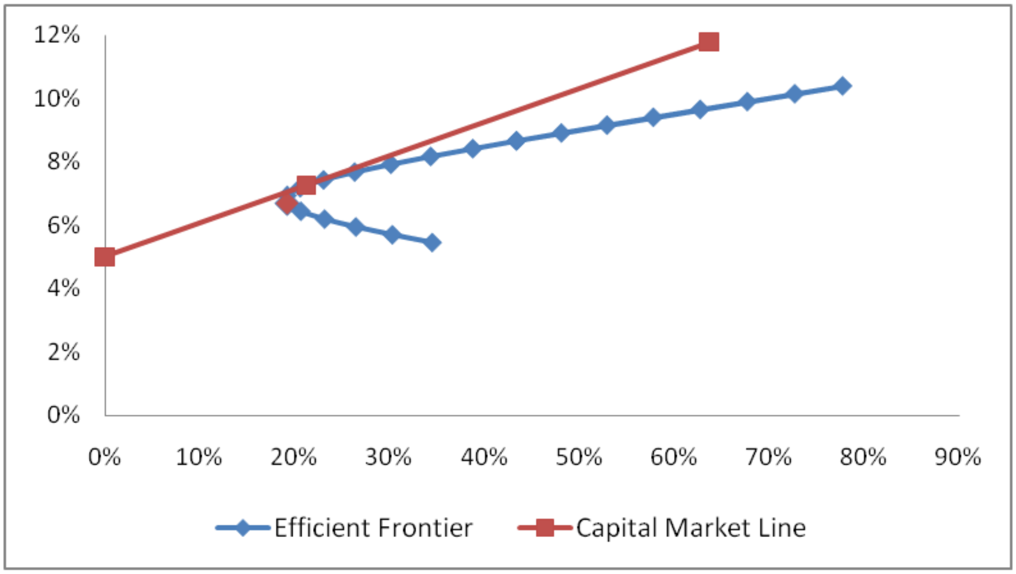
$$R - r_f \;=\; Sz$$

Now consider a convex combination of the portfolio M and the risk-free asset r_f; for example, and suppose that the weight of the risk-free asset in such a portfolio is α. It follows from the standard equations for portfolio return and σ that

$$E(r_p) = \alpha r_f + (1-\alpha)E(r_M)$$

$$\sigma_P = (1-\alpha)\sigma_m$$

The locus of all such combinations for a≥0 is known as the capital market line. It is graphed along with the efficient frontier as follows:



The portfolio M is called the market portfolio for several reasons:

- Suppose investors agree about the statistical portfolio information (i.e., the vector of expected returns R and the variance-covariance matrix S). Suppose furthermore that investors are interested only in maximizing expected portfolio return given portfolio standard deviation σ. Then it follows that all optimal portfolios will lie on the CML.
- In the case above, it further follows that *the portfolio M is the only portfolio of risky assets included in any optimal portfolio. It must therefore include all the risky assets, with each asset weighted in proportion to its market value.*

It is not difficult to find M when we know r_f: We merely have to solve for the efficient portfolio given that the constant c = r_f. When r_f changes, we get a different "market" portfolio—this is just the efficient portfolio given a constant of r_f. For example, in our numerical example, suppose that the risk-free rate is $r_f = 5$ percent. Then solving the system $R - r_f = Sz$ gives

	A	B	C	D
72				
73		Constant	5%	
74			0,015746359	0,0314
75			0,103403879	0,2059
76			0,029967007	0,0597
77			0,353052225	0,7031
78			0,50216947	
79				
80			Mean Y	0,0726
81			Var Y	0,045
82			Sigma Y	0,21214

When there is a risk-free asset, the following linear relationship (known as the SML-the **security market line**) holds:

$$E(R_x) = r_f + \beta_x(E(R_M) - r_f)$$

Where

$$\beta_x = \frac{Cov(x, M)}{\sigma_M^2}$$

Later we explore some statistical techniques for finding the SML that parallel those used by finance researchers.

Financial Modeling, 2nd ed., Simon Benninga, 2000; Chapter 9, pp.132-146
Investments, 8th ed., Zvi Bodie, Alex Kane, Alan J. Marcus, 2009; Chapter 7pp. 204-218

14. ESTIMATING BETAS AND THE SECURITY MARKET LINE

In this chapter we look at some typical capital-market data and replicate a simple test of the CAPM. We have to calculate the betas for a set of assets, and we then have to determine the equation of the security market line (SML). The test in this chapter is the simplest possible test of the CAPM. There is an enormous literature in which the possible statistical and methodological pitfalls of CAPM tests are discussed.

We illustrate the tests of the CAPM with a simple numerical example that uses the same data used in previous chapter.This example starts with rates of return on six securities and the S&P 500 portfolio. As a first step in analyzing thesedata and testing the CAPM, we calculate the mean return and the beta of each security's return, where we use thefollowing formulas:

$$Mean\ return\ for\ Security\ i = Average(Security\ i'sReturns, 1972 - 1981)$$

$$\beta_i = \frac{Covar(Security\ i'sreturn, S\&P500\ returns)}{Varp(S\&P500\ Returns)}$$

Here **Average, Covar, and Varp** are Excel functions on the column vectors of returns. Calculating these statistics gives the results in the following spreadsheet. Note that the β_{sp500} = 1, which is the way it should be if the S&P 500 is the market portfolio. Also note that instead of calculating the β using the **Covar( )** and **Varp( )** functions, we can also use Excel's **Slope( )** function.

	A	B	C	D	E	F	G	H	I
3		AMR	BS	GE	HR	MO	UK	SP500	
4	1974	-0,3505	-0,1154	-0,4246	-0,2107	-0,0758	0,2331	-0,2647	
5	1975	0,7083	0,2472	0,3719	0,2227	0,0213	0,3569	0,372	
6	1976	0,7329	0,3665	0,255	0,5815	0,1276	0,0781	0,2384	
7	1977	-0,2034	-0,4271	-0,049	-0,0938	0,0712	-0,2721	-0,0716	
8	1978	0,1663	-0,0452	-0,0573	0,2751	0,1372	-0,1346	0,0656	
9	1979	-0,2659	0,0158	0,0896	0,0793	0,0215	0,2254	0,1844	
10	1980	0,0124	0,4751	0,335	-0,1894	0,2002	0,3657	0,3242	
11	1981	-0,0264	-0,2042	-0,0275	-0,7427	0,0913	0,0479	-0,0491	
12	1982	1,0642	-0,1493	0,6968	-0,2615	0,2243	0,0456	0,2141	
13	1983	0,1942	0,368	0,311	1,8682	0,2066	0,254	0,2251	
14									
15	Mean	0,20321	0,05314	0,15009	0,15287	0,10254	0,12	0,12384	
16	Beta	1,48214	1,08388	1,31081	1,29925	0,26223	0,49095	1	
17									
18									
19									
20									
21									

=SLOPE(B4:B13,$H$4:$H$13)
=COVAR(B4:B13,H4:H13)/VARP(H4:H13)

The CAPM's security market line postulates that the mean return of each security should be linearly related to its beta. Assuming that the historic data provide an accurate description of the distribution of future returns, we postulate that $E(R_i) = \alpha + \beta_i \Pi + \varepsilon_i$. In the second step of our test of the CAPM, we examine this hypothesis by regressing the mean returns on the βs.

Excel offers us several ways of producing regression output. A simple way is to use the functions **Intercept( )**, **Slope ( )**, and **Rsq( )** to produce the basic ordinary least-squares results:

	A	B	C	D	E	F
22						
23	**Regressing the Means on Betas**					
24	Intercept	0,07519	<---INTERCEPT(B15:H15;B16:H16)			
25	Slope	0,05474	<---SLOPE(B15:H15;B16:H16)			
26	R-Square	0,28137	<---RSQ(B16:H16;B15:H15)			
27						

These results suggest that the SML is given by $E(R_i) = \alpha + \beta_i P$, where $\alpha = 0.0766$ and $P = 0.0545$. The R^2 of the regression (the percentage of the variability in the means explained by the betas) is 28 percent. We can also use **Tools|Data Analysis|Regression** to produce a new worksheet that has much more output.

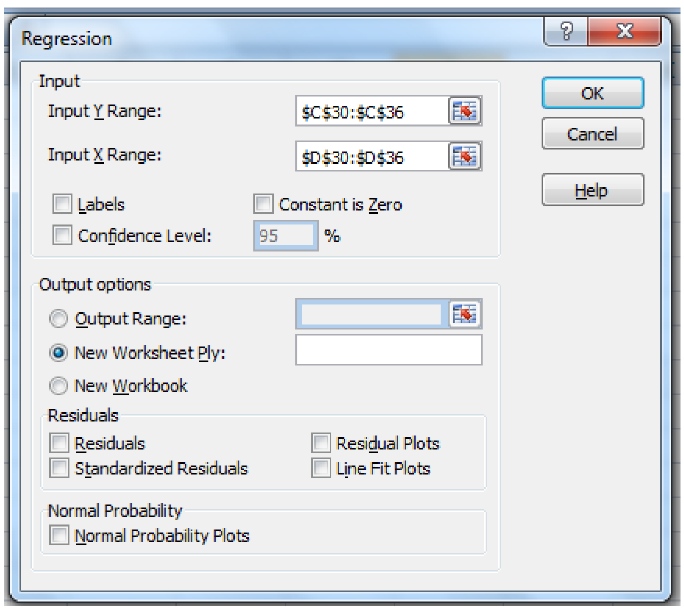
However, this tool will only work if the data are in columns, so we first rewrite the data as

	B	C	D	E
28				
29		Mean	Beta	
30		0,20321	1,482136	
31		0,05314	1,083878	
32		0,15009	1,310811	
33		0,15287	1,299254	
34		0,10254	0,262228	
35		0,12	0,490954	
36		0,12384	1	
37				

Here is some sample output:

	A	B	C	D	E	F	G	H	I
2									
3	*Regression Statistics*								
4	Multiple R	0,5304							
5	R Square	0,2814							
6	Adjusted R Square	0,1376							
7	Standard Error	0,0434							
8	Observations	7							
9									
10	ANOVA								
11		*df*	*SS*	*MS*	*F*	*Significance F*			
12	Regression	1	0,00368	0,00368	1,95766	0,22064			
13	Residual	5	0,00940	0,00188					
14	Total	6	0,01308						
15									
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95,0%*	*Upper 95,0%*
17	Intercept	0,0752	0,04206	1,78799	0,13381	-0,03291	0,18330	-0,03291	0,18330
18	X Variable 1	0,0547	0,03913	1,39916	0,22064	-0,04583	0,15532	-0,04583	0,15532

Both the standard error figures and the t-statistics show that neither α nor Π is significantly different from zero. The command that produced this output looks like this:



As discussed Excel has two functions that give the variance: **Var(array)** gives the sample variance and **Varp(array)** gives the population variance. We use the latter function here.

The previous section showed a specific numerical example in which we used some data to test the CAPM. In thissection we summarize what we did in previous section. Tests of the CAPM start with return data on a set of assets. The steps in the test are as follows:

- Determine a candidate for the market portfolio M. In the preceding example, we used the Standard & Poor's 500 Index (SP500) as a candidate for M. This is a critical step: In principle, the "true" market portfolio should contain all the market's risky assets in proportion to their value. It is clearly impossible to calculate this theoretical market portfolio, and we must therefore make do with a surrogate.

- For each of the assets in question, determine the asset beta (β).
- Regress the mean returns of the assets on their respective betas; this step should give the security market line (SML).

Our "test" yielded the following SML:

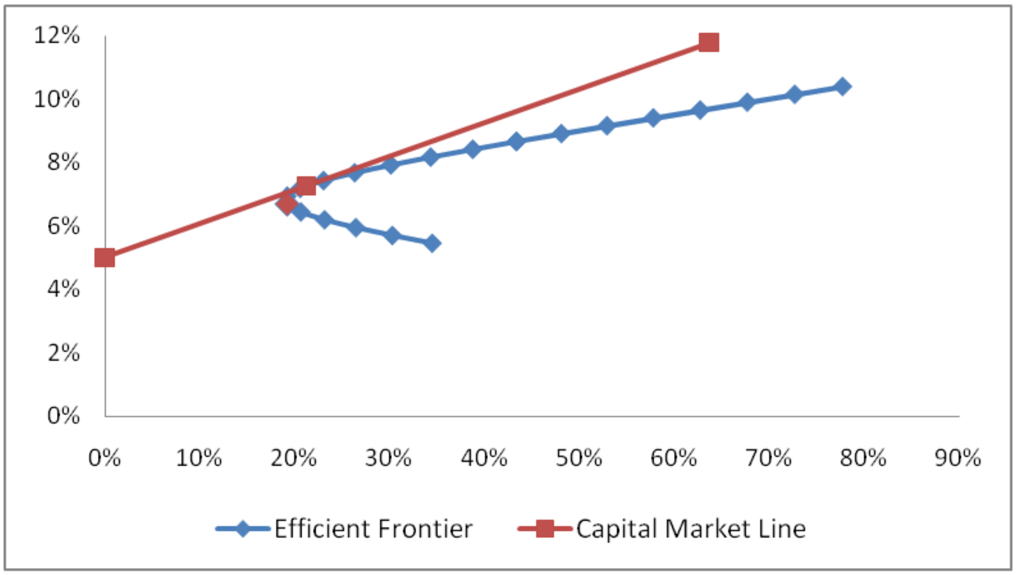
$$E(r_i) = 7.66\% + 5.45\%\beta_i$$

If the intercept is the risk-free rate (or the return on the zero-beta portfolio), then the expected return on the market is $E(r_m)$ =7.66% + 5.45% = 13.11%.

Financial Modeling, 2nd ed., Simon Benninga, 2000; Chapter 10, pp.151-163
Investments, 8th ed., Zvi Bodie, Alex Kane, Alan J. Marcus, 2009; Chapter 8 pp. 245-259; Chapter 9 pp. 283-295

15. EFFICIENT PORTFOLIOS WITHOUT SHORT SALES

Earlier we discussed the problem of finding an efficient portfolio. As shown there, this problem can be written as finding a tangent portfolio on the envelope of the feasible set of portfolios:



Our conditions for solving for such an efficient portfolio involved finding the solution to the following problem:

$$Max\Theta = \frac{E(r_x) - c}{\sigma_p}$$

such that

$$\sum_{i=1}^{N} x_i = 1$$

Where

$$E(r_x) = \sum_{i=1}^{N} x_i E(r_i) = x^T R$$

$$\sigma_p = \sqrt{x^T S x} = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}}$$

Solutions to the maximization problem allownegative portfolio proportions; when xi < 0, this is equivalent to the following assumptions:

- The ith security is sold short by the investor.
- The proceeds from this short sale become immediately available to the investor.

Reality is, of course, considerably more complicated than this academic model of short sales. In particular, it is rarefor all of the short-sale proceeds to become available to the investor at the time of investment, since brokeragehouses typically escrow some or even all of the proceeds. It may also be that the investor is completely prohibitedfrom making any short sales (indeed, most small investors seem to proceed on the assumption that short sales areimpossible).

In this chapter we investigate these problems. We show how to use Excel's Solver to find efficient portfolios of assetswhen we restrict short sales.[1] Although the solutions are not perfect (in particular, they take too much time), theyare instructive and easy to follow.

