GEORGIAN AMERICAN UNIVERSITY

Derivatives
Applications in Business

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PREFACE

The purpose of this book is to introduce the essence of derivatives and its applications in business and is intended for students, academicians, field professionals and other interested parties. The book has been written in a question – answer form and can be used to review and refresh knowledge in the field as well as for integration of academic achievements in the study process.

The book has been written based on Business Research Center’s extended seminars.

DERIVATIVES

Basic Concepts

1. What are Derivative Instruments? Why are they called "Derivatives"?

Options, futures and swaps are examples of derivatives. A derivative is financial instrument (or more precisely, an agreement between two people) that has value determined by the price of something else. For example, a bushel of corn is not a derivative; it is a commodity with a value determined by the price of corn. However, you could enter into an agreement with a friend that says: if the price of a bushel of corn is less than $3, the friend will pay you $1. This is a derivative in the sense that you have an agreement with a value derived by the price of something else (corn, in this case), that's why it is called derivative – its value is derived from the price of something else.

You might think: "that's not a derivative; that's just a bet on the price of corn." So it is: derivatives can be thought of as bets on the price of something. But don't automatically think the term "bet" is pejorative. Suppose your family grows corn and you friend's family buys corn to mill into cornmeal. The bet provides insurance: you earn $1 if your family's corn sells for a low price; this supplements you income. Your family's friend earns $1 if the corn his family buys is expensive; this offsets the high cost of corn. Viewed in this light, the bet hedges you both against unfavorable outcomes. The contract has reduced risk for both of you.

Investors could also use this kind of contract simply to speculate on the price of corn. In this case the contract is not insurance. And that is a key point: *it is not the contract itself, but how it is used, and who uses it, that determines whether or not it is risk-reducing.* Context is everything.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 1, pp. 1-2

2. Explain the concept of Forward Contract

Suppose you wish to buy a share of stock. Doing so entails at least three separate steps: (1) setting the price to be paid, (2) transferring cash from the buyer to the seller, and (3) transferring the share from the seller to the buyer.

With an outright purchase of stock, all three occur simultaneously. However, as a logical matter, a price could be set today and the transfer of shares and cash would occur at a specified date in the future.

This is in fact a definition of a forward contract: it sets today the terms at which you buy or sell an asset or commodity at a specific time in the future. A forward contract does the following:

- Specifies the quantity and exact type of the asset or commodity the seller must deliver
- Specifies delivery logistics, such as time, date, and place
- Specifies the price the buyer will pay at the time of delivery
- Obligates the seller to sell and the buyer to buy, subject to the above specifications

The time at which the contract settles is called the *expiration date*. The asset or commodity on which the forward contract is based is called the *underlying asset*. Apart from commissions and bid-ask spreads, a forward contract requires no initial payment or premium. The contractual forward price simply represents the price at which consenting adults agree today to transact in the future at which time the buyer pays the seller the forward price and the seller delivers the asset.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 2, p. 21

3. What are Future Contracts? What is the difference between Forwards and Futures Contracts?

Futures contracts are similar to forward contract in that they create an obligation to buy or sell at a predetermined price at a future date. Futures contracts are essentially exchange-traded forward contracts. Because futures are exchange-traded, they are standardized and have specified delivery dates, locations, and procedures. Each exchange has an associated *clearinghouse*. The role of the clearinghouse is to match the buys and sells that take place during the day, and to keep track of the obligations and payments required of the members of the clearinghouse, who are *clearing members*. After matching trades, the clearinghouse typically becomes the counterparty for each clearing member.

Although forwards and futures are similar in many respects, there are differences:

- Whereas forward contracts are settled at expiration, futures contracts are settled daily. The determination of who owes what to whom is called marking-to-market
- As a result of daily settlement, futures are liquid – it is possible to offset an obligation on a given date by entering into the opposite position
- Over-the-counter forward contract can be customized to suit the buyer or seller, whereas futures contracts are standardized
- Because of daily settlement, the nature of credit risk is different with the futures contract, in fact, futures contract are structures so as to minimize the effects of credit risk
- There are typically daily price limits in the futures markets (and on some stock exchanges as well). A *price limit* is a move in the futures price that triggers a temporarily halt in trading

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 5, p. 142

4. Explain the concept of Call / Put Option. List their characteristics and draw graph for long and short positions. What is the difference between American and European Options?

Whereas a forward contract obligates the buyer (the holder of the long position) to pay the forward price at expiration, even if the value of the underlying asset at expiration is less than the forward price, a **call option** is a contract where the buyer has the right to buy, but not the obligation to buy. Here are some key terms used to describe options:

Strike price: the **strike price**, or **exercise price**, of a call option is what the buyer pays for the asset

Exercise: the **exercise** of a call option is the act of paying the strike price to receive the asset

Expiration: the **expiration** of the option is the date by which the option must either be exercise or it becomes worthless

Exercise style: the **exercise style** of the option governs the time at which exercise can occur. If exercise can occur only at expiration, option is said to be **European-style option**. If the buyer has the right to exercise at any time during the life of the option, it is an **American-style option**. If the buyer can only exercise during specified periods, but not for the entire life of the option, the option is a **Bermudan-style option.**

To summarize, a European call option gives the owner of the call the right, but not the obligation, to buy the underlying asset on the expiration date by paying the strike price. The buyer is not obligated to buy the underlying, and hence will only exercise the option if the payoff is greater than zero. The algebraic expression for the *payoff* to a purchased (long) call is therefore:

$$Long\ call\ payoff = max[0, S_T - K]$$

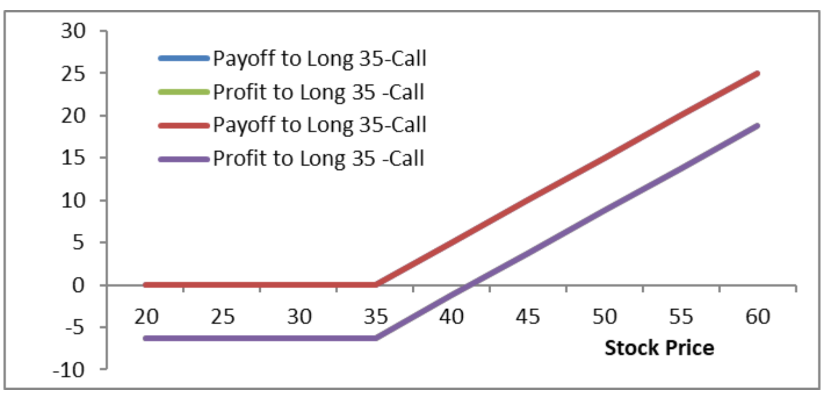
Where S_T is spot price of the underlying asset at expiration and K is the exercise price. The expression max $[a, b]$ means take the greater of the two values a and b. In computing profit at expiration, suppose we defer the premium payment (the price of an option); then by the time of expiration we accrue interest on the premium. So, the option *profit* is computed as:

$$Long\ call\ profit = max[0, S_T - K] - future\ value\ of\ option\ premium$$

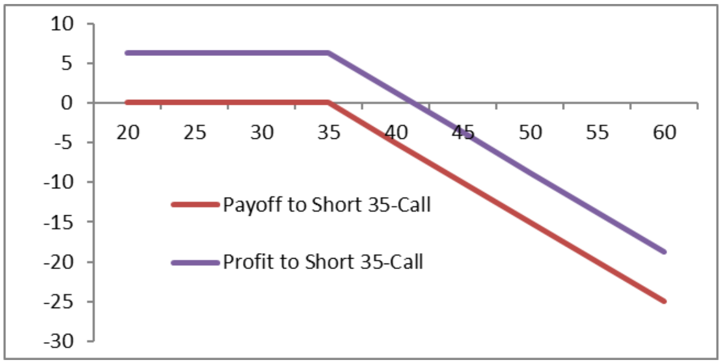
The seller is said to be the **option writer**, or to have a short position in a call option. The payoff and profit to a written call are just the opposite of those for a purchased call:

$$Short\ call\ payoff = -max[0, S_T - K]$$

$$Short\ call\ profit = -max[0, S_T - K] + future\ value\ of\ option\ premium$$



Graphically, the payoff and profit calculations for long and short call option are:



Perhaps you wondered if there could also be a contract in which the *seller* could walk away if it is not in his or her interest to sell. The answer is yes. A **put option** is a contract where the seller has the right to sell, but not the obligation. The put option gives the put buyer the right to sell the underlying asset for the strike price. Thus, the *payoff* on the put option is:

$$Long\ put\ payoff = max[0, K - S_T]$$

At the time the option is acquired, the put buyer pays the option premium to the put seller; we need to account for this in computing profit. If we borrow the premium amount, we must pay interest. The option *profit* is computed as:

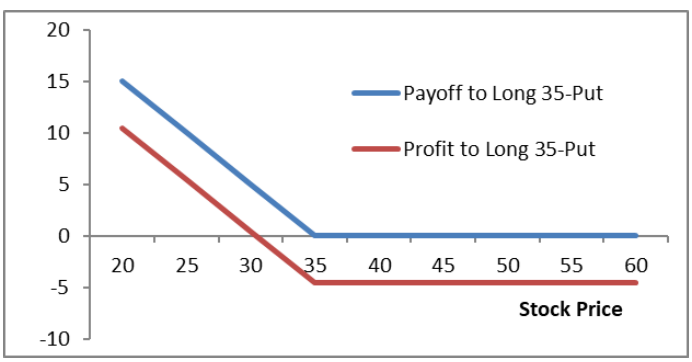
$$\text{Long put profit} = \max[0, \text{K} - \text{S}_\text{T}] - \text{future value of option premium}$$

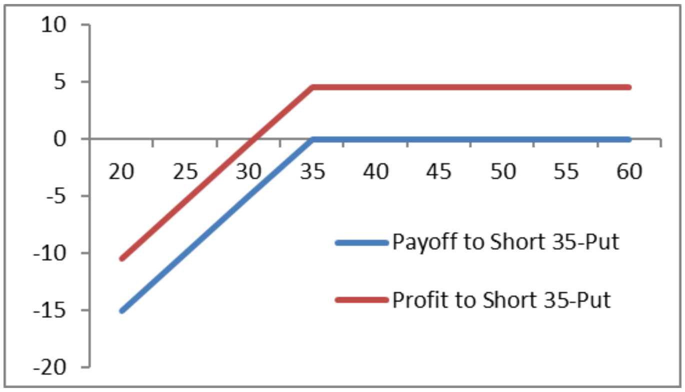
The payoff and profit for a written put are the opposite:

$$Short\ put\ payoff = -max[0, K - S_T]$$

$$Short\ put\ profit = -max[0, K - S_T] + future\ value\ of\ option\ premium$$

Graphically, the payoff and profit calculations for long and short put option are:





Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 2, pp. 31-43

5. List uses of Derivatives

Risk management Derivatives is a tool for companies and other users to reduce risks. The corn example above illustrates this in a simple way: The farmer-a seller of com-enters into a contract which makes a payment when -the price of corn is low. This contract reduces the risk of loss for the farmer, who we therefore say is hedging. It is common to think of derivatives· as forbiddingly complex, but many derivatives are simple and familiar. Every form of insurance is a derivative, for example. Automobile insurance is a bet on whether you will have an accident. If you wrap your car around a tree, your insurance is valuable; if the car remains intact, it is not.

Speculation Derivatives can serve as investment vehicles. As you will see later in the book, derivatives can provide a way to make bets that are highly leveraged (that is, the potential gain or loss on the bet can be large relative to the initial cost of making the bet) and tailored to a specific view. For example, if you want to bet that the S&P500 stock index will be between 1 300 and 1 400 one year from today, derivatives can be constructed to let you do that.

Reduced transaction costs Sometimes derivatives provide a lower-cost way to effect a particular financial transaction. For example, the manager of a mutual fund may wish to sell stocks and buy bonds. Doing this entails paying fees to brokers and paying other trading costs, such as the bid-ask spread, which we will discuss later. It is possible to trade derivatives instead and achieve the same economic effect as if stocks had actually been sold and replaced by bonds. Using the derivative might result in lower transaction costs than actually selling stocks and buying bonds.

Regulatory arbitrage it is sometimes possible to circumvent regulatory restrictions, taxes, and accounting rules by trading derivatives. Derivatives are often used, for example, to achieve the economic sale of stock (receive cash and eliminate the risk of holding the stock) while still maintaining physical possession of the stock. This transaction may allow the owner to defer taxes on the sale of the stock, or retain voting rights, without the risk of holding the stock.

These are common reasons for using derivatives. The general point is that derivatives provide an alternative to a simple sale or purchase, and thus increase the range of possibilities for an investor or manager seeking to accomplish some goal.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 1, pp. 2-3

6. What are Financial Engineering and Security Design?

One of the major ideas in derivatives-perhaps the major idea-is that it is generally possible to create a given payoff in multiple ways. The construction of a given financial product from other products is sometimes called financial engineering. The fact that this is possible has several implications.

First, since market-makers need to hedge their positions, this idea is central in understanding how market-making works. The market maker sells a contract to an end-user, and then creates an offsetting position that pays him if it is necessary to pay the customer. This creates a hedged position.

Second, the idea that a given contract can be replicated often suggests how it can be customized. The market-maker can, in effect, turn dials to change the risk, initial premium, and payment characteristics of a derivative. These changes permit the creation of a product that is more appropriate for a given situation.

Third, it is often possible to improve intuition about a given derivative by realizing that it is equivalent to something we already understand.

Finally, because there are multiple ways to create a payoff, the regulatory arbitrage discussed above can be difficult to stop. Distinctions existing in the tax code, or in regulations, may not be enforceable, since a particular security or derivative that is regulated or taxed may be easily replaced by one that is treated differently but has the same economic profile.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 1, pp. 3-4

7. List reasons for short-selling an asset?

When we buy something, we are said to have a long position in that thing. For example, if we buy the stock of XYZ, we pay cash and receive the stock. Sometime later, we sell the stock and receive cash. This transaction is lending, in the sense that we pay money today and receive money back in the future. The rate of return we receive may not be known in advance (if the stock price goes up a lot, we get a high return; if the stock price goes down, we get a negative return), but it is a kind of loan nonetheless. The opposite of a long position is a short position. A short-sale of XYZ entails borrowing shares of XYZ and then selling them, receiving the cash. Sometime later, we buy back the XYZ stock, paying cash for it, and return it to the lender. A short-sale can be viewed, then, as just a way of borrowing money. When you borrow money from a bank, you receive money today and repay it later, paying a rate of interest set in advance. This is also what happens with a short-sale, except that you don't necessarily know the rate you pay to borrow.

There are at least three reasons to short-sell:

- Speculation: A short-sale, considered in essence, makes money if the price of the stock goes down. The idea is to first sell high and then buy low. (With a long position, the idea is to first buy low and then sell high.)
- Financing: A short-sale is a way to borrow money, and it is frequently used as a form of financing. This is very common in the bond market, for example.
- Hedging: You can undertake a short-sale to offset the risk of owning the stock or a derivative on the stock. This is frequently done by market-makers and traders.

These reasons are not mutually exclusive. For example, a market-maker might use a short-sale to simultaneously hedge and finance a position.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 1, pp. 12-13

8. Explain the concept of insuring a long position

Put options are insurance against a fall in the price of an asset. Thus, if we own the S&R index, we can insure the position by buying an S&R put option. The purchase of a put option is also called a floor, because we are guaranteeing a minimum sale price for the value of the index. To examine this strategy, we want to look at the combined payoff of the index position and put. Now we add them together to see the net effect of holding both positions at the same time.

Table below summarizes the result of buying a 1000-strike put with 6 months to expiration, in conjunction with holding an index position with a current value of $ 1000.The table computes the payoff for each position and sums them to obtain the total payoff.

Payoff and profit at expiration from purchasing the S&R index and a 1000-strike put option. Payoff is the sum of the first two columns. Cost plus interest for the position is ($1000 + $74.201) × 1.02 = $1095.68. Profit is payoff less $1095.68.