We start with the problem of finding an optimal portfolio when there are no short sales allowed. The problem wesolve is similar to the maximization problem stated previously, with the addition of the short sales constraint:

$$Max\Theta = \frac{E(r_x) - c}{\sigma_p}$$

such that

$$x_i \geq 0, i = 1,2, \dots N$$

$$\sum_{i=1}^{N} x_i = 1$$

Where

$$E(r_x) = \sum_{i=1}^{N} x_i E(r_i) = x^T R$$

$$\sigma_p = \sqrt{x^T S x} = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}}$$

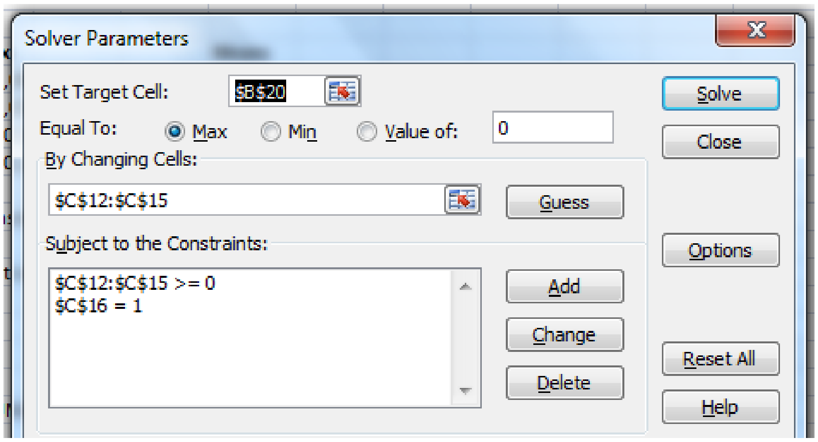
Our problem can be solved in Excel using **Tools|Solver**. We illustrate with the following numerical example, inwhich there are only four risky assets:

	B	C	D	E	F	G	H
1							
2							
3	**Variance-Covariance Matrix**					**Means**	
4	0,1	0,03	-0,08	0,05		8%	
5	0,03	0,2	0,02	0,03		9%	
6	-0,08	0,02	0,3	0,2		10%	
7	0,05	0,03	0,2	0,9		11%	

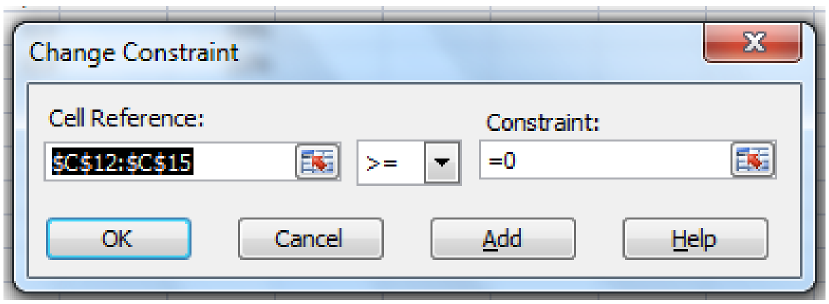
In order to solve the portfolio problem with no short sales, we set up the following spreadsheet (which also illustratesa solution to the problem for the c = 9 percent):

	A	B	C	D	E	F	G	H	I
1				**No Short-Sales**					
2									
3		**Variance-Covariance Matrix**					**Means**		
4		0,1	0,03	-0,08	0,05		8%		
5		0,03	0,2	0,02	0,03		9%		
6		-0,08	0,02	0,3	0,2		10%		
7		0,05	0,03	0,2	0,9		11%		
8									
9		c	9%	<- Constant					
10									
11		Optimal Portfolio proportions							
12		x1	0						
13		x2	0						
14		x3	0,55556						
15		x4	0,44444						
16		Total	1	<---SUM(C12:C15)					
17									
18	Portfolio Mean	10,44%	<---MMULT(TRANSPOSE(C12:C15);G4:G7)						
19	Portfolio Sigma	60,76%	<---SQRT(MMULT(TRANSPOSE(C12:C15);MMULT(B4:E7;C12:C15)))						
20	Theta	2,38%	<---(B18-C9)/B19						
21									

The solution was achieved by use the **Tools|Solver** feature of Excel. The first time we bring up the Solver, we createthe following dialogue box:



The nonnegativity constraints can be added by clicking on the **Add** button in the preceding dialogue box to bring upthe following window (shown here filled in):



The second constraint (which constrains the portfolio proportions to sum to 1) is added in a similar fashion.By changing the value of c in the spreadsheetwe can compute other portfolios; in the following example, we haveset the constant c equal to 8.50 percent.

	A	B	C	D	E	F	G	H	I
1				No Short-Sales					
2									
3		Variance-Covariance Matrix					Means		
4		0,1	0,03	-0,08	0,05		8%		
5		0,03	0,2	0,02	0,03		9%		
6		-0,08	0,02	0,3	0,2		10%		
7		0,05	0,03	0,2	0,9		11%		
8									
9		c	8,5%	<- Constant					
10									
11		Optimal Portfolio proportions							
12		x1	0						
13		x2	0,25145						
14		x3	0,48847						
15		x4	0,26008						
16		Total	1	<---SUM(C12:C15)					
17									
18	Portfolio Mean	10,01%	<---MMULT(TRANSPOSE(C12:C15);G4:G7)						
19	Portfolio Sigma	45,25%	<---SQRT(MMULT(TRANSPOSE(C12:C15);MMULT(B4:E7;C12:C15)))						
20	Theta	3,33%	<---(B18-C9)/B19						

In both examples, the short-sale restriction is effective: In the first example both x_1 and x_2 are equal to zero, whereasin the second example x_1 equals zero. However, not all values of c give portfolios in which the short-sale constraint iseffective. For example, if the constant is 8 percent, we get

	A	B	C	D	E	F	G	H	I
1				No Short-Sales					
2									
3		Variance-Covariance Matrix					Means		
4		0,1	0,03	-0,08	0,05		8%		
5		0,03	0,2	0,02	0,03		9%		
6		-0,08	0,02	0,3	0,2		10%		
7		0,05	0,03	0,2	0,9		11%		
8									
9		c	8,0%	<- Constant					
10									
11		Optimal Portfolio proportions							
12		x1	0,20041						
13		x2	0,25868						
14		x3	0,4219						
15		x4	0,11901						
16		Total	1	<---SUM(C12:C15)					
17									
18	Portfolio Mean	9,46%	<---MMULT(TRANSPOSE(C12:C15);G4:G7)						
19	Portfolio Sigma	31,91%	<---SQRT(MMULT(TRANSPOSE(C12:C15);MMULT(B4:E7;C12:C15)))						
20	Theta	4,57%	<---(B18-C9)/B19						
21									

As c gets lower, the short-sale constraint begins to be effective with respect to asset 4. For example, when c = 3percent,

	A	B	C	D	E	F	G	H	I
8									
9		c	3,0%	<- Constant					
10									
11		Optimal Portfolio proportions							
12		x1	0,5856						
13		x2	0,09654						
14		x3	0,31786						
15		x4	0						
16		Total	1	<---SUM(C12:C15)					
17									
18	Portfolio Mean	8,73%	<---MMULT(TRANSPOSE(C12:C15);G4:G7)						
19	Portfolio Sigma	20,32%	<---SQRT(MMULT(TRANSPOSE(C12:C15);MMULT(B4:E7;C12:C15)))						
20	Theta	28,21%	<---(B18-C9)/B19						

For very high cs (the next case illustrates c = 11 percent) only asset 4 is included in the maximizing portfolio:

	A	B	C	D	E	F	G	H	I
8									
9		c	11,0%	<- Constant					
10									
11		Optimal Portfolio proportions							
12		x1	0						
13		x2	0						
14		x3	0						
15		x4	1						
16		Total	1	<---SUM(C12:C15)					
17									
18	Portfolio Mean	11,00%	<---MMULT(TRANSPOSE(C12:C15);G4:G7)						
19	Portfolio Sigma	94,87%	<---SQRT(MMULT(TRANSPOSE(C12:C15);MMULT(B4:E7;C12:C15)))						
20	Theta	0,00%	<---(B18-C9)/B19						
21									

If **Tools|Solver** doesn't work, you may not have loaded the Solver add-in. To do so, go to **Tools|Add-**ins and clicknext to the **Solver Add-in**.

Financial Modeling, 2nd ed., Simon Benninga, 2000; Chapter 11, pp.164-174
Investments, 8th ed., Zvi Bodie, Alex Kane, Alan J. Marcus, 2009; Chapter 7

16. VALUE-AT-RISK

Value-at-Risk (VaR) measures the worst expected loss under normal market conditions over a specific time intervalat a given confidence level. VaR answers the question: How much can I lose with x percent probability over a presethorizon? Another way of expressing this idea is that VaR is the lowest quantile of the potential losses that can occurwithin a given portfolio during a specified time period. The basic time period T and the confidence level (the quantile)q are the two major parameters that should be chosen in a way appropriate to the overall goal of risk measurement.The time horizon can differ from a few hours for an active trading desk to a year for a pension fund. When theprimary goal is to satisfy external regulatory requirements, such as bank capital requirements, the quantile is typicallyvery small (for example, 1 percent of worst outcomes). However, for an internal risk management model used by acompany to control the risk exposure, the typical number is around 5 percent.

In the jargon of VaR, suppose that a portfolio manager has a daily VaR equal to $1 million at 1 percent. Thisstatement means that there is only one chance in 100 that a daily loss bigger than $1 million occurs under normalmarket conditions.

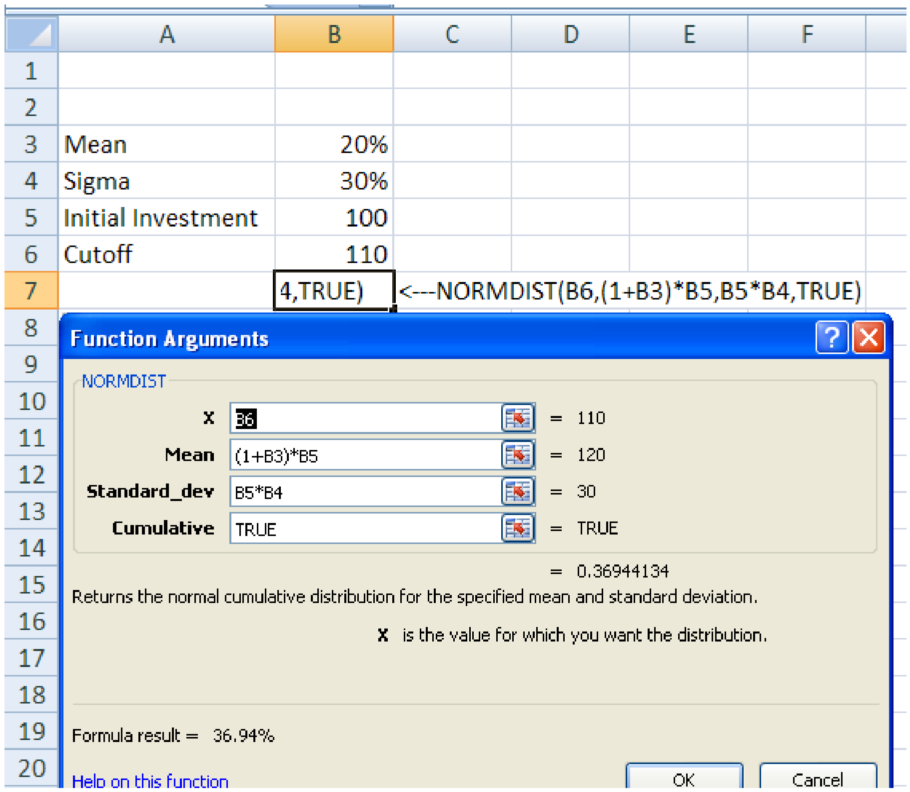
Suppose a manager has a portfolio that consists of a single asset. The return of the asset is normally distributed withmean return 20 percent and standard deviation 30 percent. The value of the portfolio today is $100 million. We wantto answer various simple questions about the end-of-year distribution of portfolio value:

- What is the distribution of the end-of-year portfolio value?
- What is the probability of a loss of more than $20 million dollars by year-end (i.e., what is the probability that the end-of-year value is less than $80 million)?
- With 1 percent probability what is the maximum loss at the end of the year? This is the VaR at 1 percent.

The probability that the end-of-year portfolio value is less than $80 million is about 37 percent. ("Million" is omitted inthe example.)

	A	B	C	D	E	F	G	H
1								
2								
3	Mean	20%						
4	Sigma	30%						
5	Initial Investment	100						
6	Cutoff	110						
7		36.94%	<---NORMDIST(B6,(1+B3)*B5,B5*B4,TRUE)					

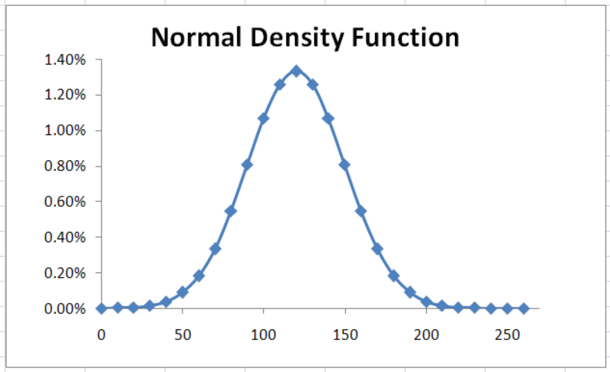
Here's the way the screen looks when we apply the **NormDist** function:



This picture shows that the Excel function **Normdist** can give both the cumulative normal distribution and theprobability mass function. Using the latter option and a data table gives the standard bell-shaped graph:

	A	B	C
1			
2			
3	Mean	20%	
4	Sigma	30%	
5	Initial Investment	100	
6	Cutoff	110	
7		36.94%	<---NORMDIST(B6,(1+B3)*B5,B5*B4,TRUE)
8			
9			
10			
11			1.26%
12		0	4.5E-06
13		10	1.6E-05
14		20	5.1E-05
15		30	0.00015
16		40	0.00038
17		50	0.00087
18		60	0.0018
19		70	0.00332
20		80	0.00547
21		90	0.00807
22		100	0.01065
23		110	0.01258

Data Table Header
=NORMDIST(B6,(1+B3)*B5,B5*B4,FALSE)

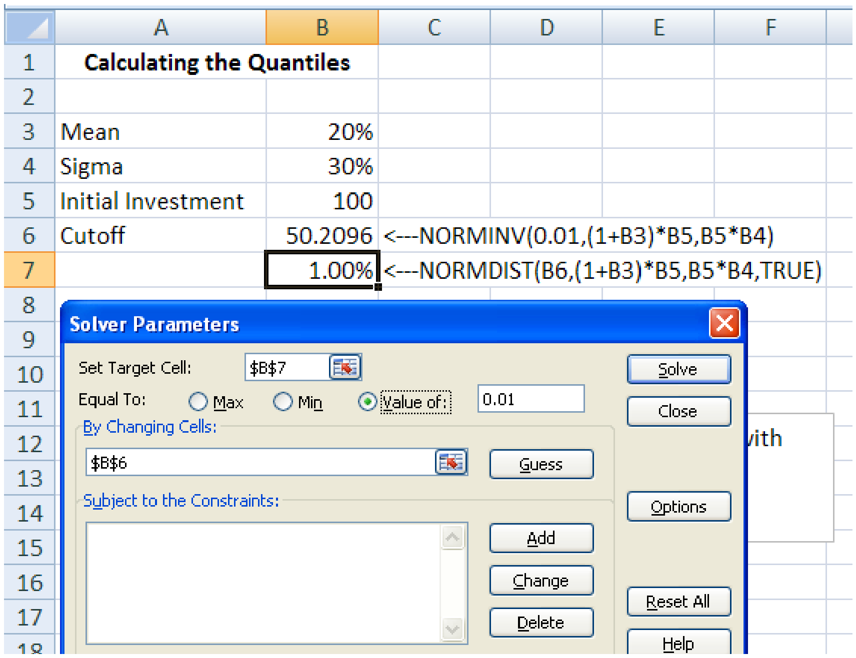


With a probability of 1 percent the end-of-year portfolio value will be less than 50.20865; thus the VaR of thedistribution is 100 − 50.20865 = 49.79135.

	A	B	C
1	**Calculating the Quantiles**		
2			
3	Mean	20%	
4	Sigma	30%	
5	Initial Investment	100	
6	Cutoff	50.2096	<---NORMINV(0.01,(1+B3)*B5,B5*B4)
7		1.00%	<---NORMDIST(B6,(1+B3)*B5,B5*B4,TRUE)
8			
9	Var at 1.00% level	49.7904	
10			
11			
12			
13			
14			
15			

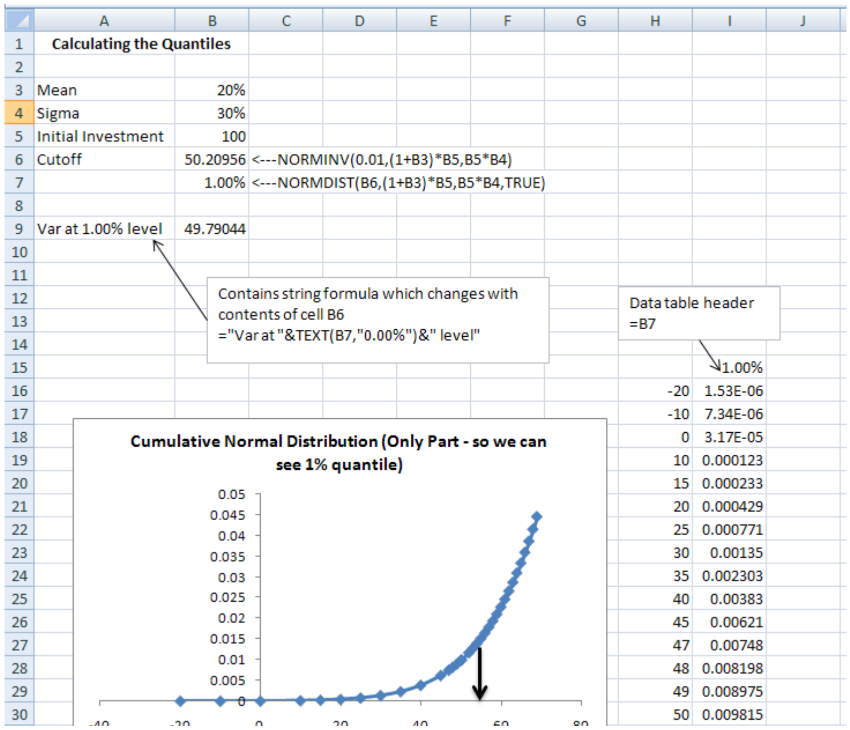
Contains string formula which changes with contents of cell B6
="Var at "&TEXT(B7,"0.00%")&" level"

The cutoff is known as the quantile of the distribution. We found this solution by using **Excel's Solver**:



We can use Solver to find the quantiles for any distribution. For Normal Distribution Excel has built in functions that find the quantile. These functions—Norminv and Normsinv—find the inverse for the normal and standard normal.