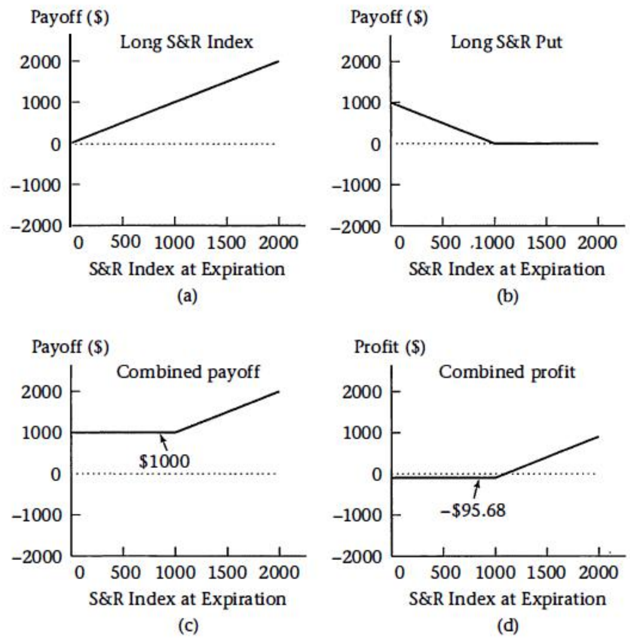
Payoff at Expiration				
S&R Index	**S&R Put**	**Payoff**	**−(Cost + Interest)**	**Profit**
$900	$100	$1000	−$1095.68	−$95.68
950	50	1000	−1095.68	−95.68
1000	0	1000	−1095.68	−95.68
1050	0	1050	−1095.68	−45.68
1100	0	1100	−1095.68	4.32
1150	0	1150	−1095.68	54.32
1200	0	1200	−1095.68	104.32

The final column takes account of financing cost by subtracting cost plus interest from the payoff to obtain profit. "Cost" here means the initial cash required to establish the position. This is positive when payment is required and negative when cash is received. We could also have computed profit separately for the put and index. For example, if the index is $900 at expiration, we have:

$$\underbrace{\$900 - (\$1000 \times 1.02)}_{\text{Profit on S\&R Index}} + \underbrace{\$100 - (\$74.201 \times 1.02)}_{\text{Profit on Put}} = -\$95.68$$

This gives the same result as the calculation performed in the Table above. The level of the floor is -$95 .68, which is the lowest possible profit. Figure below graphs the components of the Table. Panels (c) and (d) show the payoff and profit for the combined index and put positions. The combined payoff graph in panel(c) is created by adding at each index price the value of the index and put positions; this is just like summing columns 1 and 2 in the Table.

Panel (a) shows the payoff diagram for a long position in the index (column 1 in Table 3.1). Panel (b) shows the payoff diagram for a purchased index put with a strike price of $1000 (column 2 in Table 3.1). Panel (c) shows the combined payoff diagram for the index and put (column 3 in Table 3.1). Panel (d) shows the combined profit diagram for the index and put, obtained by subtracting $1095.68 from the payoff diagram in panel (c) (column 5 in Table 3.1).



Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 2, pp. 59-62

9. Explain the concept of insuring a short position

If we have a short position in the S&R index, we experience a loss when the index rises. We can insure a short position by purchasing a call option to protect against a higher price of repurchasing the index. Buying a call option is also called a cap.

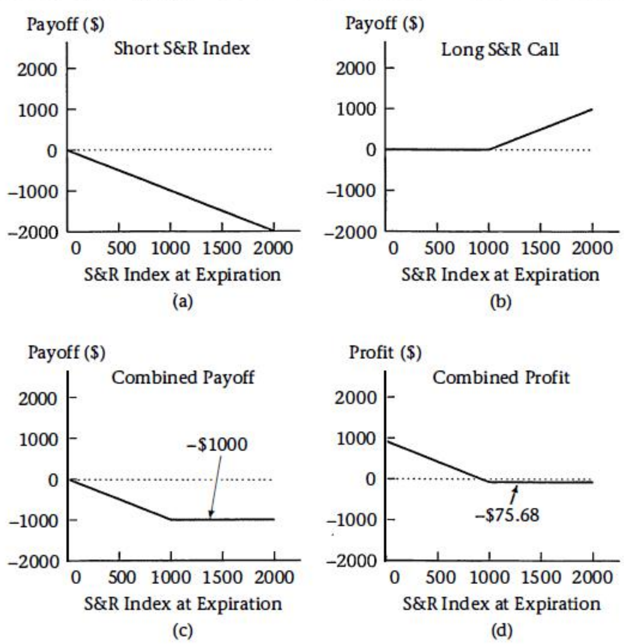
Table below presents the payoff and profit for a short position in the index coupled with a purchased call option. Because we short the index, we earn interest on the short proceeds less the cost of the call option, giving -$924.32 as the future value of the cost. Figure below graphs the columns of the Table. The payoff and profit diagrams resemble those of a purchased put. As with the insured index position, we have to be careful in dealing with cash flows. The payoff in panel (c) of the Figure is like that of a purchased put coupled with borrowing. In this case, the payoff diagram for shorting the index and buying a call is equivalent to that from buying a put and borrowing the present value of $1000 ($980.39). Since profit diagrams are unaffected by borrowing, however, the profit diagram in panel (d) is exactly the same as that for a purchased S&R

index put. Not only does the insured short position look like a put, it has the same loss as a purchased put if the price is above \$ 1000: \$75 .68, which is the future value of the \$74.201 put premium.

Payoff and profit at expiration from short-selling the S&R index and buying a 1000 strike call option at a premium of \$93.809. The payoff is the sum of the first two columns. Cost plus interest for the position is $(-\$1000 + \$93.809) \times 1.02 = -\$924.32$. Profit is payoff plus \$924.32.

Payoff at Expiration				
Short S&R Index	**S&R Call**	**Payoff**	**−(Cost + Interest)**	**Profit**
−\$900	\$0	−\$900	\$924.32	\$24.32
−950	0	−950	924.32	−25.68
−1000	0	−1000	924.32	−75.68
−1050	50	−1000	924.32	−75.68
−1100	100	−1000	924.32	−75.68
−1150	150	−1000	924.32	−75.68
−1200	200	−1000	924.32	−75.68

Panel (a) shows the payoff diagram for a short position in the index (column 1 in Table 3.2). Panel (b) shows the payoff diagram for a purchased index call with a strike price of \$1000 (column 2 in Table 3.2). Panel (c) shows the combined payoff diagram for the short index and long call (column 3 in Table 3.2). Panel (d) shows the combined profit diagram for the short index and long call, obtained by adding \$924.32 to the payoff diagram in panel (c) (column 5 in Table 3.2).



Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 1, pp. 62-64

10. List the strategies in which investors can be selling insurance

We can expect that some investors want to purchase insurance. However, for every insurance buyer there must be an insurance seller. Now, we examine strategies in which investors sell insurance.

It is possible, of course, for an investor to simply sell calls and puts. Often, however, investors also have a position in the asset when they sell insurance. Writing an option when there is a corresponding long position in the underlying asset is called covered writing, option overwriting, or selling a covered call. All three terms mean essentially the same thing. In contrast, naked writing occurs when the writer of an option does not have a position in the asset.

Covered call writing- if we own the S&R index and simultaneously sell a call option, we have written a covered call. A covered call will have limited profitability if the index increases, because an option writer is obligated to sell the index for the strike price. Should the index decrease, the loss on the index is offset by the premium earned from selling the call. A payoff with limited profit for price increases and potentially large losses for price decreases sounds like a written put.

Because the covered call looks like a written put, the maximum profit will be the same as with a written put. Suppose the index is \$1100 at expiration. The profit is:

$$\underbrace{\$1100 - (\$1000 \times 1.02)}_{\text{Profit on S\&R Index}} + \underbrace{(\$93.809 \times 1.02) - \$100}_{\text{Profit on Written Call}} = \$75.68$$

which is the future value of the premium received from writing a 1 000-strike put.

Covered puts– a covered put is achieved by writing a put against a short position on the index. The written put obligates you to buy the index-for a loss-if it goes down in price. Thus, for index prices below the strike price, the loss on the written put off sets the short stock. For index prices above the strike price, you lose on the short stock.

A position where you have a constant payoff below the strike and increasing losses above the strike sounds like a written call. In fact, shorting the index and writing a put produces a profit diagram that is exactly the same as for a written call.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 2, pp. 64-67

11. Is it possible to create synthetic forward position with options? Explain

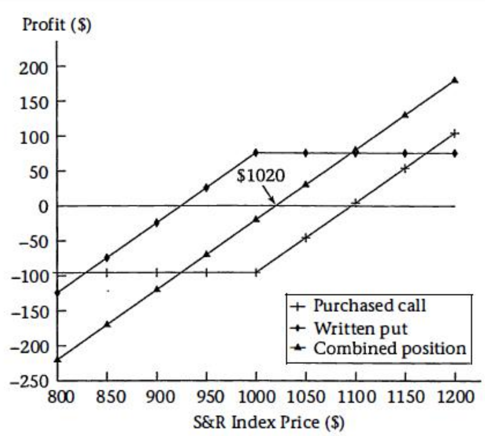
It is possible to mimic a long forward position on an asset by buying a call and selling a put, with each option having the same strike price and time to expiration. For example, we could buy the 1 000-strike S&R call and sell the 1 000-strike S&R put, each with 6months to expiration. In 6 months we will be obliged to pay $ 1 000 to buy the index, just as if we had entered into a forward contract.