Here's an example for the numbers that we've been using; this time we have written the function Norminv(0.01,(1+B3)*B5,B5*B4) in cell B6. This function finds the cutoff point for which the normal distribution with a mean of 120 and astandard deviation of 30 has probability of 1 percent. You can see this point on the following graph, which shows partof the cumulative distribution:



The lognormal distribution is a more reasonable distribution for many asset prices (which cannot become negative)than the normal distribution. Suppose that the return on the portfolio is normally distributed with annual mean μ andannual standard deviation σ. Furthermore, suppose that the current value of the portfolio is given by V_0 . Then it follows that the logarithm of the portfolio value at time T, V_T, is normally distributed:

$$\ln(V_T) \sim Normal\left[\ln(V_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

Suppose, for example, that $V_0 = 100$, μ = 10 percent, and σ = 30 percent. Thus the end-of-year log of the portfoliovalue is distributed normally:

$$\ln(V_T) \sim \mathrm{Normal}\left[\ln(100) + \left(0.10 - \frac{0.3^2}{2}\right), 0.3\right]$$

Thus a portfolio whose initial value is $100 million and whose annual returns are lognormally distributed withparameters μ = 10 percent and σ = 30 percent, has an annual VaR equal to $47.42 million at 1 percent:

	A	B	C	D	E
2					
3	Initial Value, V_0	100			
4	Mean, μ	10%			
5	Sigma, σ	30%			
6	Time Period, T	1	<-- In years		
7					
8	Parameters of Normal Distributions of $\ln(V_T)$				
9	Mean	4.66			
10	Sigma	0.3			
11					
12	Cutoff	52.57631976	<---LOGINV(0.01,B9,B10)		
13	VaR at 1% level	47.42368024	<---B3-B12		

Most VaR calculations are not concerned with annual value at risk. The main regulatory and management concern iswith loss of portfolio value over a much shorter time period (typically several days or perhaps weeks). It is clear thatthe distribution formula:

$$\ln(V_T) \sim \text{Normal}\left[\ln(V_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

can be used to calculate the VaR over any horizon. Recall that T is measured in annual terms; if there are 250business days in a year, then the daily VaR corresponds to T = 1/250 (for many fixed-income instruments one shoulduse 1/360, 1/365, or 1/365.25 depending on the market convention).

As can be seen from the preceding examples, VaR is not—in principle, at least—a very complicated concept. In theimplementation of VaR, however, there are two big practical problems:

1. The first problem is the estimation of the parameters of asset return distributions. In "real-world" applicationsof VaR, it is necessary to estimate means, variances, and correlations of returns. This is a not-inconsiderableproblem! In this section we illustrate the importance of the correlations between asset returns. In the followingsection we give a highly simplified example of the estimation of return distributions from market data. Forexample, you can imagine that a long position in euros and a short position in U.S. dollars is less risky than aposition in only one of the currencies, because of a high probability that profits of one position will be mainlyoffset by losses of another.
2. The second problem is the actual calculation of position sizes. A large financial institution may havethousands of loans outstanding. The database of these loans may not classify them by their riskiness, noreven by their term to maturity. Or—to give a second example—a bank may have offsetting positions in foreigncurrencies at different branches in different locations. A long position in deutschemarks in New York may beoffset by a

short position in deutschemarks in Geneva; the bank's risk—which we intend to measure by VaR—is based on the net position.We start with the problem of correlations between asset returns.

We continue the previous example, but assume thatthere are three risky assets. As before, the parameters of the distributions of the asset returns are known: all themeans, μ1, μ2, μ3, as well as the variance-covariance matrix of the returns:

$$S = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

The matrix S is of course symmetric, with μ_i the variance of the ith asset's return and σ_{ij} the covariance of the returnsof assets i and j (if i = j, σij is the variance of asset i's return).

Suppose that the total portfolio value today is $100 million, with $30 million invested in asset 1, $25 million in asset 2,and $45 million in asset 3. Then the return distribution of the portfolio is given by

$$Mean\ Return = \{x_1 \quad x_2 \quad x_3\}\begin{Bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{Bmatrix} = x_1\mu_1 + x_2\mu_2 + x_3\mu_3$$

$$Variance\ of\ Return = \{x_1 \quad x_2 \quad x_3\}S\{x_1 \quad x_2 \quad x_3\}^T$$

Where x = {x1, x2, x3} = {0.3, 0.25, 0.45} is the vector of proportions invested in each of the three assets. Assumingthat the returns are normally distributed (meaning that prices are lognormally distributed), we may calculate the VaRas in the following spreadsheet fragment:

	A	B	C	D	E	F	G	H	I
1		VaR for Three Asset Portfolio							
2									
3		Mean		Variance-Covariance				Portfolio	
4		Returns		Matrix				Proportions	
5	Asset 1	10%		0.1	0.04	0.03		0.3	
6	Asset 2	12%		0.04	0.2	-0.04		0.25	
7	Asset 3	13%		0.03	-0.04	0.6		0.45	
8									
9	Initial Investment	100							
10	Mean Return	0.1185	<---MMULT(TRANSPOSE(B5:B7),H5:H7)						
11	Portfolio Sigma	0.38484	<---SQRT(MMULT(MMULT(TRANSPOSE(H5:H7),D5:F7),H5:H7))						
12									
13	Mean Investment Value	11.85							
14	Sigma of Invesvestment Value	38.4838							
15									
16	Cutoff	22.3234	<---NORMINV(0.01,(1+B10)*B9,B11*B9)						
17	Cumulative PDF	0.01	<---NORMDIST(B16,B13,B14,0)						
18	VaR at 1,00% Level	77.6766	<---B9-B16						

Sometimes it helps to simulate data. In this section we give an example. Suppose that the current date is February10, 1997, and consider a firm that has an investment in two assets:

- It is long two units of an index fund. The fund's current market price is 293, so that the investment in the index fund is worth 2 * 293 = 586.

- It is short a foreign bond denominated in rubles. The bond is a zero-coupon bond (i.e., pays no interest), has face value of 100 rubles and maturity of May 8, 2000. If the current ruble interest rate is 5.30 percent, then the February 10, 1997, ruble value of the bond is −100 * exp [−5.30 percent * (May 8, 2000 − Feb. 10, 1997)/365] = −84.2166

In dollars, the value of the bond is −84.2166 * 3.40 = −286.3365, so that the net portfolio value is 586 − 286.3365 =299.66.

This example is illustrated in the following display:

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3	Units of Index Held	2									
4	Bond Maturity	5/8/2000									
5											
6				Ruble	Ruble		Total	Ruble	Dollar		
7			Index	Interest	Exchange		Index	Bond	Bond	Portfolio	
8		Date	Value	Rate	Rate		Value	Value	Value	Value	
9		2/10/1997	293	5.30%	3.40		586	-84.21660441	-286.336455	299.66355	
10											
11							B3*C9		H9*E9		G9+I9
12											
13							-100*EXP(-5.3%*(B4-B9)/365)				
14											

Now suppose we have exchange-rate and index data.

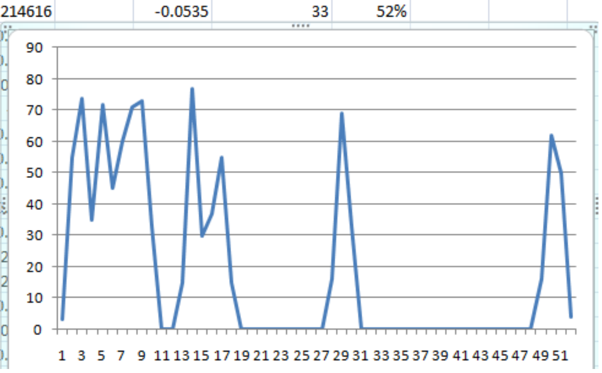
	A	B	C	D	E	F	G
4							
5	Day	Index	Foreing Interest Ra	Exchange Rate		Portfolio Value	
6	01.02.1997	462.71	5.28%	3.5		634.62	
7	01.03.1997	514.71	5.26%	3.47		739.75	
8	01.04.1997	456.5	5.23%	3.46		622.58	
9	01.05.1997	487.39	5.24%	3.45		684.04	
10	01.06.1997	470.42	5.25%	3.45		648.90	
11	02.08.1997	467.14	5.31%	3.44		641.11	
12	02.09.1997	562.06	5.32%	3.41		832.28	
13	02.10.1997	481.61	5.30%	3.4		670.79	
14							

We want to use these data as a basis for generating "random" return data. We illustrate one technique for doing so,called bootstrapping. This term refers to random reshufflings of the data. For eachiteration, we reorder the series ofindex prices, interest rates, and exchange rates and calculate the return on the portfolio for some date in future. We need to iterate random numbers for each variable and lookup value in first table.

	A	B	C	D	E	F	G	H	I	J
1	**Bootstraping Return Distribution**									
2	Units of Index Held	2								
3	Bond Maturity	05.08.2000								
4										
5		Day	Index	Foreing Int	Exchange Rate		Portfolio Value			
6	1	01.02.1997	462.71	5.28%	3.5		634.62			
7	2	01.03.1997	514.71	5.26%	3.47		739.75 <-- C7*$B$2+(-100*EXP(-($B$3-B7)/365*D7)*E7)			
8	3	01.04.1997	456.5	5.23%	3.46		622.58			
9	4	01.05.1997	487.39	5.24%	3.45		684.04			
10	5	01.06.1997	470.42	5.25%	3.45		648.90			
11	6	02.08.1997	467.14	5.31%	3.44		641.11			
12	7	02.09.1997	562.06	5.32%	3.41		832.28			
13	8	02.10.1997	481.61	5.30%	3.4		670.79			
14										
15										
16										
17	Date	Index Rand	Index	Foreing Interest Rate Rand	Foreing Interest Rate	Exchange Rate Rand	Exchange Rate		Portfolio Value	
18	02.10.1998	6	467.14	1	0.0528	5	3.45		621.28	
19	02.10.1998	7	562.06	6	0.0531	3	3.46		810.39	
20	02.10.1998	3	456.5	3	0.0523	3	3.46		598.81	
21	02.10.1998	3	456.5	8	0.053	6	3.44		601.03	

The bootstrapped return data look like this:

	H	I	J	K	L	M	N	O	P	Q
12										
13										
14						Min Value	-0.112726278			
15						Max Value	0.216307166			
16										
17		Portfolio Value		Return		Bin	Frequency			
18		621.28		-0.0738		-0.11273	3	0%		
19		810.39		0.208112		-0.10615	55	6%		
20		598.81		-0.10731		-0.09956	74	13%		
21		601.03		-0.104		-0.09298	35	17%		
22		815.77		0.216138		-0.0864	72	24%		
23		812.03		0.210558		-0.07982	45	28%		
24		625.91		-0.0669		-0.07324	60	34%		
25		619.18		-0.07694		-0.06666	71	42%		
26		597.90		-0.10867		-0.06008	73	49%		
27		814.75		0.214616		-0.0535	33	52%		
28		599.72		-0.						
29		623.19		-0.						
30		716.54		0.0						
31		666.09								
32		619.18		-0.						
33		595.23		-0.						
34		660.99		-0.						
35		600.12		-(						
36		626.65		-0.						
37		809.02		0.2						
38		810.04		0.2						
39		616.28		-0.						
40		720.22		0.0						
41		631.30		-0.						



The graph on the right indicates the return distribution, which is far from normal. From columns M and O, you cantell that the 5 percent VaR is about −10 percent, meaning that with a probability of 5 percent, the firm could lose 10 percent of its investment.

Financial Modeling, 2nd ed., Simon Benninga, 2000; Chapter 12, pp.175-184

17. THE BINOMIAL OPTION-PRICING MODEL

Here we discuss the binomial option pricingmodel, with which we can compute the price of an option, given the characteristics ofthe stock or other underlying asset.

The binomial option pricing model assumes that, over a period of time, the price ofthe underlying asset can move only up or down by a specified amount-that is, the assetprice follows a binomial distribution. Given this assumption, it is possible to determinea no-arbitrage price for the option. Surprisingly, this approach, which appears at firstglance to be overly simplistic, can be used to price options, and it conveys much of theintuition underlying more complex (and seemingly more realistic) option pricing models.

Because of its usefulnesswe will see how the binomial model works and use it to price bothEuropean and American call and put options.As part of the pricing analysis, we will also see how market-makers can create optionssynthetically using the underlying asset and risk-free bonds.

Next to the Black-Scholes model, the binomial option-pricing model is probably the most widely used option-pricing model. It has many advantages: It is a simple model that is easily programmed and adapted to numerous, and often quite complicated, option-pricing problems. In addition, it gives many insights into option pricing. When extended to many periods, the binomial model becomes one of the most powerful ways of valuing securities like options whose payoffs are contingent on the market prices of other assets.

AOne-Period Binomial Tree

To illustrate the use of the binomial model, we start with the following very simple example:

- There are two dates: date 0 represents today, and date 1 is one year from now
- There are two "fundamental" assets: a stock and a bond. There is also a call option written on the stock
- The stock price today is \$50. At date 1 it will either be 55 or 48.5
- The one-period interest rate, r is 6 percent
- The call option matures at date 1 and has exercise price K = \$50

Here is a picture from a spreadsheet that embodies this model:

	A	B	C	D	E	F	G	H	I	J	K	L	M
1			A ONE-PERIOD BINOMIAL TREE										
2													
3	Up factor	1.1	<= F4/E5		Stock Price				Bond Price				
4	Down Factor	0.97	<= F6/E5				55				1.06	<= I5*(1+rate)	
5					50				1				
6	Initial Stock Price	50					48.5				1.06	<= I5*(1+rate)	
7	Interest Rate	6%											
8	Exercise Price	50			Call Option								
9							5	<= MAX (0,G4-K)					
10					???								

We wish to price the call option. We do so by showing that there is a *combination of the bonds and stocks that exactly replicates the call option's payoffs*. To show this fact, we use some basic linear algebra; suppose we find Δshares of the stock and *B* bonds such that

$$55\Delta + 1.06B = 5$$

$$48.5\,\Delta + 1.06B = 0$$

This system of equations solves to give Δ= 0.769231, *B* = −35.1959. Thus purchasing 0.77 of a share of the stock and borrowing $35.20 at 6 percent for one period will give payoffs of $5 if the stock price goes up and $0 if the stock price goes down — the payoffs of the call option. It follows that the price of the option must be equal to the cost of replicating its payoffs; that is,

$$\text{Call option price} = 0.7692 * \$50 - \$35.1959 = \$3.2656$$

Here is a picture from a spreadsheet that embodies this result:

	A	B	C	D	E	F	G	H	I	J
12										
13	Δ	0.769231	<= (G9-G11)/(G4-G6)							
14	B	-35.1959	<= 1/(1+B7)*(B3*G11-B4*G9)/(B3-B4)							
15										
16		Call Option				Δ Stock + Bond				
17				5				5	<= Δ*G4+B*K4	
18	F18 =>	3.2656			Δ*S+B =>	3.2656				
19				0				0	<= Δ*G6+B*K6	
20										

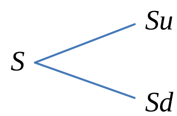
This logic is called *pricing by arbitrage:* If two assets or sets of assets (in our case—the call option and the portfolio of 0.77 of the stock and −$35.20 of the bonds) have the same payoffs, they must have the same market price.

The Binomial Solution

In the preceding example, how did we know that buying 0.77 of a share of stock and borrowing $35.20 would replicate a call option?

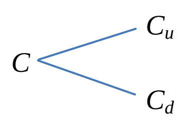
We have two instruments to use in replicating a call option: shares of stock and aposition in bonds (i.e., borrowing or lending). To find the replicating portfolio, we needto find a combination of stock and bonds such that the portfolio mimics the option.To be specific, we wish to find a portfolio consisting of *Δ* shares of stock and adollar amount *B* in lending, such that the portfolio imitates the option whether the stock rises or falls.

We can write the stock price as *Su* when the stock goes up and as *Sd* when theprice goes down and represent the stock price tree as follows:



In this tree *u*(the up factor) is interpreted as one plus the rate of capital gain on the stock if it goes up, and *d*(the down factor) is one plus the rate of capital loss if it goes down.

Let C_u, and C_d represent the value of the option when the stock goes up or down, respectively. The tree for the stock implies a corresponding tree for the value of theoption:



If the length of a period is *T*, the interest factor per period is e^{rT}. The problem is to solvefor Δ and *B* such that our portfolio of Δ shares and *B* in lending duplicates the optionpayoff. The value of the replicating portfolio at time *T*, with stock price S_T, is

$$\Delta S_T + e^{rT} B$$

And a successful replicating portfolio will satisfy

$$\Delta Su + e^{rT} B = C_u$$
$$\Delta Sd + e^{rT} B = C_d$$

Solving for Δ and *B* gives

$$\Delta = \frac{C_u - C_d}{S(u-d)}$$

$$B = e^{rT} \frac{uC_d - dC_u}{u-d}$$

Thus, the cost of the option is Δ*S* + *B*

$$C = \Delta S + B = e^{-rT} \left(C_u \frac{e^{rT} - d}{u-d} + C_d \frac{u - e^{rT}}{u-d} \right)$$

As a fact

$$\frac{e^{rT} - d}{u-d} + \frac{u - e^{rT}}{u-d} = 1$$

So, by assigning

$$p^* = \frac{e^{rT} - d}{u-d} \text{ and } q^* = \frac{u - e^{rT}}{u-d} = 1 - p^*$$

We can rewrite option pricing formula using assigned probabilities as:

$$C = \Delta S + B = e^{-rT}(p^*C_u + q^*C_d)$$

Thus, the option price is the present value of the expected value of option next period using assigned probabilities *p** and *q**.

Arbitraging a Mispriced Option

The following examples illustrate that if the option price is anything other than $\Delta S + B$, arbitrage is possible.

The option is overpriced

Suppose that the market price for the option is $4, instead of $3.26.If we simultaneously sell the actual option and buy thesynthetic, the initial cash flow is

$$\underbrace{\$4}_{Receive\ option\ premium} - \underbrace{0.7692 \times \$50}_{Cost\ of\ shares} + \underbrace{\$35.20}_{Borrowing} = \$0.74$$

We earn $0.74, the amount by which the option is mispriced.Now we verify that there is no risk at expiration. We have

	Stock Price in 1 Year (S_1)	
	$48.5	$55
Written call	0	-5
0.77 purchased shares	37.31	42.31
Repay loan on 35.20	-37.31	-37.31
Total Payoff	**0**	**0**

By hedging the written option, we eliminate risk.

The option is underpriced

Now suppose that the market price of the option is $3.We can hedge by selling a syntheticoption. We accomplish this by reversing the position for a synthetic purchased call: Weshort 0.77 shares and invest $35.20 of the proceeds in Treasury bills. The cost of this is

$$\underbrace{-\$3}_{\text{Option premium}} + \underbrace{0.7692 \times \$50}_{\text{Short}-\text{sale proceeds}} - \underbrace{\$35.20}_{\text{Invest in T}-\text{bills}} = \$0.26$$

At expiration we have

	Stock Price in 1 Year (S_1)	
	\$48.5	\$55
Purchased call	0	+5
0.77 short-sold shares	-37.31	-42.31
Sell T-bill of 35.20	+37.31	+37.31
Total Payoff	**0**	**0**

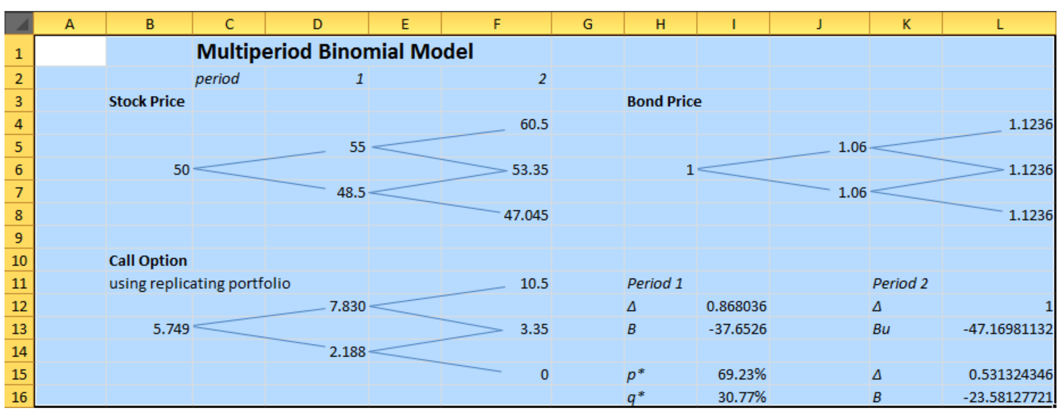
We have earned \$0.26, the amount by which the option was mispriced and hedged the risk associated with buying the option.

Multiperiod Binomial Model

The binomial model can easily be extended to more than two periods. Consider, for example, a three-period binomialmodel that has the following characteristics:

- In each period the stock price goes up by u = 1.1 or down by d = 0.97 from what it was in the previous period.
- In each period the interest rate is 6 percent.

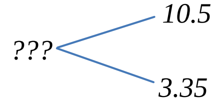
We can now use these characteristics to price a call option written on the stock after two periods. As before, we assumethat the stock price is \$50 initially and that the call exercise price K is 50 after two periods. These assumptions givethe following picture:



How was the call option price of 5.749 determined? To make this determination, we go backward, starting at period2:

At date 2: At the end of two periods the stock price is either \$60.50 (corresponding to two "up" movements in theprice), \$53.35 (one "up" and one "down" movement), or \$47.05 (two "down" movements in the price). Given the exercise price of K = 50, therefore, the terminal option payoff in period 2 is either \$10.50, \$3.35, or \$0.

At date 1: At date 1, there are two possibilities: The first is that we have reached an "up" state, in which case thecurrent stock price is \$55 and the option will pay off \$10.50 or \$3.35 in the next period:

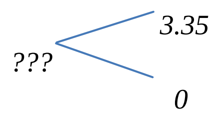


Using either assigned probabilities $p*$ and $q*$or replicating portfolio with Δ shares and B bonds, the option price at “up” state is:

$$period\ 1\ option\ price\ at\ "up"\ state = \frac{1}{(1+6\%)}(10.5 \times 69.23\% + 3.35 \times 30.77\%)$$
$$= 7.83$$

$$period\ 1\ option\ price\ at\ "up"\ state = 1 \times 55 + 47.17 = 7.83$$

The second possibility is that we are in the "down" state of period 1:

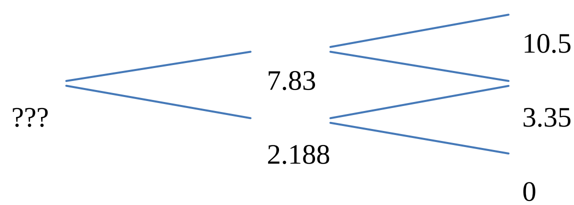


Similarly:

$$period\ 1\ option\ price\ at\ \mathrm{down}state = \frac{1}{(1+6\%)}(3.35 \times 69.23\% + 0 \times 30.77\%) = 7.83$$

$$period\ 1\ option\ price\ at\ "down"\ state = 0.53132 \times 48.5 - 23.58127 = 2.188$$

At date 0: Going backwards in this way, we've now filled in the following picture:



Thus at period 0 the buyer of an option owns a security that will be worth \$7.83 if the underlying stock has an "up"movement in its return and that will be worth \$2.188 if the stock has a "down" movement in its return. We can again use the expected value or replicating portfolio principle to value this option:

$$period\ 0\ option\ price = \frac{1}{(1+6\%)}(7.83 \times 69.23\% + 2.188 \times 30.77\%) = 7.83$$

$$period\ 0\ option\ price = 0.86803 \times 50 - 37.6526 = 5.749$$

It is clear that the logic of the **example** can be extended tomany periods.

Pricing American Options Using the Binomial Pricing Model

An American option can be exercised at or prior to the maturity date of the option. This provides the practical need for binomial tree methods which easily accommodate possibleearly exercise.The binomial tree method provides the value of the American option by comparing thevalue of the European version of the option with the value of intermediate exercise foreach node in the tree.

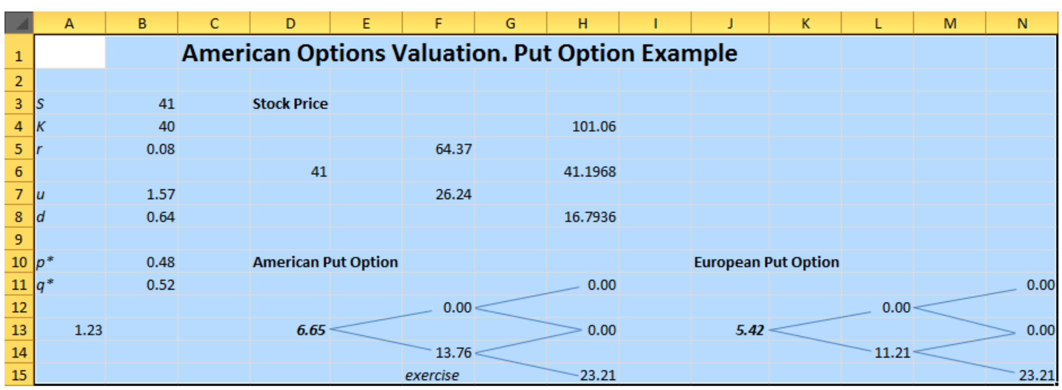
The value of the option if it is left"alive" (i.e., unexercised, often called "option to wait") is given by the value of holding it for another period. The value of the option if it is exercised is given by max(*O*, *S* - *K*) if it is a calland max(*O*, *K* - *S*) if it is a put.Thus, for an American option, the value at a node is given by:

$$\max\left(exercise\ value, unexercised\ value\right)$$

As an example, we will consider 2-period binomial tree for pricing an American putoption that assumes: stock price, S = \$41, exercise price, K = 40, "up" factor, u = 1.57, "down" factor, d = 0.64, risk-free rate, r = 0.08 and time to expiration, T = 2 years implying 1 year binomial periods. For an American put, the value of the option, P at each node is given by:

$$\mathrm{P} = \max\left(\mathrm{K} - \mathrm{S}, \mathrm{e}^{-\mathrm{r}}\left(\mathrm{p}^{*}\mathrm{P}_{\mathrm{u}} + \mathrm{q}^{*}\mathrm{P}_{\mathrm{d}}\right)\right)$$

Below is the binomial tree for American put compared to European put under the same assumptions:



As expected American option is more expensive than European and this difference occurs due to early exercise option value. In the above example, early exercise occurs when stock price decreases to 26.14 in the next period. At that node, *Sd* value of American put option is 13.86 compared to European put option value of 11.22; this makes the difference in prices of American and European put of 1.23 (6.67 – 5.4).

Financial Modeling, Second Edition, Simon Benninga; MIT Press 2000, pp. 210-220

Derivatives Markets, Second Edition, Robert L. McDonald; Pearson Education Inc. 2006, pp. 313-321

Advanced Modeling in Finance using Excel and VBA, Mary Jackson and Mike Staunton;John Wiley & Sons, Ltd 2001, pp. 167-184

18. THE BLACK-SCHOLES MODEL

In a famous paper published in 1973, Fisher Black and Myron Scholes proved a formula for pricing European call and put options on non-dividend-paying stocks. Their model is probably the most famous model of modern finance. The Black-Scholes formula is relatively easy to use, and it is often an adequate approximation to the price of more complicated options.

Here we'll make no pretense at a full-blown development of the model; this would require knowledge of stochastic processes and a not-inconsiderable mathematical investment. Instead, we shall describe the mechanics of the model and show how to implement it in Excel. The Black-Scholes formula coincides with the binominal option-pricing model formula when:

a. The length of a typical period → 0,
b. The "up" and the "down" moves in the binominal model converge to a lognormal price processes, and
c. The term structure of interest rates is flat.

The Black-Scholes Formula

Consider a stock whose price is lognormally distributed. The Black-Scholes model uses the following formula to pricecalls on the stock:

$$C = SN(d_1) - Ke^{-rT}N(d_2)$$

Where

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Here *C* denotes the price of a call, *S* is the price of the underlying stock, *K* is the exercise price of the call, T is thecall's time to exercise, *r* is the interest rate, and σ is the standard deviation of the logarithm of the stock's return. *N( )*denotes a value of the standard normal distribution. It is assumed that the stock will pay no dividends before date *T*.By the put-call parity, a put with the same exercise date *T* and exercise price *K* written onthe same stock will have price $P = C - S + X\,e^{-rT}$. Substituting for *C* in this equation and doing some algebra givesthe Black-Scholes put-pricing formula:

$$P = Ke^{-rT}N(-d_2) - SN(-d_1)$$

Implementing the Black-Scholes Formulas in a Spreadsheet

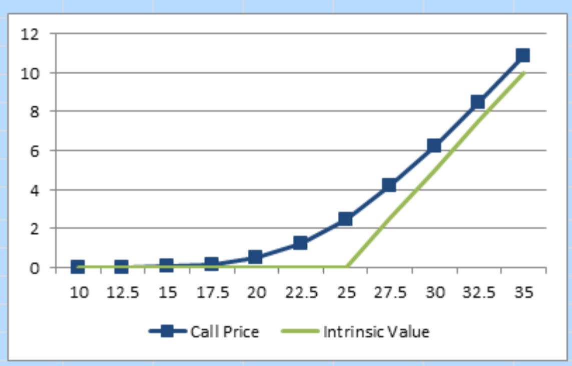
The Black-Scholes formulas for call and put pricing are easily implemented in a spreadsheet. The following exampleshows how to calculate the price of a call option written on a stock whose current $S = 25$, when the exercise price $K = 25$, the annualized interest rate r = 6 percent, and σ = 30 percent. The option has $T = 0.5$ years to exercise. Notethat all three of the parameters, T, r, and σ are assumed to be in annual terms.

	A	B	C	D
1		**Black-Scholes Option-Pricing Formula**		
2				
3	*S*	25		current stock price
4	*K*	25		exercise price
5	*r*	0.06		risk-free rate of interest
6	*T*	0.5		time to maturity of option (in years)
7	*σ*	0.3		stock volatility
8				
9	*d1*	0.2475	<=	(LN(S/K) + (r + 0.5*σ^2)*T)/(σ*SQRT(T))
10	*d2*	0.0354	<=	d1 - σ*SQRT(T)
11				
12	*N (d1)*	0.597734	<=	normsdist(d1)
13	*N (d2)*	0.514102	<=	normsdist(d2)
14				
15	***Call Price***	2.47	<=	S*N(d1)-K*EXP(-r*T)*N(d2)
16	***Put Price***	1.73	<=	C - S + K*EXP(-r*T); by Put-Call Parity
17		1.73	<=	K*EXP(-r*T)*N(-d2)-S*N(-d1)

Note that we have calculated the put price twice: Once by using put-call parity, the second time by the direct Black – Scholes formula.

We can use this spreadsheet to do the usual sensitivity analysis. For example, the following Data/Table gives—as the stock price S varies—the Black-Scholes value of the call compared to its intrinsic value[max (S − K, 0)].

	I	J	K
1			
2	***Stock Price***	***Call Price***	***Intrinsic Value***
3			
4	10	0.0000105	0
5	12.5	0.0008936	0
6	15	0.0162566	0
7	17.5	0.1163958	0
8	20	0.4587518	0
9	22.5	1.2171838	0
10	25	2.4706668	0
11	27.5	4.1742808	2.5
12	30	6.2131674	5
13	32.5	8.4670675	7.5
14	35	10.8440840	10
15			
22			



Option Greeks

Option Greeks are formulas that express the change in the option price when an inputto the formula changes, taking as fixed all the other inputs.Specifically, the Greeks are mathematical derivatives of the option price formula with respect to theinputs.

The Black–Scholes formula has as its inputs the current share price *S*, the interest rate*r*, the option life, and the volatility σ amongst other factors. One way to quantify theimpact of changes in the inputs on the option value is to calculate the so-called option'greeks' or hedge parameters. The most commonly calculated hedge parameters are thefirst-order derivatives: delta (Δ) (for change in share price), rho (ρ) (for change in interest rate),theta (Θ) (for change in option life) and vega (for change in volatility). The second-orderderivative with respect to share price, called gamma (Γ), is also calculated. Apart from theta,the hedge parameters are represented by straightforward formulas. The Black–Scholespartial differential equation links theta with the option value, its delta and its gamma.

One important use ofGreek measures is to assess risk exposure. For example, a market-making bank with a portfolio of options would want to understand its exposure to stock price changes, interest rates, volatility, etc. A portfolio manager wants to know what happens to thevalue of a portfolio of stock index options if there is a change in the level of the stockindex. An options investor would like to know how interest rate changes and volatility changes affect profit and loss.

Keep in mind that the Greek measures by assumption change only one input at atime. In real life, we would expect interest rates and stock prices, for example, to change together. The Greeks answer the question, what happens when one and only one input changes.