For example, suppose the index in 6 months is at 900. We will not exercise the call, but we have written a put. The put buyer will exercise the right to sell the index for $1000; therefore we are obligated to buy the index at $1000. If instead the index is at $1100, the put is not exercised, but we exercise the call, buying the index for $1000. Thus, whether the index rises or falls, when the options expire we buy the index for the strike price of the options, $1000.

The purchased call, written put, and combined positions are shown in Figure below. The purchase of a call and sale of a put creates a synthetic long forward contract, which has two minor differences from the actual forward:

1. The forward contract has a zero premium, while the synthetic forward requires that we pay the net option premium.
2. With the forward contract we pay the forward price, while with the synthetic forward we pay the strike price.

Purchase of a 1000-strike S&R call, sale of a 1000-strike S&R put, and the combined position. The combined position resembles the profit on a long forward contract.



Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 2, pp. 64-67

12. Define put-call parity relationship in options

The net cost of buying the index using options must equal the net cost of buying the index using a forward contract. If at time 0we enter into a long forward position expiring at time *T*, we obligate ourselves to buying the index at the forward price, F_0^T. The present value of buying the index in the future is just the present value of the forward price, $PV(F_0^T)$.

If instead we buy a call and sell a put today to guarantee the purchase price for the index in the future, the present value of the cost is the net option premium for buying the call and selling the put, *Call(K, T) – Put(K, T)*, plus the present value of the strike price, *PV(K)*. (The notations "*Call(K, T)*" and "*Put(K, T)*" denote the premiums of options with strike price *K* and with I periods until expiration)

Equating the costs of the alternative ways to buy the index at time *t* gives us:

$$PV(F_0^T) = [Call(K, T) - Put(K, T)] + PV(K)$$

We can rewrite this as:

$$\boxed{\mathrm{Call}(K, T) - \mathrm{Put}(K, T) = \mathrm{PV}(F_{0,T} - K)}$$

In words, the present value of the bargain element from buying the index at the strike price[the right-hand side of the equation)] must be offset by the initial net option premium[the left-hand side of the equation]. Equation is known as put-call parity, and one of the most important relations in options.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 3, pp. 68-70

Simple trading strategies

13. What are Bull and Bear Spreads?

An option spread is a position consisting of only calls or only puts, in which some options are purchased and some written. Spreads are a common strategy.

Suppose you believe a stock will appreciate. You can enter into either a long forward contract or a call option to benefit from price increase. Long forward costs zero but expose you to the risk of loss, while call option cannot cause a negative payoff though it has a premium. On the other hand, spread strategies essentially serve as tools for lowering the cost of your strategy if you are willing to reduce your profit should the stock appreciate. You can do this by selling a call at a higher strike price. The owner of this second call buys appreciation above the higher strike price and pays you a premium. You achieve a lower cost by giving up some portion of profit. A position in which you buy a call and sell an otherwise identical call with a higher strike price is an example of a bull spread.

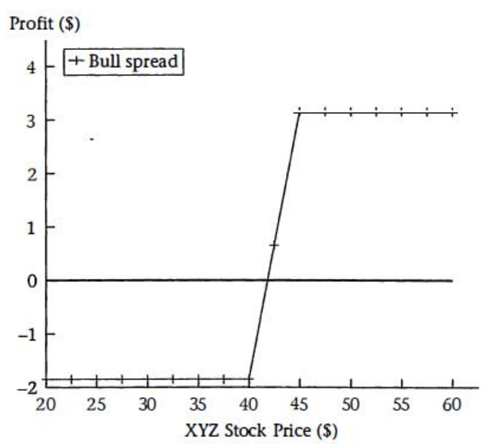
Bull spreads can also be constructed using puts. Perhaps surprisingly, you can achieve the same result either by buying a low-strike call and selling a high-strike call, or by buying a low-strike put and selling a high-strike put.

Spreads constructed with either calls or puts are sometimes called vertical spreads. The terminology stems from the way option prices are typically presented, with strikes arrayed vertically (as in Table below).

Profit at expiration from purchase of 40-strike call and sale of 45-strike call.

Stock Price at Expiration	Purchased 40-Call	Written 45-Call	Premium Plus Interest	Total
$35.0	$0.0	$0.0	−$1.85	−$1.85
37.5	0.0	0.0	−1.85	−1.85
40.0	0.0	0.0	−1.85	−1.85
42.5	2.5	0.0	−1.85	0.65
45.0	5.0	0.0	−1.85	3.15
47.5	7.5	−2.5	−1.85	3.15
50.0	10.0	−5.0	−1.85	3.15

Profit diagram for a 40–45 bull spread: buying a 40-strike call and selling a 45-strike call.



The opposite of a bull spread is a bear spread. Using the options from the above example, we could create a bear spread by selling the 40-strike call and buying the45-strike call. The profit diagram would be exactly the opposite of the graph above.

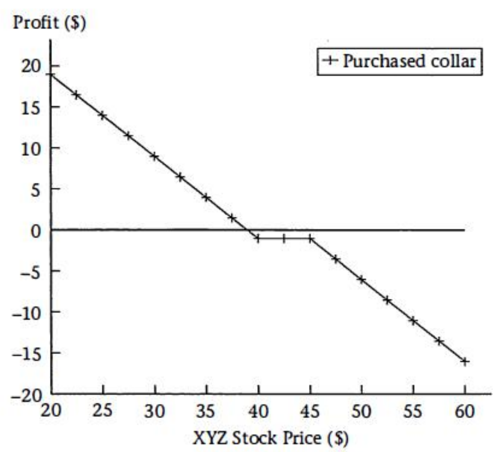
Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 3, pp. 71-72

14. What is a collar?

A collar is the purchase of a put option and the sale of a call option with a higher strike price, with both option shaving the same underlying asset and having the same expiration date. If the position is reversed (sale of a put and purchase of a call), the collar is written. The collar width is the difference between the call and put strikes.

Example: Suppose we sell a 45-strike call with a $0.97 premium and buy a 40-strike put with a $1.99premium. This collar is shown in Figure below. Because the purchased put has a higher premium than the written call, the position requires investment of $1.02.

Profit diagram of a purchased collar constructed by selling a 45-strike call and buying a 40-strike put.



If you hold this book at a distance and squint at Figure 3.8, the collar resembles a short forward contract. Economically, it is like a short forward contract in that it is fundamentally a short position: The position benefits from price decreases in the underlying asset and suffers losses from price increases. A collar differs from a short forward contract in having a range between the strikes in which the expiration payoff is unaffected by changes in the value of the underlying asset.

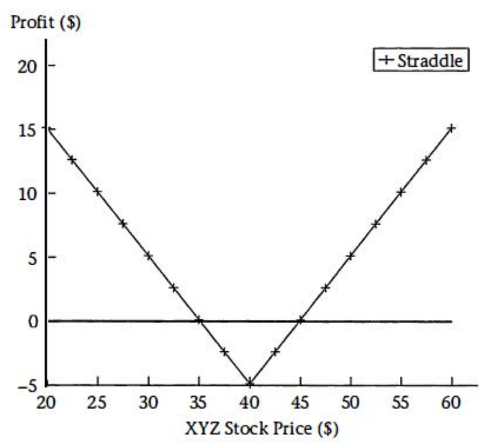
In practice collars are frequently used to implement insurance strategies-for example, by buying a collar when we own the stock. This position, which we will call a collared stock, entails buying the stock, buying a put, and selling a call. It is an insured position because we own the asset and buy a put. The sale of a call helps to pay for the purchase of the put. The collared stock looks like a bull spread; however, it arises from a different set of transactions. The bull spread is created by buying one option and selling another. The collared stock begins with a position in the underlying asset that is coupled with a collar.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 1, pp. 73-78

15. What are the strategies for speculation on volatility?

Besides directional positions like a bull spread or a collar which is a bet that the price of the underlying asset will increase, options can also be used to create positions that are non-directional with respect to the underlying asset. With anon-directional position, the holder does not care whether the stock goes up or down, but only how much it moves.