Delta (Δ)

Delta measures the change in the option price for a $1 change in the stock price:

$$Call\ delta\ = \frac{\partial C(S,K,\sigma,r,T-t,\delta)}{\partial S} = e^{-\delta(T-t)}N(d_1)$$

$$Put\ delta\ = \frac{\partial P(S,K,\sigma,r,T-t,\delta)}{\partial S} = -e^{-\delta(T-t)}N(-d_1)$$

Gamma (Γ)

Gamma measures the change in delta when the stock price changes:

$$Call\ gamma = \frac{\partial^2 C(S,K,\sigma,r,T-t,\delta)}{\partial^2 S^2} = \frac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}}$$

$$Put\ gamma = \frac{\partial^2 P(S,K,\sigma,r,T-t,\delta)}{\partial^2 S^2} = Call\ Gamma$$

The second equation follows from put-call parity.

Theta (Θ)

Theta measures the change in the option price with respect to calendar time (t), holdingfixed time to expiration (T):

$$Call\ theta = \frac{\partial C(S, K, \sigma, r, T-t, \delta)}{\partial t}$$

$$= \delta S e^{-\delta(T-t)} N(d_1) - rKe^{-r(T-t)} N(d_2) - \frac{Ke^{-r(T-t)} N'(d_2)\sigma}{2\sqrt{T-t}}$$

$$Call\ theta = \frac{\partial P(S, K, \sigma, r, T-t, \delta)}{\partial t} = Call\ theta + rKe^{-r(T-t)} - \delta S e^{-\delta(T-t)}$$

If time to expiration is measured in years, theta will be the annualized change in theoption value. To obtain a per-day theta, divide by 365.

Vega

Vega measures the change in the option price when volatility changes. Some writers also use the terms lambda or kappa to refer to this measure:

$$Call\ vega = \frac{\partial C(S, K, \sigma, r, T-t, \delta)}{\partial \sigma} = Se^{-\delta(T-t)} N'(d_1)\sqrt{T-t}$$

$$Put\ vega = \frac{\partial P(S, K, \sigma, r, T-t, \delta)}{\partial \sigma} = Call\ Vega$$

It is common to report Vega as the change in the option price per percentage point change in the volatility. This requires dividing the Vega formula above by 100.

Rho (ρ)

Rho is the partial derivative of the option price with respect to the interest rate:

$$Call\ rho = \frac{\partial C(S, K, \sigma, r, T-t, \delta)}{\partial r} = (T-t)Ke^{-r(T-t)} N(d_2)$$

$$Put\ rho = \frac{\partial P(S, K, \sigma, r, T-t, \delta)}{\partial r} = -(T-t)Ke^{-r(T-t)} N(-d_2)$$

These expressions for rho assume a change in r of 1.0. We are typically interested in evaluating the effect of a change of 0.01 (100 basis points) or 0.0001 (1 basis point). Toreport rho as a change per percentage point inthe interest rate, divide this measure by100. To interpret it as a change per basis point, divide by 10,000.

In above example, where S = 25, K= 25, r = 0.06, and σ = 0.30 and T = 0.5, we get the following greeks:

	I	J	K
16		***Call***	***Put***
17	delta	0.5977345	-0.4022655
18	gamma	0.0729564	0.0729564
19	rho	0.0623635	-0.0589422
20	theta	-0.0076719	-0.0036838
21	vega	0.0683966	0.0683966

\Note that *N'(x)* is:

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Financial Modeling, Second Edition, Simon Benninga; MIT Press 2000, pp. 247-257
Derivatives Markets, Second Edition, Robert L. McDonald; Pearson Education Inc. 2006, pp. 410-412

19. REAL OPTIONS

The standard net present values (NPV) analysis of capital budgeting values a project by discounting its expectedcash flows at a risk-adjusted cost of capital. This technique is by far the most widely used technique for evaluatingcapital projects. However, standard NPV analysis doesnot take account of the flexibility inherent in the capital budgeting process. Part of the complexity of the capitalbudgeting process is that we can change our decision dynamically, depending on the circumstances.

Many of the most important decisions that firms make concern *real assets*, a term that broadly encompasses factories,mines, office buildings, research and development, and other nonfinancial firm assets. We will see that it is possible to analyze investment and operating decisions for real assets using pricing models developed for financial options.To illustrate how it can be possible to evaluate an investment decision as an option,consider a firm that is deciding whether or not to build a factory. Compare the followingtwo descriptions:

- A *call option* is the right to pay a *strike price* to receive the present value of a stream of future cash flows (represented by the *price of underlying asset*).
- An *investment project* is the right to pay an *investment cost* to receive the present value of a stream of future cash flows (represented by the *present value of the project*).

So we have:

Investment Project		**Call Option**
Investment Cost	=	Strike Price
Present Value of The Project	=	Price of Underlying Asset

This comparison suggests that we can view any investment project as a call option, with the investment cost equal to the strike price and the present value of cash flows equalto the asset price. The exploitation of this and other analogies between real investmentprojects and financial options has come to be called **real options**, which we define asthe application of derivatives theory to the operation and valuation of real investment projects. Note the phrase "operation *and* valuation." We will see in this chapter that you cannot value a real asset without also understanding how you will operate it. Wehave encountered this link before: You cannot value any option without understandingwhen you will exercise it.

Option to Choose

Suppose a large manufacturing firm decides to hedge itself through the use of strategic options. Specifically it has the option to choose among threestrategies: expanding its current manufacturing operations, contracting its manufacturing operations, or completely abandoning its business unit at any time within the next five years.

Suppose the firm has a current operatingstructure whose static valuation of future profitability using a discounted cashflow model (that is, the present value of the future cash flows discounted at an appropriate market risk-adjusted discount rate) is found to be \$100 million. Possible strategic decisions that the firm can choose in the future can be summarized as the following real options:

- *Option to Expand:* The expansion option will increase the firm's operations by 30 percent with a \$20 million implementation cost.
- *Option to Contract:* The firm has the option to contract 10 percent of its current operations at any time over the next five years, thereby creating an additional \$25 million in savings after this contraction.
- *Option to Abandon:* By abandoning its operations, the firm can sell its business for \$100 million.

The most commonly used model for valuing real options is binomial lattice valuation constructed by Cox-Ross-Rubinstein method. Suppose we come up with the following parameters needed for construction of binomial tree:

S_0 – current price of underlying asset, i.e. current value of the firm, \$100 million
σ – volatility of the logarithmic returns on the projected future cash flows, 15%
r – risk free rate for the next five years, 5%
T – expiration of the options, 5 years
δt – the length of one binomial step, we will use $\delta t = 1$ and have a total of 5 binomial steps

Besides, the model requires computation of additional parameters – up factor, u, down factor, d, and the risk-neutral probability, p^*. According to Cox-Ross-Rubinstein:

$$u = e^{+\sigma\sqrt{t}} = 1.1618 \text{ and } d = e^{-\sigma\sqrt{t}} = \frac{1}{u} = 0.8607$$

$$p^* = \frac{e^{rt} - d}{u - d} = 0.633$$

Note should be made how to calculate the volatility of the logarithmic returns on the projected future cash flows ("lognormal sigma"). Most commonly used methods for calculating lognormal sigma are Logarithmic Stock Price Returns Approach and Logarithmic Present Value Returns Approach.

Logarithmic Stock Price Returns Approachcalculates the volatility using historical stock prices and their corresponding logarithmicreturns. Starting with a series of past stock prices, convert them into relative returns. Then take the natural logarithms of these relative returns. And the standard deviation of thesenatural logarithm returns is the volatility of the cash flow series used in areal options analysis. This method apparently cannot be used for firms without stocks outstanding.

Logarithmic Present Value Returns Approach is used for start-ups and firms without outstanding stock. This method can be divided into 4 steps:

- calculating present value of the projected future cash flows at times $t = 0$ and $t = 1$, i.e. S_0 and S_1
- taking natural logarithm of S_1 / S_0, i.e. $ln(S_1/S_0)$
- simulating the values of $ln(S_1/S_0)$ using Monte Carlo Simulation
- taking standard deviation of simulated values

Thus, according to this method "lognormal sigma" used in the binomial tree will be the standard deviation of simulated logarithmic return on the present values of the projected future cash flows from time *0* to *1*. In the spreadsheet below is an example illustrating the calculation of "lognormal sigma" by Logarithmic Present Value Returns Approach using 10 simulations (number of simulations is taken only for illustrative purposes):

	A	B	C	D	E	F	G	H	I	J	K	L	M
1			Calculating Lognormal Sigma										
2	Cash Flows Forecast												
3	μ	15	<=	*average cash flow per year*									
4	σ	5	<=	*annual standard deviation of cash flow*									
5	*r*	0.05											
6			*t = 0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
7				8.89	16.31	17.72	25.55	19.22	7.55	14.34	10.15	14.59	15.25
8	*S0*	$116.11											
9	*S1*	$113.03		***lognormal sigma***									
10	*ln(S1/S0)*	-0.02689		15.00%									
11	1	-0.00082											
12	2	-0.08004											
13	3	-0.05499											
14	4	-0.11118											
15	5	-0.04617											
16	6	-0.15375											
17	7	-0.17651											
18	8	-0.10485											
19	9	-0.05936											
20	10	-0.0926											

Having estimated all required parameters, we can easily model the evolution of the value of the firm (underlying asset) over time using binomial tree. As a result (spreadsheet is shown below), binomial tree with $\delta t = 1$ and $T = 5$ shows the possible values of the firm per year over the next 5 years. Knowing possible combinations of firm's value over time, we can also calculate the value of strategic options at all nodes of the binomial tree using backward induction technique. Let's again define each of this strategic option more deeply.

Option to Expand

Expansion option implies that the firm has growth opportunities in the future like extending its current operations, building new branches, entering new markets, acquiring its competitor, etc. In our example, the firm has future possible opportunity to expand its business by 30% with the cost of 20 million for this expansion.

Thus, the firm's decision to expand will increase its current value by 30% and require additional investment of 20 million. Value of option to expand if exercised is:

$$
\begin{aligned}
option\ to\ expand &= (1 + expansion) \times value\ of\ the\ firm - expansion\ cost \\
&= 1.3 \times S - 20
\end{aligned}
$$

Option to Contract

Contraction option is the strategic decision to contract firm's current operations in exchange for the fixed payment. In this way, firm suspends some part of its business but receives guaranteed savings from contracting party. An example of contraction option would be renting part of the firms building to another company and receiving rent payments.

In our example, the firm has option to contract 10% of its current operations and receive 25 million in savings. Option to contract will be valued as:

$$
\begin{aligned}
option\ to\ contract &= (1 - contraction) \times value\ of\ the\ firm + savings \\
&= 0.9 \times S + 25
\end{aligned}
$$

Option to Abandon

Abandonment option is the right to exit the business and sell its intellectual property, patents or licenses, etc. for the fixed payment. In other words, it is an option to abandon firm's current operations and receive whatever possible in its sale, in worst case, even the salvage value might more than continuing firm's current business.

We assume that the firm can abandon its business at any time over the next 5 years in exchange for 100 million. Thus,

$$option\ to\ abandon = 100$$

Eventually, the *option to choose* is the greatest of all options that firm can pursue at a point in time. Here is a general implication of chooser option:

A chooser option implies that management has the flexibility to choose among several strategies, including the option to expand, abandon, switch, contract, and so forth.

Real options valuation is similar to financial options valuation, the difference is in the payoff formulas. So, following binomial option pricing, we first model the evolution of underlying asset and then value options at all possible nodes of binomial tree representing the future possible values of underlying. As mentioned above, we need to use backward induction to value options on the binomial tree. This implies that:

- At the terminal node: chooser option is the greatest of expansion, contraction, abandonment options – *max (expand, contract, abandon)*
- At intermediate nodes: chooser option acquires continuation option as well, often called option to wait, and becomes - *max (expand, contract, abandon, continue)*

From binomial option pricing model, option to wait or continuation option is valued as the present value of the weighted average of option prices at the next node with weights being risk-neutral probability of up and down moves. With C^+ being the option value at *up* state and C^- being the option value at *down* state, we have:

$$\text{option to wait} = e^{-r\delta t}(p^* \times C^+ + (1 - p^*)C^-)$$

The following spreadsheet shows both the binomial evolution of firm's value and the option valuation tree for chooser option:

	A	B	C	D	E	F	G	H	I	J	K	L	M
1			**Option to Choose**										
2	Expand = expansion factor x firm value - expansion cost = 1.3 x firm value - 20												
3	Contract = contraction factor x firm value + savings = 0.9 x firm value + 25												
4	Abandon = Salvage Value = 100 receipt												
5	Lattice Evolution of the Underlying							Option Valuation Lattice					
6								*Max among Abandonment, Expanding,Contracting or Continue*					
7	Given:	*S* =	100										
8		*σ* =	0.15				*Cell M15:*	Abandonment Salvage Value = $100M					
9		*T* =	5					Expanding = 1.3 (211.7) - 20 = $255.21					
10		*rf* =	0.05					Contracting = 0.9 (211.7) + 25 = $215.53					
11		*δt* =	1										
12							*Cell J21:*	Keeping Option Open = [p*(185.8)+(1-p*)(134.4)]exp(-rf*dt) = $158.8					
13	*u* =	1.1618		*d* =	0.8607			Abandonment Salvage Value = $100M					
14								Expanding = 1.3 (135) - 20 = $155.48					
15		*p** =	0.633			211.7		Contracting = 0.9 (135) + 25 = $146.49					255.2
16						Su^5							expand
17					182.2							217.9	
18					Su^4							open	
19				156.8		156.8					185.8		183.9
20				Su^3		Sdu^4					open		expand
21			135.0		135.0		*Option to Choose*	***19.0291***		158.8		156.5	
22			Su^2		Sdu^3					open		open	
23		116.2		116.2		116.2			136.5		134.3		131.0
24		Su		Sdu^2		Su^3d^2			open		open		expand
25	100.0		100.0		100.0			119.0291		117.2		115.0	
26	S		Sud		Su^2d^2			open		open		contract	
27		86.1		86.1		86.1			105.5		104.2		102.5
28		Sd		Sud^2		Su^2d^3			open		open		contract
29			74.1		74.1					100.0		100.0	
30			Sd^2		Sud^3					abandon		abandon	
31				63.8		63.8					100.0		100.0
32				Sd^3		Sud^4					abandon		abandon
33					54.9							100.0	
34					Sd^4							abandon	
35						47.2							100.0
36						Sd^5							abandon

Using five binomial steps,the real options valueis calculated as $19.03 million. Let's consider several nodes of real option valuation tree.

In spreadsheet above, we see that the sample *Cell M15* reveals a value of $255.2, which can be obtained through the value maximization of expansion, contraction, abandonment, and continuation. At the end of five years, the firm has the option to choose how it wishes to continue its existing operations through these options. Obviously, management will choose the strategy that maximizes profitability. The value of abandoning the firm's business unit is $100 million. The value of expansion is 1.3($211.7) – $20 = $255.2 million. The value of contracting 10 percent of its operations is equivalent to 90 percent of its existing operations plus the $25 million in savings. Hence, the value of contracting the firm's operations is 0.9($211.7) + $25 = $215.5 million. The value of continuing with existing business operations can be found in lattice evolution of the underlying at the same node (S_0u^5), which is \$211.7 million. The profit-maximizing decision is to expand the firm's current level of operations at \$255.2 million on that node (*Cell M15*).

This is intuitive because if the underlying asset value of pursuing existing business operations is such that it is very high based on current marketdemand (*Cell M15*), then it is wise to expand the firm's current levels of operation.Otherwise, if circumstances force the value of the firm's operations down to such a low level as specified by node Su^2d^3(*Cell M27*), then it is more optimal tocontract the existing business by 10 percent. At any time below level K, forinstance, at node M, it is better to abandon the business unit all together.

Moving on to the intermediate nodes, we see that node Su^2 (*Cell J21*) is calculatedas \$158.8 million. At this particular node, the firm again has four options:to expand, contract, abandon its operations, or not execute anything, thuskeeping these options open for the future. The value of contracting at that node is 0.9(\$134.9) + \$25 = \$146.49 million. The value of abandoningthe business unit is \$100.0 million. The value of expanding is1.3(\$134.99) – \$20 = \$155.48 million. The value of continuing is simply the discounted weighted average of potential future option values using therisk-neutral probability. As the risk adjustment is performed on the probabilitiesof future option cash flows, the discounting can be done using the risk – free rate. That is, for the value of keeping the option alive and open,we have $e^{-r\delta t}\,(p^* \times \$185.8 + (1-p^*)\$134.3)$ =\$158.8 million,which is the maximum value. Using thisbackward induction technique, the lattice is calculated back to the startingpoint to obtain the value of \$119.03 million. As the present value of the underlying is \$100 million, the real options value is \$19.03 million.

Financial Modeling, Second Edition, Simon Benninga; MIT Press 2000, pp. 276-285

Derivatives Markets, Second Edition, Robert L. McDonald; Pearson Education Inc. 2006, pp. 547-558

Real Options Analysis, Johnathan Mun; John Wiley & Sons, Inc. 2002, pp. 171-181

20. DURATION

Duration is a measure of the sensitivity of the price of a bond to changes in the interest rate at which the bond is discounted. It is widely used as a risk measure for bonds (i.e., the higher a bond's duration, the more risky it is). We will consider a basic duration measure—Macauley duration—which is defined for the case when the termstructure is flat.

Introduction

Consider a bond with payments C_t, where t = 1, ...,N. Ordinarily, the first $N-1$ payments will be interest payments, and C_N will be the sum of the repayment of principal and the last interest payment. If the term structure is flat and the discount rate for all of the payments is r, then the bond's market price today will be

$$P = \sum_{t=1}^{N} \frac{C_t}{(1+r)^t}$$

The Macauley duration measure is defined as:

$$D = \frac{1}{P} \sum_{t=1}^{N} \frac{tC_t}{(1+r)^t}$$

Before considering the meaning of this formula, we will show how to calculate theduration in Excel.

Two Examples

Consider two par bonds. Bond A has just been issued. Its face value is $1,000, it bears the current market interest rateof 7 percent, and it will mature in 10 years. Bond B was issued five years ago, when interest rates were higher. Thisbond has $1,000 face value and bears a 13 percent coupon rate. When issued, this bond had a 15-year maturity, soits remaining maturity is 10 years. Since the current market rate of interest is 7 percent, bond B's market price is given by:

$$\$1{,}421.41 = \sum_{t=1}^{10} \frac{\$130}{(1.07)^t} + \frac{\$1{,}000}{(1.07)^{10}}$$

It is worthwhile calculating the duration of each of the two bonds the long way. We set up a table inExcel:

	A	B	C	D	E	F	G
1			**Basic Duration Calculation**				
2	**YTM**	**7%**					
3	***Bond A***				***Bond B***		
4	**Year**	**Ct**	**(1/*P*)*(*t***Ct*)/(1+ytm)^*t***		**Year**	**Ct**	**(1/*P*)*(*t***Ct*)/(1+ytm)^*t***
5	1	70	0.0654		1	130	0.0855
6	2	70	0.1223		2	130	0.1598
7	3	70	0.1714		3	130	0.2240
8	4	70	0.2136		4	130	0.2791
9	5	70	0.2495		5	130	0.3260
10	6	70	0.2799		6	130	0.3657
11	7	70	0.3051		7	130	0.3987
12	8	70	0.3259		8	130	0.4258
13	9	70	0.3427		9	130	0.4477
14	10	1070	5.4393		10	1130	4.0413
15		**Bond Price**	**Duration**			**Bond Price**	**Duration**
16		$1,000.00	7.5152			$1,421.41	6.7535
17							
18			NPV = (YTM,B5:B14)			SUM = (G5:G14)	

As might be expected, the duration of bond *A* is longer than that of bond *B*, since the average payoff of bond *A*takeslonger than that of bond *B*. To look at this relationship another way, the net present value of bond *A*'s first-year payoff($70) represents 6.54 percent of the bond's price, whereas the net present value of bond *B*'s first-year payoff ($130) is 8.55 percent of its price. The figures for the second-year payoffs are 6.11 percent and 7.99 percent, respectively.(For the second-year figures, you have to divide the appropriate line of the preceding table by 2, since in the durationformula each payoff is weighted by the period in which it is received)

Using an Excel Formula

Excel has two duration formulas, *Duration ( )* and *MDuration()*. MDuration –Modified Macauley duration by Excel – is defined as:

$$MDuration = \frac{Duration}{\left(1 + \frac{yield\ \ to\ maturity}{number\ \ of\ coupon\ \ payments\ \ per\ \ year}\right)}$$

Both formulas have the same syntax; for example, for *Duration( )* the syntax is as follows:

Duration (settlement, maturity, coupon, yield, frequency, basis),

where

- **settlement** is the settlement date (i.e., the purchase date) of the bond
- **maturity** is the bond's maturity date
- **coupon** is the bond's coupon
- **yield** is the bond's yield to maturity
- **frequency** is the number of coupon payments per year

- **basis** is the "day count basis" (i.e., the number of days in a year). This is a code between 0 and 4:

0 or omitted	US (NASD) 30 / 360
1	Actual / Actual
2	Actual / 360
3	Actual / 360
4	European 30/360

The **Duration** formula gives the standard Macauley duration. The **MDuration** formula is modified duration measure that can be used in calculating theprice volatility of the bond.

Both duration formulas may require a bit of trickery to implement because they demand a date serial number for boththe settlement and the maturity. In the preceding spreadsheet picture, the Excel formula is implemented by arbitrarily choosingthe dates. The last parameter of the Excel durationformula, which gives the basis, is optional and could be omitted.

What Does Duration Mean?

In this section we present three different meanings of duration. Each is interesting and important in its own right.

Duration as the Weighted Average of the Bond's Payments

As originally defined by Macauley (1938), duration is a weighted average of the bond's payments. Rewrite theduration formula as follows:

$$D = \frac{1}{P}\sum_{t=1}^{N}\frac{tC_t}{(1+r)^t} = \sum_{t=1}^{N}\left[\frac{C_t/P}{(1+r)^t}\right]\times t$$

Note that the bracketed terms $\left[\frac{C_t/P}{(1+r)^t}\right]$sum to 1. This fact follows from the definition of the bond price; eachof these terms is the proportion of the bond's price represented by the payment at time *t*. In the duration formula, each of the terms $\left[\frac{C_t/P}{(1+r)^t}\right]$ is multiplied by its time of occurrence: Thus *the duration is the time-weighted average of the bond's discounted payments as a proportion of the bond's price*.

Duration as the Bond's Price Elasticity with Respect to Its Discount Rate

Viewing duration another way – as the bond's price elasticity with respect to its discount rate – explains why theduration measure can be used to measure the bond's price volatility; it also shows why duration is often used as arisk measure for bonds. To derive this interpretation, we take the derivative of the bond's price with respect to thecurrent interest rate:

$$\frac{dP}{dr} = \sum_{t=1}^{N} \frac{-tC_t}{(1+r)^{t+1}}$$

A little algebra shows that

$$\frac{dP}{dr} = \sum_{t=1}^{N} \frac{-tC_t}{(1+r)^{t+1}} = -\frac{DP}{(1+r)}$$

This formula transforms into two useful interpretations of duration:

- First, duration can be regarded as the *discount-factor elasticity of the bond price*, where by "discount factor" we mean 1 + *r*:

$$\frac{dP/P}{dr/(1+r)} = \frac{Percent\ Change\ in\ Bond\ Price}{Percent\ Change\ in\ Discount\ Factor} = -D$$

- Second, we can use duration to measure the *price volatility* of a bond by rewriting the previous equation as:

$$\frac{dP}{P} = -D\frac{dr}{1+r}$$

Let's go back to the examples of the previous section. Suppose that the market interest rate rises by 10 percent,from 7 percent to 7.7 percent. What will happen to the bond prices? The price of bond *A* will be

$$\$952.39 = \sum_{t=1}^{10} \frac{\$70}{(1.077)^t} + \frac{\$1,000}{(1.077)^{10}}$$

A similar calculation shows the price of bond *B* to be

$$\$1360.50 = \sum_{t=1}^{10} \frac{\$130}{(1.077)^t} + \frac{\$1,000}{(1.077)^{10}}$$

As predicted by the price-volatility formula, the changes in the bond prices are approximated by $\Delta P \cong -DP\Delta r/(1+r)$.To see this relationship, work out the numbers for each bond:

	A	B	C	D	E	F	G	H	I	J
1		**Approximating Price Changes Using Duration**								
2										
3		**Actual**								
4	*Bond*	*ΔP*	*D*	*P*	*Δr*	*-DP Δr / (1+r)*				
5	A	-47.61	7.515232	$1,000.00	0.007	-49.17				
6	B	-60.92	6.753539	$1,421.41	0.007	-62.80				
7										
8										
9		(-PV(7.7%,10,130)+1000/(1.077)^10)-(-PV(7%,10,130)+1000/(1.07)^10)								

Note that instead of using the Excel Duration function and multiplying by $\Delta r/(1+r)$, we could have used theMDuration function and multiplied by Δr.

	A	B	C	D	E
1		**Using MDuration**			
2	***Bond A***				
3		*MDuration*	7.0236	<=	MDURATION(DATE(2000,1,1),DATE(2010,1,1),7%,ytm,1)
4		*Δr*	0.007		
5		*Bond Price*	$1,000.00		
6		*-MD P Δr*	-49.17		
7	***Bond B***				
8		*MDuration*	6.3117	<=	MDURATION(DATE(2000,1,1),DATE(2010,1,1),13%,ytm,1)
9		*Δr*	0.007		
10		*Bond Price*	$1,421.41		
11		*-MD P Δr*	-62.80		

Babcock's Formula: Duration as the Convex Combination of Bond Yields

A third interpretation of duration is Babcock's (1985) formula, which shows that duration is a weighted average of twofactors:

$$D = N\left(1 - \frac{y}{r}\right) + \frac{y}{r} PVIF(r, N) \times (1 + r)$$

Where the "current yield" of the bond is

$$y = \frac{Bond\ Coupon}{Bond\ Price}$$

And the present value of an N = period annuity is

$$PVIF(r, N) = \sum_{i=1}^{N} \frac{1}{(1+r)^i}$$

This formula gives two useful insights into the duration measure:

- Duration is a weighted average of the maturity of the bond and of $(1 + r)$ times the PVIF associated with the bond. (Note that the PVIF is given by the Excel formula **PV(*r*, *N*, −1)**)

- In many cases the current yield of the bond, y is not greatly different from its yield to maturity r. In these cases, duration is not very different from $(1 + r)$ PVIF.

Unlike the two previous interpretations, Babcock's formula holds only for the case of a bond with constant couponpayments and single repayment of principal at time N; that is, the formula does not extend to the case where thepayments C_t differ over time.Here's an implementation of Babcock's formula for bond *B*:

Babcock's Formula			
N	10	<=	*Bond Maturity*
r	7%	<=	*Current Market Interest Rate*
C	13%	<=	*Bond Coupon*
Price	1,421.41	<=	-PV(K5,K4,K6*1000)+1000/(1+K5)^K4
y	9.15%	<=	*Current Yield*
PVIF(r,N)	7.02	<=	PV(K5,K4,-1)
D	6.753539	<=	K4*(1-K8/K5)+K8/K5*K9*(1+K5)

Duration Patterns

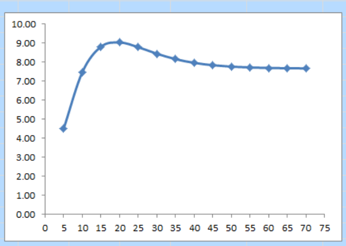
Intuitively we would expect that duration is an increasing function of a bond's maturity and a decreasing function of a bond's coupon. However, as the following examples show, this expectation is not always fulfilled. As the coupon increases, the bond's duration unequivocally decreases. The following tables and graphs (basedon the previous example) show those effects:

Effect of Coupon and Maturity on Duration

Current Date	1/1/2000		
N	21		
Maturity Date	1/1/2021		
YTM	15%		
Coupon	4%		
Face Value	1000		
Duration	9.0110	<=	DURATION(B3,B5,B7,B6,1)

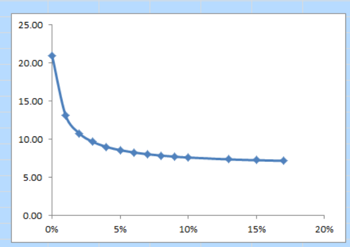
Effect of Maturity on Duration

N	*D*
5	4.5163
10	7.4827
15	8.8148
20	9.0398
25	8.7881
30	8.4461
35	8.1633
40	7.9669
45	7.8421
50	7.7668
55	7.7228
60	7.6977
65	7.6837
70	7.6759



Effect of Coupon on Duration

N	*D*
0%	21.0000
1%	13.1204
2%	10.7865
3%	9.6677
4%	9.0110
5%	8.5792
6%	8.2736
7%	8.0459
8%	7.8697
9%	7.7294
10%	7.6149
13%	7.3707
15%	7.2593
17%	7.1729



Duration of a Bond with Uneven Payments

Consider a bond with N payments, the first of which occurs at time $\alpha < 1$, and the rest of which are evenly spaced. Inthe derivation that follows, we show that the duration of such a bond is given by the sum of two terms:

- The duration of a bond with N payments spaced at even intervals (i.e., the standard duration discussed earlier)
- $\alpha - 1$

The derivation is relatively simple. Denote the payments on the bond by C_{α}, $C_{\alpha+1}$, $C_{\alpha+2}$... $C_{\alpha+N-1}$, where $0 < \alpha < 1$. The price of the bond is given by:

$$P = \sum_{t=1}^{N} \frac{C_{\alpha+t-1}}{(1+r)^{\alpha+t-1}} = (1+r)^{1-\alpha} \sum_{t=1}^{N} \frac{C_{\alpha+t-1}}{(1+r)^{t}}$$

The duration of this bond is given by

$$D = \frac{1}{P} \sum_{t=1}^{N} \frac{(\alpha+t-1)C_{\alpha+t-1}}{(1+r)^{\alpha+t-1}}$$

Here is an example of the calculation of the duration of a bond with uneven periods.

	A	B	C	D	E	F	G	H
1			Duration of Bond with Uneven Periods					
2								
3	*alpha*	0.3	time till first coupon payment (in years)					
4	*N*	5	number of payments					
5	*YTM*	6%						
6	*Coupon*	100						
7	*Face*	1000						
8	*Bond Price*	1,217.14	<= NPV(B5,C11:C15)*(1+B5)^(1-B3)					
9								
10		*Period*	*Payment*					
11		0.3	100	0.0242				
12		1.3	100	0.0990				
13		2.3	100	0.1653				
14		3.3	100	0.2237				
15		4.3	1100	3.0249				
16		Duration		3.5371	<= SUM(D11:D15)			

Calculating the YTM for Uneven Periods

As the preceding discussion shows, the calculation of duration requires us to know the bond's yield to maturity

(YTM); this YTM is just the internal rate of return of the bond's payments and its initial price. Often the YTM is given,but when it is not, we can run into a problem that requires us to make an adjustment to the Excel **IRR** function. Theproblem has to do with unevenly spaced bond payments. This section gives a simple example of this problem andshows how a small trick can solve it.

Consider a bond that currently costs \$1,123 and that pays a coupon of \$89 on January 1 of each year. On January 1,2001, the bond will pay \$1,089, the sum of its annual coupon and its face value. The current date is October 3,1996. The problem in finding the YTM of this bond is that while most of the bond payments are spaced one year apart, there is only 0.2466 of a year until the first coupon payment: 0.2466 = [Date(1997, 1, 1) – Date(1996, 10,3)]/365. Thus we wish to use Excel to solve the following equation:

$$-1{,}123 + \sum_{t=0}^{3} \frac{89}{(1+YTM)^{t+0.2466}} + \frac{1{,}089}{(1+YTM)^{4.2466}} = 0$$

To solve this problem, we can use the Excel function **XIRR**. To use the **XIRR** function, you first have to make sure that the Analysis ToolPak is loaded into Excel.

	A	B	C	D
1			Using XIRR to Calculate The IRR with Uneven Payments	
2				
3	*Current Date*	3-Oct-96		
4	*Annual Coupon*	89	Paid January 1 for each of the next five years	
5	*Maturity*	1-Jan-01		
6	*Face Value*	1000		
7	*Price of Bond*	1123		
8				
9	*Time to First Payment*	0.24658		
10				
11		*Date*	*Payment*	
12		3-Oct-96	-1123	
13		1-Jan-97	89	
14		1-Jan-98	89	
15		1-Jan-99	89	
16		1-Jan-00	89	
17		1-Jan-01	1089	
18				
19		**YTM**	7.300%	<= XIRR(C12:C17,B12:B17)
20				

There is also a function **XNPV** for finding the present value of a series of payments paid out at uneven dates.

Nonflat Term Structures and Duration

In a general model of the term structure, payments at time t are discounted by rate r_t, so that the value of a bond isgiven by

$$P = \sum_{t=1}^{N} \frac{C_t}{(1+r_t)^t}$$

The duration measure discussed above assumes either a flat term structure (i.e., $r_t = r$ for all t) or a term structure that shifts in a parallel fashion. When the term structure exhibits parallel shifts, we can write the bond price as

$$P = \sum_{t=1}^{N} \frac{C_t}{(1 + r_t + \Delta t)^t}$$

And then derive a measure of duration by taking the derivative with respect to Δt. A general model of the term structure should explain how the discount rate r_t for time – t payments comes about, and how the rates at time $t$change.