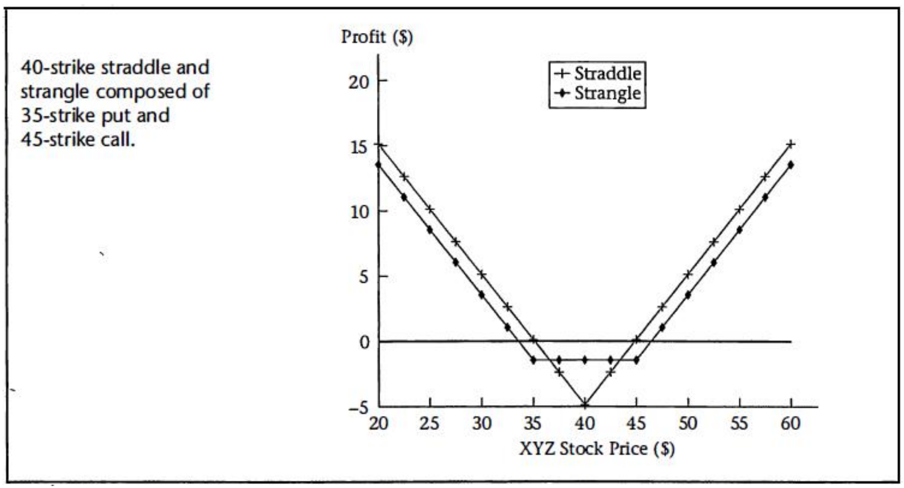
Combined profit diagram for a purchased 40-strike straddle—i.e., purchase of one 40-strike call option and one 40-strike put option.



Straddles Consider the strategy of buying a call and a put with the same strike price and time to expiration: This strategy is called a straddle. The general idea of a straddle is simple: If the stock price rises, there will be a profit on the purchased call, and if the stock price declines there will be a profit on the purchased put. Thus, the advantage of a straddle is that it can profit from stock price moves in both directions. The disadvantage to a straddle is that it has a high premium because it requires purchasing two options. Figure below demonstrates that a straddle is a bet that volatility will be high: The buyer of an at-the-money straddle is hoping that the stock price will move but does not care about the direction of the move.

Strangle The disadvantage of a straddle is the high premium cost. To reduce the premium, you can buy out-of-the-money options rather than at-the-money options. Such a position is called a strangle. For example, consider buying a 35-strike put and a 45-strike call, for a total premium of $1.41, with a future value of $1.44. These transactions reduce your maximum loss if the options expire with the stock near $40, but they also increase the stock-price move required for a profit.

Figure below shows the 40-strike straddle graphed against the 35-45 strangle. This comparison illustrates a key point: In comparing any two fairly priced option positions, there will always be a region where each outperforms the other. Indeed, this is necessary to have a fairly priced position.



Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 3, pp. 78-83

16. Explain basic risk management strategies from the producer's perspective / the buyer's perspective?

Business, like life, is inherently risky. Firms convert inputs such labor, raw materials, and machines into goods and services. A firm is profitable if the cost of what it produces exceeds the cost of the inputs. Prices can change, however, and what appears to be a profitable activity today may not be profitable tomorrow. Many instruments are available that permit firms to hedge various risks, ranging from commodity prices to weather. A firm that actively uses derivatives and other techniques to alter its risk and protect its profitability is engaging in risk management.

From producer's perspective risk management considers protection from price decline. Let's consider gold mining company. Suppose the gold price is \$350/oz. If gold miners produce no gold, the firms lose its fixed cost, \$330/oz. If they do produce gold, the firms have fixed cost of \$330/oz. and variable cost of \$50/oz., and so they lose \$350-(\$330+\$50) = -\$30/oz. It is better to lose only \$30, so gold will continue to be mined even when net income is negative. If the gold price were to fall below the variable cost of \$50, then it would make sense to stop producing.

Gold mining company can lock in a price for gold in 1 year by entering into a short forward contract, agreeing today to sell its gold for delivery in 1 year. Profit calculations when gold miners are hedged by forward contract are summarized in below table.

Golddiggers's net income one year from today, hedged with a forward sale of gold.

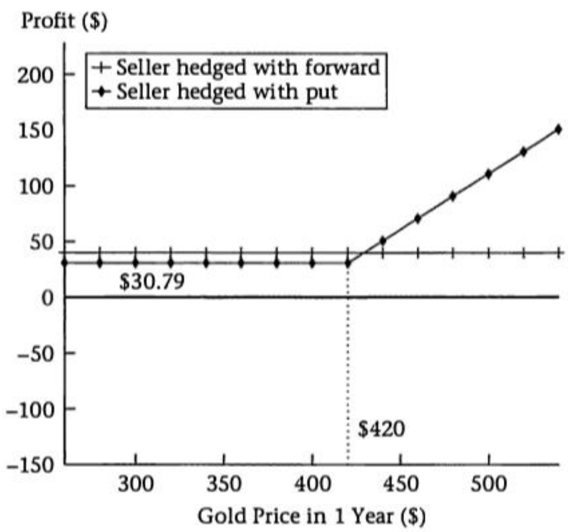
Gold Price in One Year	Fixed Cost	Variable Cost	Profit on Short Forward	Net Income on Hedged Position
\$350	−\$330	−\$50	\$70	\$40
\$400	−\$330	−\$50	\$20	\$40
\$450	−\$330	−\$50	−\$30	\$40
\$500	−\$330	−\$50	−\$80	\$40

A possible objection to hedging with a forward contract is that if gold prices do rise, gold miners will still receive only \$420/oz. There is no prospect for greater profit. Gold insurance with a put option-provides a way to have higher profits at high gold prices while still being protected against low prices. Suppose that the market price for a 420strike put is \$8.77/oz. This put provides a floor on the price. Since the put premium is paid 1 year prior to the option payoff, we must take into account interest cost when we compute profit in 1 year. The future value of the premium is \$8.77 x 1.05 = \$9.21. Table below shows the result of buying this put:

Golddiggers's net income 1 year from today, hedged with a 420-strike put option.

Gold Price in One Year	Fixed Cost	Variable Cost	Profit on Put Option	Net Income
$350	−$330	−$50	$60.79	$30.79
$400	−$330	−$50	$10.79	$30.79
$450	−$330	−$50	−$9.21	$60.79
$500	−$330	−$50	−$9.21	$110.79

Figure below compares the profit from the two protective strategies we have examined: Selling a forward contract and buying a put. As you would expect, neither strategy is clearly preferable; rather, there are trade-offs, with each contract outperforming the other for some range of prices:



From buyer's perspective risk management considers protection against price increase. For example, Auric Enterprises is a manufacturer of widgets, a product that uses gold as an input. We will suppose for simplicity that the price of gold is the only uncertainty Auric faces. In particular, we assume that:

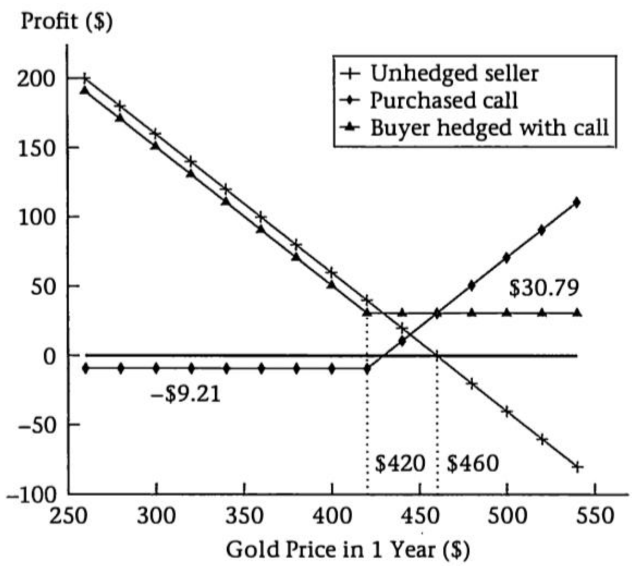
- Auric sells each widget for a fixed price of $800, a price known in advance.
- The fixed cost per widget is $340.
- The manufacture of each widget requires 1 oz. of gold as an input.
- The non-gold variable cost per widget is zero.
- The quantity of widgets to be sold is known in advance.

Because Auric makes a greater profit if the price of gold falls, Auric's gold position is implicitly short. The forward price is $420 as before. Auric can lock in a profit by entering into a long forward contract. Auric thereby guarantees a profit of

Profit = $800 – $340 – $420 = $40

Note that whereas gold miners were selling in the forward market, Auric is buying in the forward market. Thus, gold miners and Auric are natural counterparties in an economic sense.

Rather than lock in a price unconditionally, Auric might like to pay $420/oz. if the gold price is greater than $420/oz. but pay the market price if it is less. Auric can accomplish this by buying a call option. As a future buyer, Auric is naturally short; hence, a call is insurance. Suppose the call has a premium of $8.77/oz. The future value of the premium is $8.77 x 1.05 = $9.21. Figure below compares the profit from the two protective strategies we have examined for Auric: buying a forward contract and buying a call option:



Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 4, pp. 91-108

17. What is cross-hedging?

In the buyer's perspective example with Auric we assumed that widget prices are fixed. However, since gold is used to produce widgets, widget prices might vary with gold prices. If widget and gold prices vary one-for-one, Auric's profits would be independent of the price of gold and Auric would have no need to hedge.

More realistically, the price of widgets could change with the price of gold, but not one-for-one; other factors could affect widget prices as well. In this case, Auric might find it helpful to use gold derivatives to hedge the price of the widgets it sells as well as the price of the gold it buys. Using gold to hedge widgets would be an example of **cross hedging:** the use of a derivative on one asset to hedge another asset. Cross-hedging arises in many different contexts.

The hedging problem for Auric is to hedge the difference in the price of widgets and gold. Conceptually, we can think of hedging widgets and gold separately, and then combining those separate hedges into one net hedge. The ability to cross-hedge depends upon the correlation between the hedging instrument and the asset being hedged. We can determine the hedging amount as a regression coefficient. The same analysis is used with stock index futures contracts to cross-hedge a stock portfolio.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 4, pp. 114-116

18. Describe alternative ways to buy a stock?

The purchase of a share of XYZ stock has three components: (1) fixing the price, (2) the buyer making payment to the seller, and (3) the seller transferring share ownership to the buyer. If we allow for the possibility that payment and physical receipt can occur at different times, say time *0* and time *T*, then once the price is fixed there are four logically possible purchasing arrangements:

1. **Outright purchase:** The typical way to think about buying stock. You simultaneously pay the stock price in cash and receive ownership of the stock.
2. **Fully leveraged purchase:** A purchase in which you borrow the entire purchase price of the security. Suppose you borrow the share price, S_0, and agree to repay the borrowed amount at time T. If the continuously compounded interest rate is r, at time T you would owe e^{rT} per dollar borrowed, or S_0e^{rT}.
3. **Prepaid forward contract:** An arrangement in which you pay for the stock today and receive the stock at an agreed-upon future date. The difference between a prepaid forward contract and an outright purchase is that with the former, you receive the stock at time T.

4. **Forward contract:** An arrangement in which you both pay for the stock and receive it at time *T*, with the time *T* price specified at time *0*.

Table below depicts these four possibilities. It is clear that you pay interest when you defer payment. The interesting question is how deferring the physical receipt of the stock affects the price; this deferral occurs with both the forward and prepaid forward contracts.

Four different ways to buy a share of stock that has price S_0 at time 0. At time 0 you agree to a price, which is paid either today or at time T. The shares are received either at 0 or T. The interest rate is r.

Description	Pay at Time	Receive Security at Time	Payment
Outright purchase	0	0	S_0 at time 0
Fully leveraged purchase	T	0	$S_0 e^{rT}$ at time T
Prepaid forward contract	0	T	?
Forward contract	T	T	$? \times e^{rT}$

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 5, pp. 127-128

Forwards and Futures

19. Explain the concept of forward pricing. Write no-arbitrage bounds of forward prices. Give an interpretation of the forward pricing formula

Forwarding pricing is derived from analyzing prepaid forward prices and then adjusting for the only difference of timing of the stock. Thus, the forward price is just the future value of the prepaid forward.

A prepaid forward contract entails paying today to receive something – stocks, a foreign currency, or bonds – in the future. The sale of a prepaid forward contract permits the owner to sell an asset while retaining physical possession for a period of time. The prepaid forward price can be derived using three different methods: pricing by analogy, pricing by present value, and pricing by arbitrage.

Here are forward prices for stocks with discrete and continuous dividends:

Discrete dividends:

$$F_{0,T} = S_0 e^{rT} - \sum_{i=1}^{n} e^{r(T-t_i)} D_{t_i}$$

Continuous dividends:

$$F_{0,T} = S_0 e^{(r-\delta)T}$$

Tables below illustrate strategies used by market – makers and arbitrageurs to offset the risk of holding counter position with their customers.

Transactions and cash flows for a cash-and-carry: A market-maker is short a forward contract and long a synthetic forward contract.

	Cash Flows	
Transaction	**Time 0**	**Time *T* (expiration)**
Buy tailed position in stock, paying $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$+S_T$
Borrow $S_0e^{-\delta T}$	$+S_0e^{-\delta T}$	$-S_0e^{(r-\delta)T}$
Short forward	0	$F_{0,T} - S_T$
Total	0	$F_{0,T} - S_0e^{(r-\delta)T}$

Transactions and cash flows for a reverse cash-and-carry: A market-maker is long a forward contract and short a synthetic forward contract.

	Cash Flows	
Transaction	**Time 0**	**Time *T* (expiration)**
Short tailed position in stock, receiving $S_0e^{-\delta T}$	$+S_0e^{-\delta T}$	$-S_T$
Lend $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$+S_0e^{(r-\delta)T}$
Long forward	0	$S_T - F_{0,T}$
Total	0	$S_0e^{(r-\delta)T} - F_{0,T}$

A transaction in which you buy the underlying asset and short the offsetting forward contract is called a **cash-and-carry**. A cash-and-carry has no risk: You have an obligation to deliver the asset but also own the asset. The market-maker offsets the short forward position with a cash-and-carry. An arbitrage that involves buying the underlying asset and selling it forward is called a **cash-and-carry arbitrage**. As you might guess, a **reverse cash-and-carry** entails short-selling the index and entering into a long forward position

Tables above demonstrate that an arbitrageur can make a costless profit if FT $\neq S_0 e^{(r-\delta)T}$. This analysis ignores transaction costs. In practice an arbitrageur will face trading fees, bid-ask spreads, different interest rates for borrowing and lending, and the possibility that buying or selling in large quantities will cause prices to change. The effect of such costs will be that, rather than there being a single no-arbitrage price, there will be a no-arbitrage bound: a lower price F^- and an upper price F^+ such that arbitrage will not be profitable when the forward price is between these bounds.

Suppose that the stock and forward have bid and ask prices of $S^b < S^a$ and $F^b < F^a$, a trader faces a cost k of transacting in the stock or forward, and the interest rates for borrowing and lending are $r^b > r^1$. Incorporating above in arbitrageur's strategies, we get no-arbitrage bounds of forward prices:

$$F_{0,T} > F^+ = (S_0^a + 2k)e^{r^b T}$$

$$F_{0,T} < F^- = (S_0^b - 2k)e^{r^l T}$$

The forward pricing formula for a stock index, equation $S_0 e^{(r-\delta)T}$ depends on $r - \delta$, the difference between the risk – free rate and the dividend yield. This difference is called the cost of carry. Here is an interpretation of the forward pricing formula:

$$\text{Forward price} = \text{Spot price} + \underbrace{\text{Interest to carry the asset} - \text{Asset lease rate}}_{\text{Cost of carry}}$$

The forward contract, unlike the stock, requires no investment and makes no payouts and therefore has a zero cost of carry. One way to interpret the forward pricing formula is that, to the extent the forward contract saves our having to pay the cost of carry, we are willing to pay a higher price.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 5, pp. 128-141

20. List main differences between forwards and futures

Futures contracts are essentially exchange-traded forward contracts. As with forwards, futures contracts represent a commitment to buy or sell an underlying asset at some future date. Because futures are exchange-traded, they are standardized and have specified delivery dates, locations, and procedures. Futures may be traded either electronically or in trading pits, with buyers and sellers shouting orders to one another (this is called **open outcry**). Each exchange has an associated **clearinghouse**. The role of the clearinghouse is to match the buys and sells that take place during the day, and to keep track of the obligations and payments required of the members of the clearinghouse, who are called *clearing members*. After matching trades, the clearinghouse typically becomes the counterparty for each clearing member.

Although forwards and futures are similar in many respects, there are differences.

- Whereas forward contracts are settled at expiration, futures contracts are settled daily. The determination of who owes what to whom is called marking-to-market. Frequent marking-to-market and settlement of a futures contract can lead to pricing differences between the futures and an otherwise identical forward.
- As a result of daily settlement, futures contracts are liquid – it is possible to offset an obligation on a given date by entering into the opposite position. For example, if you are long the September S&P 500 futures contract, you can cancel your obligation to buy by entering into an offsetting obligation to sell the September S&P 500 contract. If you use the same broker to buy and to sell, your obligation is officially cancelled.
- Over-the-counter forward contracts can be customized to suit the buyer or seller, whereas futures contracts are standardized. For example, available futures contracts may permit delivery of 250 units of a particular index in March or June. A forward contract could specify April delivery of 300 units of the index.
- Because of daily settlement, the nature of credit risk is different with the futures contract. In fact, futures contracts are structured so as to minimize the effects of credit risk through **initial margin** and **maintenance margin** requirements from counterparties.
- There are typically daily price limits in futures markets (and on some stock exchanges as well). A **price limit** is a move in the futures price that triggers a temporary halt in trading. For example, there is an initial 5% limit on *down* moves in the S&P 500 futures contract. An offer to sell exceeding this limit can trigger a temporary trading halt, after which time a 10% price limit is in effect. If that is exceeded, there are subsequent 15% and 20% limits. The rules can be complicated, but it is important to be aware that such rules exist.