Financial Modeling, Second Edition, Simon Benninga; MIT Press 2000, pp. 303-315

Derivatives Markets, Second Edition, Robert L. McDonald; Pearson Education Inc. 2006, pp. 224-230

Fundamentals of Financial Markets, Volf Frishling; National Australia Bank 2007, pp. 12-15

21. Immunization Strategies

A bond portfolio's value in the future depends on the interest-rate structure prevailing up to and including the date at which the portfolio is liquidated. If a portfolio has the same payoff at some specific future date, no matter what interest-rate structure prevails, then it is said to be immunized. Here we will discuss immunization strategies, which are closely related to the concept of duration discussed earlier in the text. Immunization strategies have been discussed for many concepts of duration, but this chapter is restricted to the simplest duration concept, that of Macauley.

A Basic Simple Model of Immunization

Consider the following situation: A firm has a known future obligation, Q.the discounted value of this obligation is:

$$V_0 = \frac{Q}{(1+r)^N}$$

Suppose that this future obligation is currently hedged by a bond held by the firm. That is, the firm currently holds abond whose value V_B is equal to the discounted value of the future obligation V_0. If P_1, $P_2 \ldots P_M$ is the stream ofanticipated payments made by the bond, then the bond's present value is given by

$$V_B = \sum_{t=1}^{M} \frac{P_t}{(1+r)^t}$$

Now suppose that the underlying interest rate, r, changes to $r + \Delta r$. Using a first-order linear approximation, we findthat the new value of the future obligation is given by

$$V_0 + \Delta V_0 \approx V_0 + \frac{dV_0}{dr}\Delta r = V_0 + \Delta r\left[\frac{-NQ}{(1+r)^{N+1}}\right]$$

However, the new value of the bond is given by

$$V_B + \Delta V_B \approx V_B + \frac{dV_B}{dr}\Delta r = V_B + \Delta r \sum_{t=1}^{N} \frac{-tP}{(1+r)^{t+1}}$$

If these two expressions are equal, a change in r will not affect the hedging properties of the company's portfolio.Setting the expressions equal gives us the condition

$$V_B + \Delta r \sum_{t=1}^{N} \frac{-tP}{(1+r)^{t+1}} = V_0 + \Delta r\left[\frac{-NQ}{(1+r)^{N+1}}\right]$$

Recalling that

$$V_B = V_0 = \frac{Q}{(1+r)^N}$$

We can simplify this expression to get

$$\frac{1}{V_B}\sum_{t=1}^{M}\frac{tP_t}{(1+r)^t} = N$$

That is $D_B = N$, or as duration of single future obligation (i.e., zero-coupon bond) is equal to its maturity we can rewrite it as:

$$D_B = D_Q$$

Where D_B is the duration of the bond,
D_Q is the duration of the obligation.

This statement is worth restating as a formal proposition:Suppose that the term structure of interest rates is always flat (that is, the discount rate for cash flows occurring at all future times is the same) or that the term structure moves up or down in parallel movements. Then a necessary and sufficient condition that the market value of an asset be equal under all changes of the discount rate r to the market value of a future obligation Q is that the duration of the asset equal the duration of the obligation. Here weunderstand the word "equal" to mean equal in the sense of a first-order approximation.An obligation against which an asset of this type is held is said to be immunized.The preceding statement has two critical limitations:

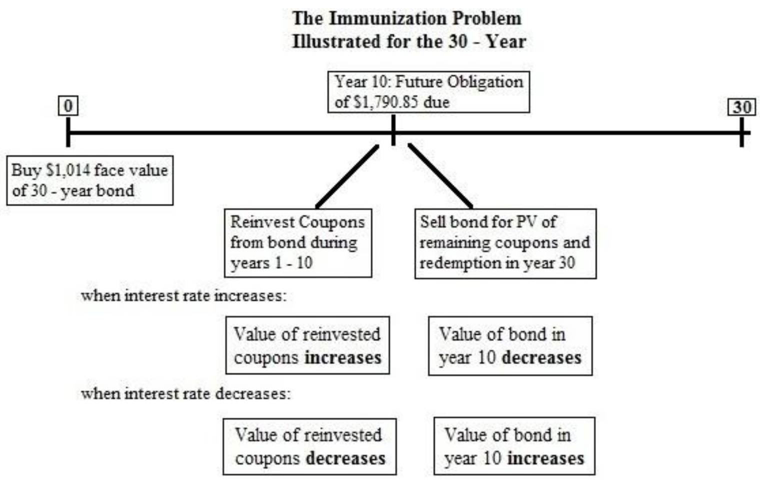
- The immunization discussed applies only to first-order approximations. When we get to a numerical example in the succeeding sections, we shall see that there is a big difference between first-order equality and "true" equality
- We have assumed either that the term structure is flat or that the term structure moves up or down in parallel movements. At best, this assumption might be considered to be a poor approximation to reality

A Numerical Example

In this section we consider a basic numerical immunization example. Suppose you are trying to immunize a 10 – year obligation whose present value is $1,000 (for example, at a current interest rate of 6 percent, its future value would be $1,000 × 1.0610 = $1,790.85). You intend to immunize the obligation by purchasing $1,000 worth of a bond or a combination of bonds. You consider three bonds:

1. Bond 1 has 10 years remaining until maturity, a coupon rate of 6.7 percent, and a face value of $1,000.
2. Bond 2 has 15 years until maturity, a coupon rate of 6.988 percent, and a face value of $1,000.
3. Bond 3 has 30 years until maturity, a coupon rate of 5.9 percent, and a face value of $1,000.

At the existing yield to maturity of 6 percent, the prices of the bonds differ. Bond 1, for example is worth \$1,051.52 $=\sum_{t}^{10} \frac{67}{(1.06)^t} + \frac{1,000}{(1.06)^{10}}$; thus, in order to purchase \$1,000 worth of this bond, you have to purchase \$951 = \$1,000/\$1,051.52 of *face value* of the bond.However, Bond 3 is currently worth \$986.24, so that in order to buy \$1,000 of market value of this bond, you will have to buy \$1,013.96 of face value. If you intend to use this bond to finance a \$1,790.85 obligation 10 years fromnow, following is a schematic of the problem you face.



As we will see, the 30-year bond will exactly finance the future obligation of \$1,790.85 only for the case in which thecurrent market interest rate of 6 percent remains unchanged. Here is a summary of price and duration information for the three bonds:

	A	B	C	D	E	F	G	H	I
1			Basic Immunization Example With Three Bonds						
2									
3	***YTM***	6%							
4									
5		**Bond 1**	**Bond 2**	**Bond 3**					
6	***Coupon Rate***	6.70%	6.988%	5.90%					
7	***Maturity***	10	15	30					
8	***Face Value***	1000	1000	1000					
9									
10	***Bond Price***	\$1,051.52	\$1,095.96	\$986.24					
11	***Face Value equal to \$1,000 of market value***	\$951.00	\$912.44	\$1,013.96					
12									
13	***Duration***	7.6655	10.0000	14.6361					
14									

If the yield to maturity does not change, then you will be able to reinvest each coupon at 6 percent. Thus, bond 2, forexample, will give a terminal wealth at the end of 10 years of

$$\sum_{t=1}^{9} 69.88 \times (1.06)^t + \left[\sum_{t=1}^{5} \frac{69.88}{(1.06)^t} + \frac{1,000}{(1.06)^5}\right] = 921.07 + 1,041.62 = 1,962.69$$

The first term in this expression is the sum of the reinvested coupons. The second term represents the market value of the bond maturing in year 10, when the bond has five more years until maturity. Since we will be buying only $912.44 of face value of this bond, we have, at the end of 10 years, 0.91244 × $1,962.69 = $1,790.85. This is exactly the amount we wanted to have at this date. The results of this calculation forall three bonds, provided there is no change in the yield to maturity, are given in the following table:

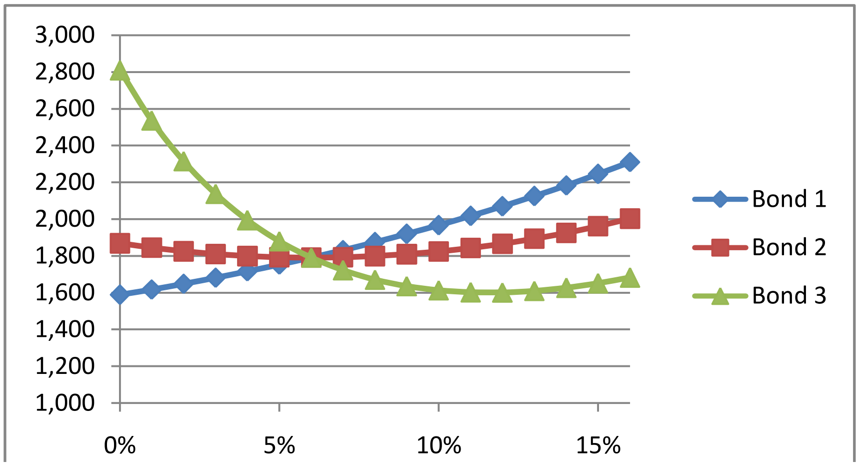
	A	B	C	D	E	F	G	H	I
1			Basic Immunization Example With Three Bonds						
2									
3	***YTM***	6%							
4		**Bond 1**	**Bond 2**	**Bond 3**					
5	***Coupon Rate***	6.70%	6.988%	5.90%					
6	***Maturity***	10	15	30					
7	***Face Value***	1000	1000	1000					
8									
9	***Bond Price***	$1,051.52	$1,095.96	$986.24					
10	***Face Value equal to $1,000 of market value***	$951.00	$912.44	$1,013.96					
11									
12	***Duration***	7.6655	10.0000	14.6361					
13									
14		**Bond 1**	**Bond 2**	**Bond 3**					
15	*Bond Price*	$1,000.00	$1,041.62	$988.53	<= PV($B$3,D7-10,-D6*D8)+D8/(1+$B$3)^(D7-10)				
16	*Reinvested Coupons*	$883.11	$921.07	$777.67	<= FV($B$3,10,-D6*D8)				
17	***Total***	$1,883.11	$1,962.69	$1,766.20					
18									
19	*Percent of face value bought*	95.10%	91.24%	101.40%	<= 1000/D9				
20	***Terminal Wealth***	$1,790.85	$1,790.85	$1,790.85					

The upshot of this table is that purchasing $1,000 of any of the three bonds will provide – 10 years from now – funding for your future obligation of $1,790.85, provided the market interest rate of 6 percent doesn't change.

Now suppose that, immediately after you purchase the bonds, the yield to maturity changes to some new value and stays there. This change will obviously affect the calculation we already did. For example, if the yield falls to 5percent, the table will now look as follows:

	A	B	C	D
22	***New YTM***	5%		
23		**Bond 1**	**Bond 2**	**Bond 3**
24	*Bond Price*	$1,000.00	$1,086.07	$1,112.16
25	*Reinvested Coupons*	$842.72	$878.94	$742.10
26	***Total***	$1,842.72	$1,965.01	$1,854.26
27				
28	*Percent of face value bought*	95.10%	91.24%	101.40%
29	***Terminal Wealth***	$1,752.43	$1,792.97	$1,880.14

Thus, if the yield falls, bond 1 will no longer fund our obligation, whereas bond 3 will overfund it. Bond 2's ability to fund the obligation – not surprisingly, in view of the fact that its duration is exactly 10 years – hardly changes. We canrepeat this calculation for any new yield to maturity. The results are shown in the following figure:



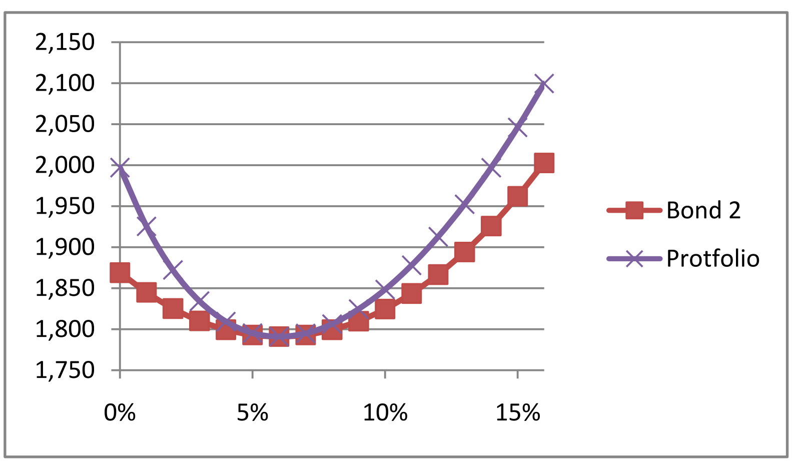
Clearly, if you want an immunized strategy, you should buy bond 2! As its duration is equal to the duration of the obligation we want to hedge.

Convexity: A Continuation of Our Immunization Experiment

The duration of a portfolio is the weighted average duration of the assets in the portfolio. As a result, there is anotherway to get a bond investment with duration of 10: If we invest \$665.09 in bond 1 and \$ 344.91 in bond 3, the resulting portfolio also has duration of 10. These weights are calculated as follows:

$$\lambda \times D_{bond\ 1} + (1-\lambda) \times D_{bond\ 3} = 7.665\lambda + 14.636(1-\lambda) = 10 \rightarrow \lambda = 0.66509$$

Suppose we repeat our experiment with this portfolio of bonds of the previous section (varying the YTM), but add in the portfolio of bond 1 and bond 3.Building a data table based on this experiment and graphing the results shows that the portfolio's performance isbetter than that of bond 2 by itself.



Look again at the graph: Notice that, while the terminal value is somewhat convex in the yield to maturity for bothbond 2 and the bond portfolio, the terminal value of the portfolio is *more convex* than that of the single bond.Redington (1952), one of the influential propagators of the concept of duration and immunization, thought thisconvexity very desirable, and we can see why: No matter what the change in the yield to maturity, the portfolio ofbonds provides *more overfunding* of the future obligation than the single bond. This is obviously a desirable propertyfor an immunized portfolio, and it leads us to formulate the following rule:

In a comparison between two immunized portfolios, both of which are to fund a known future obligation, the portfolio whose terminal value is more convex with respect to changes in the yield to maturity is preferable.

General Look at Immunization Strategies

Consider a situation when a fund manager owns n_1 units of bond B_1. The value of the holding will obviously be changing with the change in the level of the yield curve. If the yield goes up the value of the holding will fall. The fund manager wishes to protect the portfolio against this unfavorable movement.

One of the ways of doing so is to add to the holding n_2 units of bond B_2 so that the any change in the value of bond B_1 will be exactly offset by the change in the value of bond B_2.

Let the value of the portfolio be $V = n_1B_1(y_1) + n_2B_2(y_2)$. Here y_1 and y_2 are the yields of the bonds respectively. If the yields are shifted by the same amount Δy the new value of the portfolio will be $V_\Delta = n_1B_1(y_1 + \Delta y) + n_2B_2(y_2 + \Delta y)$ and the change in the value of the portfolio will be:

$$\begin{aligned}\Delta V = V_\Delta - V &= n_1\big(B_1(y_1+\Delta y) - B_1(y_1)\big) + n_2\big(B_2(y_2+\Delta y) - B_2(y_2)\big)\\ &\approx n_1D_1(y_1)\Delta y + n_2D_2(y_2)\Delta y\end{aligned}$$

WhereD_1 and D_2 are respective bond durations; for an immunized portfolio the value change should be zero, so:

$$n_1D_1(y_1)\Delta y + n_2D_2(y_2)\Delta y = 0 \to n_2 = -n_1\frac{D_1}{D_2}$$

Several things are to be noted here.

1. This type of portfolio immunization may not be allowed, as many funds are under restriction of not being able to hold ‘short’ bonds that is bonds that are nominally sold. There sometimes are ways around this restriction. Funds may, for example, ‘borrow’ bonds from other parties.
2. Even if bonds can be sold short this solution may be quite expensive by the way of losing the coupon income, transaction costs and liquidity issues.
3. This approach will immunize the portfolio only against parallel shocks, but not changes in the slope and/or curvature of the curve.

4. By immunizing the portfolio we at the same time forgo the potential benefits, for example, the increase in the value of the portfolio if the yields go down.
5. As durations (and yield) change with time, portfolio needs to be periodically rebalanced.

The method above will protect against small changes of the yields. If the market gaps (yield drop or go up unexpectedly) the price convexity will generate profit or loss on the portfolio. To immunize against large movements of the yield the convexity must be considered. Using Taylor's decomposition of our portfolio to the second order we obtain:

$$\Delta V = V_\Delta - V = n_1\big(B_1(y_1+\Delta y) - B_1(y_1)\big) + n_2\big(B_2(y_2+\Delta y) - B_2(y_2)\big)$$
$$\approx n_1 D_1(y_1)\Delta y + \frac{1}{2} n_1 C_1(y_1)\Delta y^2 + n_2 D_2(y_2)\Delta y + \frac{1}{2} n_2 C_2(y_2)\Delta y^2$$

Where $C_i$are the convexities of respective bonds; by equating the value increment to zero and solving for $n_2$the portfolio composition can be easily obtained.

An Example

Consider a fund manager who needs to meet a liability of $1,000,000 in 4 years. He has two bonds available to him to hedge his liability. The information is summarized in the table.