Figure below shows a quotation for the S&P 500 index futures contract with its specifications:

Specifications for the S&P 500 index futures contract.	Underlying	S&P 500 index
	Where traded	Chicago Mercantile Exchange
	Size	$250 × S&P 500 index
	Months	Mar, Jun, Sep, Dec
	Trading ends	Business day prior to determination of settlement price
	Settlement	Cash-settled, based upon opening price of S&P 500 on third Friday of expiration month

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 5, pp. 142-143

21. What are margins and marking-to-market?

Suppose the futures price is 1100 and you wish to acquire a $2.2 million position in the S&P 500 index. The notional value of one contract is $250 x 1100 = $275,000; this represents the amount you are agreeing to pay at expiration per futures contract. To go long $2.2 million of the index, you would enter into $2.2 million / $0.275 million = 8 long futures contracts. The notional value of 8 contracts is 8 x $250 x 1100 = $2, 000 x 1100 = $2.2 million.

A broker executes your buy order. For every buyer there is a seller, which means that one or more investors must be found who simultaneously agree to sell forward the same number of units of the index. The total number of open positions (buy/sell pairs) is called the open interest of the contract.

Both buyers and sellers are required to post a performance bond with the broker to ensure that they can cover a specified loss on the position. This deposit, which can earn interest, is called **margin** and is intended to protect the counterparty against your failure to meet your obligations. The margin is a performance bond, not a premium. Hence, futures contracts are costless (not counting, of course, commissions and the bid-ask spread).

To understand the role of margin, suppose that there is 10% margin and weekly settlement (in practice, settlement is daily). The margin on futures contracts with a notional value of $2.2 million is $220,000. If the S&P 500 futures price drops by 1, to 1099, we lose $2000 on our futures position. The reason is that 8 long contracts obligate us to pay $2000 x 1100 to buy2000 units of the index which we could now sell for only $2000 x 1099. Thus, we lose (1099-1100) x $2000 = -$2000.

We have a choice of either paying this loss directly; or allowing it to be taken out of the margin balance. It doesn't matter which we do since we can recover the unused margin balance plus interest at any time by selling our position.

The decline in the margin balance means the broker has significantly less protection should we default. For this reason, participants are required to maintain the margin at a minimum level, called the maintenance margin. This is often set at 70% to 80% of the initial margin level. Shall the margin balance decrease below maintenance, we would have to post additional margin. The broker would make a margin call, requesting additional margin. If we failed to post additional margin, the broker would close the position by selling 2000 units of the index, and return to us the remaining margin. In practice, marking-to-market and settling up are performed at least daily.

Since margin you post is the broker's protection against your default, a major determinant of margin levels is the volatility of the underlying asset. The minimum margin on the S&P 500 contract has generally been less than the 10% that we assume above. In August 2004, for example, the minimum margin on the S&P 500 futures contract was about 6% of the notional value of the contract.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 5, pp. 144-147

22. Explain asset allocation use of index futures

An index futures contract is economically like borrowing to buy the index. Why use an index futures contract if you can synthesize one? One answer is that index futures can permit trading the index at a lower transaction cost than actually trading a basket of the stocks that make up the index. If you are taking a temporary position in the index, either for investing or hedging, the transaction cost saving could be significant.

Asset allocation strategies involve switching investments among asset classes, such as stocks, money market instruments, and bonds. Trading the individual securities, such as the stocks in an index can be expensive. The practical implication of synthetic positions is that a portfolio manager can invest in a stock index without holding stocks, commodities without holding physical commodities, and so on.

As an example of asset allocation, suppose that we have an investment in the S&P 500 index and we wish to temporarily invest in T – bills instead of the index. Instead of selling all 500 stocks and investing in T – bills, we can simply keep our stock portfolio and take a short forward position in the S&P 500 index. This converts our cash investment in the index into a cash-and-carry, creating a synthetic T – bill. When we wish to revert to investing in stocks, we simply offset the forward position.

To illustrate this, suppose that the current index price, S_0, is \$100, and the effective 1-year risk-free rate is 10%. The forward price is therefore \$110. Suppose that in 1 year, the index price could be either \$80 or \$130. If we sell the index and invest in T – bills, we will have \$110 in 1 year. Table below shows that if, instead of selling, we keep the stock and short the forward contract, we earn a 10% return no matter what happens to the value of the stock. In this example 10% is the rate of return implied by the forward premium. If there is no arbitrage, this return will be equal to the risk-free rate.

Effect of owning the stock and selling forward, assuming that $S_0 = \$100$ and $F_{0,1} = \$110$.

	Cash Flows		
Transaction	**Today**	**1 year, $S_1 = \$80$**	**1 year, $S_1 = \$130$**
Own stock @ \$100	−\$100	\$80	\$130
Short forward @ \$110	0	\$110 − \$80	\$110 − \$130
Total	−\$100	\$110	\$110

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 5, pp. 150-151

23. Describe pricing of currency forwards

Currency futures and forwards are widely used to hedge against changes in exchange rates. The pricing of currency contracts is a straightforward application of the principles used in pricing prepaid forwards on stocks. Many corporations use currency futures and forwards for short-term hedging. An importer of consumer electronics, for example, may have an obligation to pay the manufacturer ¥150 million 90 days in the future. The dollar revenues from selling these products are likely known in the short run, so the importer bears pure exchange risk due to the payable being fixed in yen. By buying ¥150 million forward 90 days, the importer locks in a dollar price to pay for the yen, which will then be delivered to the manufacturer.

Suppose that 1 year from today you want to have ¥1. A prepaid forward allows you to pay dollars today to acquire ¥1 in 1 year. What is the prepaid forward price? Suppose the yen-denominated interest rate is r_y and the exchange rate today (\$/¥) is x_0. We can work backward. If we want ¥1 in 1 year, we must have e^{-r_y} in yen today. To obtain that many yen today, we must exchange $x_0 e^{-r_y}$ dollars into yen.

Thus, the prepaid forward price for a yen is:

$$F_{0,T}^{P} = x_0 e^{-r_y T}$$

The economic principle governing the pricing of a prepaid forward on currency is the same as that for a prepaid forward on stock. By deferring delivery of the underlying asset, you lose income. In the case of currency, if you received the currency immediately, you could buy a bond denominated in that currency and earn interest. The prepaid forward price reflects the loss of interest from deferring delivery, just as the prepaid forward price for stock reflects the loss of dividend income.

The prepaid forward price is the *dollar* cost of obtaining 1 yen in the future. Thus, to obtain the forward price, compute the future value using the dollar-denominated interest rate, r:

$$F_{0,T} = x_0 e^{(r-r_y)T}$$

The forward currency rate will exceed the current exchange rate when the domestic risk-free rate is higher than the foreign risk-free rate.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 5, pp. 154-156

24. What is the lease rate for a commodity?

Here is how a lender will think about a commodity loan: "If I lend the commodity, I am giving up possession of a unit worth S_0. At time T, I will receive a unit worth S_T. I am effectively making an investment of S_0 in order to receive the random amount S_T."