	FV	Maturity	Coupon	Yield	Price	Modified Duration
Bond	$100,000.00	3	5%	6.1%	$97,025.48	2.74
Bond	$100,000.00	7	6%	6.4%	$97,771.28	5.62
Liability	$1,000,000.00	4		6.3%	$780,272.21	4.00

The fund manager needs to buy n_1 units of the first bond and n_2 of the second bond so that:

1. Match the value of the liability
2. Match the duration of the liability

This leads to the following simultaneous equations:

$$97025.48 n_1 + 97771.28 n_2 = 780272.21$$
$$2.74 \times 97025.48 \times n_1 + 5.62 \times 97771.28 \times n_2 = 4 \times 780272.21$$

The second equation requires some explanation. Remember, that modified duration measures the change in the value for a unit price. So, in order to measure the actual change (in $values) the modified duration needs to be multiplied by the value.Solving these equations we obtain n_1= 4.5 and n_2 = 3.5.

Financial Modeling, Second Edition, Simon Benninga; MIT Press 2000, pp. 317-326

Fundamentals of Financial Markets, Volf Frishling; National Australia Bank 2007, pp. 15-17

22. Calculating Default-Adjusted Expected Bond Returns

Here we discuss the effects of default risk on the returns from holding bonds to maturity. The expectedreturn on a bond that may possibly default is different from the bond's promised return. The latter is defined as thebond's yield to maturity, the internal rate of return calculated from the bond's current market price and its promisedcoupon payments and promised eventual return of principal in the future. The bond's expected return is less easilycalculated: We need to take into account both the bond's probability of future default and the percentage of itsprincipal that holders can expect to recover in the case of default. To complicate matters still further, default canhappen in stages, through the gradual degradation of the issuing company's creditworthiness.

We'll use a Markov model to solve for the expected return on a risky bond. Our adjustment proceduretakes into account all three of the factors mentioned: the probability of default, the transition of the issuer from onestate of creditworthiness to another, and the percentage recovery of face value when the bond defaults. Afterillustrating the problem and using Excel to solve a small-scale problem, we use some publicly available statistics toprogram a fuller spreadsheet model. Finally, we show that this model can be used to derive bond betas, the CAPM'srisk measure for securities.

Some Preliminaries

Before proceeding, we define a number of terms:

- A bond is issued with a given amount of principal or face value. When the bond matures, the bondholder is promised the return of this principal. If the bond is issued at par, then it is sold for the principal amount
- A bond bears an interest rate called the coupon rate. The periodic payment promised to the bondholders is the product of the coupon rate times the bond's face value
- At any given moment, a bond will be sold in the market for a market price. This price may differ from the bond's coupon rate
- The bond's yield to maturity (YTM) is the internal rate of return of the bond, assuming that it is held to maturity and that it does not default

American corporate bonds are rated by various agencies on the basis of the bond issuer's ability to make repaymenton the bonds. The classification scheme for two of the major rating agencies, Standard & Poor's (S&P) and Moody'sis given in the following table:

Long-Term Senior Debt Ratings

Investment – Grade Ratings			***Speculative – Grade Ratings***		
S&P	***Moody's***	***Interpretation***	***S&P***	***Moody's***	***Interpretation***
AAA	Aaa	Highest Quality	BB+	Bal	Likely to fulfill obligations; ongoing uncertainty
			BB	Ba2	
			BB-	Ba3	
AA+	Aa1	High Quality	B+	B1	High – risk obligations
AA	Aa2		B	B2	
AA-	Aa3		B-	B3	
A+	A1	Strong Payment Capacity	CCC+	Caa	Current vulnerability to default
A	A2		CCC		
A-	A3		CCC-		
BBB+	Baa1	Adequate payment capacity	C	Ca	In bankruptcy or default, or other marked shortcomings
BBB	Baa2		D	D	
BBB-	Baa3				

When a bond defaults, its holders will typically receive some payoff, thoughless than the promised bond coupon rateand return of principal. We refer to the percent of face value paid off in default as the *recovery percentage*.

- Besides default risk, bonds are also subject to term-structure risk: The prices of bonds may show significant variations over time as a result of changing term structure. This statement will be especially true for long-term bonds. We will abstract from term-structure risk, confining ourselves only to a discussion of the effects of default risk on bond expected returns.
- Just to complicate matters, in the United States the convention is to add to a bond's listed price the *proratedcoupon* between the time of the last coupon payment and the purchase date. The sum of these two is termed the *invoice price* of the bond; the invoice price is the actual cost at any moment to a purchaser of buying the bond. In our discussion we use the term *market price* to denote the invoice price.

Calculating the Expected Return in a One-Period Framework

The bond's yield to maturity is not its expected return: It is clear that both a bond's rating and the anticipated payoffto bondholders in the case of bond default should affect its expected return. All other things being equal, we wouldexpect that if two newly issued bonds have the same term to maturity, then the lower-rated bond (having the higherdefault probability) should have a higher coupon rate. Similarly, we would expect that an issued and traded bondwhose rating has been lowered would experience a decrease in price. We might also expect that the lower is theanticipated payoff in the case of default; the lower will be the bond's expected return.

As a simple illustration, we calculate the expected return of a one-year bondthat can default at maturity. We use thefollowing symbols:

F = face value of the bond
P = price of bond
c = annual coupon rate of the bond
π = probability that the bond will not default at end of year
λ = fraction of face value that bondholders collect upon default
ER = expected return

This bond's expected end-of-year cash flow is $\pi \times (1 + c) \times F + (1 - \pi) \times \lambda \times F$ and its expected return is given by:

$$ER = \frac{\pi(1+c)+(1-\pi)\lambda F}{P} - 1$$

This calculation is illustrated in the following spreadsheet:

	A	B	C	D	E	F	G	H
1		Expected Return on A One-Year Bond						
2	With An Adjustment For Default Probability							
3								
4	*F*	100						
5	*P*	98						
6	*c*	16%						
7	*π*	90%						
8	*λ*	80%						
9								
10	***Expected Cash Flow***	112.40	<=	B7*(1+B6)*B4+(1-B7)*B8*B4				
11	***ER***	14.69%	<=	B10/B5-1				

A Multiperiod, Multistate Markov Chain Problem

We now introduce multiple periods into the problem. In this section we definea basic model using a very simple set ofratings. We suppose that at any date there are four possible bond "ratings":

- A – The highest rating.
- B – The next highest rating.
- D – The bond is in default for the first time (and hence pays off Π of the face value).
- E – The bond was in default in the previous period; it therefore pays off 0 in the current period and in any future periods.

The *transition probability* matrix Π is given by:

$$\Pi = \begin{bmatrix} \pi_{AA} & \pi_{AB} & \pi_{AD} & 0 \\ \pi_{BA} & \pi_{BB} & \pi_{BD} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The probabilities π_{ij} indicate the probability that in one period the bond will go from a rating of *i* to a rating of *j*. We will use the following transition probability matrix:

$$\Pi = \begin{bmatrix} .99 & .01 & 0 & 0 \\ .03 & .96 & .01 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What does this matrix Π mean?

- If a bond is rated A in the current period, there is a probability of 0.99 that it will still be rated A in the next period. There is a probability 0.01 that it will be rated B in the next period, but it is impossible for the bond to be rated A today and D or E in the subsequent period. While it is possible to go from ratings A and B to any of ratings A, B, and D; it is *not* possible to go from A or B to E. This statement is true because E denotes that default took place in the previous period.
- In the example of Π, a bond that starts off with a rating of B can – in a subsequent period – be rated A (with a probability of 0.03); be rated B (with a probability of 0.96); or rated D (and hence in default) with a probability of .01
- A bond that is currently in state D (i.e., first-time default); will necessarily be in E in the next period. Thus the third row of our matrix Π will always be [0 0 0 1]
- Once the rating is in E, it remains there permanently. Therefore, the fourth row of the matrix Π also will always be [0 0 0 1]

The Multiperiod Transition Matrix

The matrix P defines the transition probabilities over one period. The two-period transition probabilities are given bythe matrix product Π × Π:

$$\Pi \times \Pi = \begin{bmatrix} .99 & .01 & 0 & 0 \\ .03 & .96 & .01 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} .99 & .01 & 0 & 0 \\ .03 & .96 & .01 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .9804 & .0195 & .0001 & 0 \\ .0585 & .9219 & .0096 & .0100 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus if a bond is rated B today, there is a probability of 5.85 percent that in two periods it will be rated A, a probabilityof 92.19 percent that in two periods it will be rated B, a probability of 0.96 percent that in two periods it will default(and hence be rated D), and a probability of 1 percent that in two periods it will be rated E. The last rating means, ofcourse, that the bond went into default in the first period.

We can use the array function **MMult** function of Excel to calculate multiyear transition probabilitymatrices:

	G	H	I	J	K	L	M	N	O	P	Q	R
4	One - Period Transition Matrix				Two - Period Transition Matrix				Three - Period Transition Matrix			
5	0.99	0.01	0	0	0.9804	0.0195	0.0001	0	0.9712	0.0285	0.0002	0.0001
6	0.03	0.96	0.01	0	0.0585	0.9219	0.0096	0.01	0.0856	0.8856	0.0092	0.0196
7	0	0	0	1	0	0	0	1	0.0000	0.0000	0.0000	1.0000
8	0	0	0	1	0	0	0	1	0.0000	0.0000	0.0000	1.0000

In general, the year – t transition matrix is given by the matrix power Π^t. Calculating these matrix powers by theprocedure that we have illustrated is cumbersome, so we first define a VBA function that can compute powers ofmatrices:

```
Function matrixpower (matrix, n)
      If n = 1 Then

            matrixpower = matrix
      Else
      matrixpower =
               Application.MMult(matrixpower(matrix,  n  -
            1) , matrix)

      End If
End Function
```

The use of this function is illustrated in the following spreadsheet. The function **Matrixpower** allows a one-stepcomputation of the power of any transition matrix:

	A	B	C	D	E	F	G
1		Using Function Matrixpower					
2							
3		One-period transition matrix					
4		0.99	0.01	0	0		
5		0.03	0.96	0.01	0		
6		0	0	0	1		
7		0	0	0	1		
8							
9	*t*	10		{=MATRIXPOWER(B4:E7,B9)}			
10							
11		0.915901	0.080152	0.000739	0.003209		
12		0.240456	0.675445	0.007013	0.077087		
13		0	0	0	1		
14		0	0	0	1		
15							

From this example it follows that if a bond started out with an A rating, there is a probability of 0.3 percent that thebond will be in default at the end of ten periods, and there is a probability of 0.07 percent that it will default before thetenth period.

Bond Payoff Vector

Recall that c denotes the bond's coupon rate and λ denotes the percentage payoff of face value if the bond defaults.The payoff vector of the bond depends on whether the bond is currently in its last period N or whether $t<N$:

$$Payoff(t) = \begin{cases} \begin{bmatrix} c \\ c \\ \lambda \\ 0 \end{bmatrix} & if\ t < N \\ \begin{bmatrix} 1+c \\ 1+c \\ \lambda \\ 0 \end{bmatrix} & if\ t = N \end{cases}$$

The first two elements of each vector denote the payoff in nondefaulted states, the third element λ is the payoff if therating is D, and the fourth element 0 is the payoff if the bond rating is E. The distinction between the two vectorsdepends, of course, on the repayment of principal in the terminal period.

Before we can define the expected payoffs, we need to define one furthervector, which will denote the *initial state of the bond*. This current-state vector is a vector with a 1 for the current rating of the bond and zeros elsewhere. For example, if the bond has rating A at date 0, then Initial = [1 0 0 0]; if it has date – 0 rating of B, then Initial = [0 1 0 0].We can now define the expected bond payoff in period t:

$$E[Payoff(t)] = Initial \times \Pi^t \times Payoff(t)$$

A Numerical Example

We continue using the numerical λ from the previous section, and we further suppose that λ = 0.8, meaning that adefaulted bond will pay off 80 percent of face value in the first period of default. We consider a bond having thefollowing characteristics:

- The bond is currently rated B
- Its coupon rate c = 7 percent
- The bond has five more years to maturity
- The bond's current market price is 98 percent of its face value

	A	B	C	D	E	F	G	H
1			Calculating The Expected Bond Return					
2								
3	*P*	98%		*Payoff (t < N)*			*Payoff (t = N)*	
4	*c*	7%		7%			107%	
5	*λ*	80%		7%			107%	
6	*N*	5		80%			80%	
7	*Rating*	B		0			0	
8								
9			*A*	*B*	*C*	*D*		
10	*Transition Matrix*	*A*	0.99	0.01	0	0		
11		*B*	0.03	0.96	0.01	0		
12		*C*	0	0	0	1		
13		*D*	0	0	0	1		
14								
15	*Initial Vector*		0	1	0	0		
16								
17	*Year*	0	1	2	3	4	5	
18	*Expected Payoffs*	-0.98	0.0773	0.076308	0.07535787	0.074447616	1.027389845	
19	*Expected Yield*	**7.24%**						
20								

The spreadsheet above shows the facts in the preceding list as well as the payoff vectors of the bond at dates before maturity (in cells D4:D7) and on the maturity date (cells G4:G7). The transition matrix is given in cells C10:F13,and the initial vector is given in C15:F15.

The expected bond payoffs are given in cells B19:G19. Before we explain how they were calculated, we note the important economic fact that – ifthe expected payoffs are as given – then the bond's expected return is calculated by IRR(B18:G18). As cell B19 shows, this expected return is 7.245 percent.

How to Calculate the Expected Bond Payoffs

As indicated in the previous section, the period – *t* expected bond payoff is given by the following formula$E[Payoff(t)] = Initial \times \Pi^t \times Payoff(t)$. *The formula in row 18 uses IF statement* to implement this formula as:

$$if\left(\begin{array}{c} t_i = N, \\ mmult\big(initial\ vector, mmult(matrixpower(transition\ matrix, t_i), payoff(t_i = N))\big), \\ mmult\big(initial\ vector, mmult(matrixpower(transition\ matrix, t_i), payoff(t_i < N))\big) \end{array}\right)$$

Here's what these statement mean:

- If the current year, ti is equal to the bond term N, then the expected payoff on the bond is $\mathrm{mmult(C15{:}F15, mmult(matrixpower(C10{:}F13, C17), G4{:}G7))}$
- If the current year, ti is less than the bond term, then the expected payoff on the bond is $\mathrm{mmult(C15{:}F15, mmult(matrixpower(C10{:}F13, C17), D4{:}D7))}$

Copying this formula gives the whole vector of expected bond payoffs.The actual formula in Cell B19 is IRR(B18:G18), i.e. default-adjusted expected bond return.

Computing Bond Betas

A vexatious problem in corporate finance is the computation of bond betas. The model presented here canbe easily used to compute the beta of a bond. Recall that the capital asset pricing model's *security market line* (SML)is given by:

$$E(r_d) = r_f + \beta_d\big(E(R_m) - r_f\big)$$

where $E(r_d)$ is expected return on debt, r_f is return on riskless debt, and $E(r_m)$ is return on equity market portfolio.

If we know expected return on debt, we can calculate $\beta$of the debt. Provided we know the risk – free rate and theexpected rate of return on the market. Suppose, for example, that the market risk premium is 8.4percent, and that risk – free rate is5 percent. Then a bond having an expected return of 8 percent will have a β of .357 as illustrated below:

	A	B	C	D	E	F
1			Calculating A Bond's Beta			
2						
3	*Market Risk Premium*	$E\ (r_m)$	8.40%			
4	*risk-free rate*	r_f	5%			
5	*expected bond return*	$E\ (r_d)$	8%			
6	*implied bond beta*	β	**0.357**	<= (C5-C4)/C3		
7						

Financial Modeling, Second Edition, Simon Benninga; MIT Press 2000, pp. 334-348

REFERENCES

1. R. Barret, M.Zieger, K. Byleen, Finite Mathematics for Business,Economics,Life Science and Social Sciences (international edition).
2. S. Benninga, Financial Modeling,Second Edition, Massachusetts Inst. of Technology 2000.
3. Z. Bodie, A. Kane, A. J. Marcus, Investments, 8^{th} ed., 2009.
4. V. Frishling; Fundamentals of Financial Markets, National Australia Bank 2007.
5. M. Jackson and M. Staunton, Advanced Modeling in Finance using Excel and VBA, John Wiley & Sons, Ltd 2001.
6. R. L. McDonald, Derivatives Markets, Second Edition, Pearson Education Inc. 2006.
7. J. Mun, Real Options Analysis, John Wiley & Sons, Inc. 2002.
8. P. Newbold, W.Carlson B.Thorne, Statics for Business and Economics,Sixth Edition.
9. R.Steiner, Mastering Financial Calculations, Prentis Hall
10. W.Winston, Microsoft Excel Data Analysis and Business Modeling.
11. W. L. Winston, Data Analyses and Business Modeling, H.B. Fenn and Company Ltd 2004