How would you analyze this investment? Suppose that α is the expected return on a stock that has the same risk as the commodity; α is therefore the appropriate discount rate for the cash flow S_T. The NPV of the investment is:

$$\text{NPV} = E_0(S_T)e^{-\alpha T} - S_0$$

Suppose that we expect the commodity price to increase at the rate g, so that:

$$E_0(S_T) = S_0 e^{gT}$$

Then the NPV of the commodity loan, without payments, is:

$$\text{NPV} = S_0 e^{(g-\alpha)T} - S_0$$

If $g<\alpha$, the commodity loan has a negative NPV. However, suppose the lender demands that the borrower return $e^{(\alpha - g)T}$ units of the commodity for each unit borrowed. If one unit is loaned, $e^{(\alpha - g)T}$ units will be returned. This is like a continuous proportional lease payment of $\alpha - g$ to the lender. Thus, the lease rate is the difference between the commodity discount rate and the expected growth rate of the commodity price, or:

$$\delta = \alpha - g$$

With this payment, the NPV of a commodity loan is:

$$\text{NPV} = S_0 e^{(\alpha-g)T} e^{(g-\alpha)T} - S_0 = 0$$

Now the commodity loan is a fair deal for the lender. The commodity lender must be compensated by the borrower for the opportunity cost associated with lending.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 6, pp. 178-179

25. How are implied forward rates calculated?

The implied forward rate is an implicit rate that can be earned from year 1 to year 2 that must be consistent with the other two rates. Suppose we could today guarantee a rate we could earn from year 1 to year 2. We know that $1 invested for 1 year earns [1 + r_0 *(0,1)*] and $1 invested for 2 years earns [1 + $r_0(0, 2)$]^2. Thus, the time 0 forward rate from year 1 to year 2, $r_0(1, 2)$, should satisfy:

$$[1 + r_0(0, 1)][1 + r_0(1, 2)] = [1 + r_0(0, 2)]^2$$

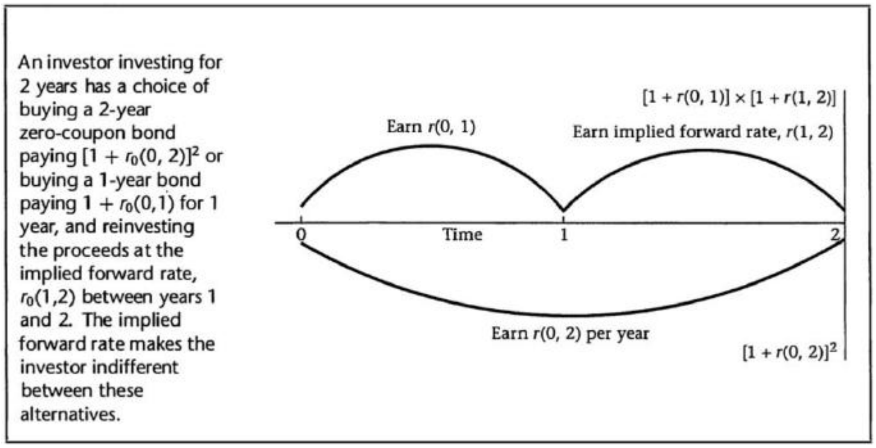
or

$$1 + r_0(1, 2) = \frac{[1 + r_0(0, 2)]^2}{1 + r_0(0, 1)}$$

In general, we have:

$$[1 + r_0(t_1, t_2)]^{t_2 - t_1} = \frac{[1 + r_0(0, t_2)]^{t_2}}{[1 + r_0(0, t_1)]^{t_1}} = \frac{P(0, t_1)}{P(0, t_2)}$$

Figure below shows graphically how the implied forward rate is related to 1-and 2-year yields:



Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 7, pp. 208-210

26. How coupon rate of par bond is calculated?

We can also compute the par coupon rate at which a bond will be priced at par. To describe a coupon bond, we need to know the date at which the bond is being priced, the start and end date of the bond payments, the number and amount of the payments, and the amount of principal.

`We will let $B_t(t_1, t_2, c, n)$ denote the time t price of a bond that is issued at t_1, matures at t_2, pays a coupon of c per dollar of maturity payment, and makes n evenly spaced payments over the life of the bond, beginning at time $t_1 + (t_2 - t_1) / n$. We will assume the maturity payment is \$1. If the maturity payment is different than \$1, we can just multiply all payments by that amount.

Since the price of a bond is the present value of its payments, at issuance time t the price of a bond maturing at T must satisfy:

$$B_t(t, T, c, n) = \sum_{i=1}^{n} c P_t(t, t_i) + P_t(t, T)$$

Where t_i = t + I (T – t)/n, with i being the index in the summation. Now, we can solve for the coupon as:

$$c = \frac{B_t(t, T, c, n) - P_t(t, T)}{\sum_{i=1}^{n} P_t(t, t_i)}$$

A par bond has $B_t = 1$, so the coupon on a par bond is given by:

$$c = \frac{1 - P_t(t, T)}{\sum_{i=1}^{n} P_t(t, t_i)}$$

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 7, pp. 210-211

27. Define and give an example of forward rate agreement (FRA)/ synthetic FRAs are created?

Let's consider the problem of a borrower who wishes to hedge against increases in the cost of borrowing. We consider a firm expecting to borrow $100 million for 91 days, beginning 120 days from today, in June. This is the borrowing date. The loan will be repaid in September on the loan repayment date. In the examples we will suppose that the effective quarterly interest rate at that time can be either 1.5% or 2%, and that the implied June 91-day forward rate (the rate from June to September) is 1.8%. Here is the risk faced by the borrower, assuming no hedging:

	120 days	211 days	
		$r_{\text{quarterly}} = 1.5\%$	$r_{\text{quarterly}} = 2\%$
Borrow $100m	+100m	−101.5m	−102.0m

Depending upon the interest rate, there is a variation of $0.5m in the borrowing cost. A forward rate agreement (FRA) is an over-the-counter contract that guarantees a borrowing or lending rate on a given notional principal amount. FRAs can be settled either at the initiation or maturity of the borrowing or lending transaction. If settled at maturity, we will say the FRA is settled in arrears. In the example above, the FRA could be settled on day 120, the point at which the borrowing rate becomes known and the borrowing takes place, or settled in arrears on day 211, when the loan is repaid.

FRAs are a forward contract based on the interest rate and as such does not entail the actual lending of money. Rather, the borrower who enters an FRA is paid if a reference rate is above the FRA rate, and the borrower pays if the reference rate is below the FRA rate. The actual borrowing is conducted by the borrower independently of the FRA. We will suppose that the reference rate used in the FRA is the same as the actual borrowing cost of the borrower.

FRA settlement in arrears: First consider what happens if the FRA is settled in September, on day 211, the loan repayment day. In that case, the payment to the borrower should be:

$$\left(r_{\text{quarterly}} - r_{\text{F A}}\right) \times \text{notional principal}$$

Thus, if the borrowing rate is 1.5%, the payment under the FRA should be:

(0.015-0.018) x \$100m= -\$300,000

Since the rate is lower than the FRA rate, the borrower pays the FRA counterparty.

Similarly, if the borrowing rate turns out to be 2.0%, the payment under the FRA should be:

(0.02-0.018) x \$100m= \$200,000

Settling the FRA in arrears is simple and seems like the obvious way for the contract to work. However, settlement can also occur at the time of borrowing.

FRA settlement in advance: If the FRA is settled in June, at the time the money is borrowed, payments will be less than when settled in arrears because the borrower has time to earn interest on the FRA settlement. In practice, therefore, the FRA settlement is tailed by the reference rate prevailing on the settlement (borrowing) date. (Tailing in this context means that we reduce the payment to reflect the interest earned between June and September.) Thus, the payment for a borrower is:

$$\text{Notional principal} \times \frac{\left(r_{\text{quarterly}} - r_{\text{FRA}}\right)}{1 + r_{\text{quarterly}}}$$

If $r_{quarterly}$ = 1.5%, the payment in June is:

$$\frac{-\$300{,}000}{1+0.015} = -\$295{,}566.50$$

By definition, the future value of this is -\$300,000. In order to make this payment, the borrower can borrow an extra \$295,566.50, which results in an extra \$300,000 loan payment in September. If on the other hand $r_{quarterly}$ = 2.0%, the payment is:

$$\frac{\$200{,}000}{1+0.02} = \$196{,}078.43$$

The borrower can invest this amount, which gives $200,000 in September, an amount that offsets the extra borrowing cost.

If the forward rate agreement covers a borrowing period other than 91 days, we simply use the appropriate rate instead of the 91-day rate in the above calculations.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 7, pp. 214-215

28. List specifications of Eurodollar futures

Eurodollar futures contracts are similar to FRAs in that they can be used to guarantee a borrowing rate. There are subtle differences between FRAs and Eurodollar contracts; however, these differences are important to understand. Figure below describes the Eurodollar contract based on a $1 million 3month deposit earning LIBOR (the London Interbank Offer Rate), which is the average borrowing rate faced by large international London banks. Suppose that current LIBOR is 1.5% over 3 months. By convention, this is annualized by multiplying by 4, so the quoted LIBOR rate is 6%. Assuming a bank borrows $1 million for 3 months, a change in annualized LIBOR of 0.01% (one basis point) would raise its borrowing cost by 0.0001/4 x $1 million= $25.

The Eurodollar futures price at expiration of the contract is:

100 – Annualized 3-month LIBOR

Specifications for the Eurodollar futures contract.

Where traded	Chicago Mercantile Exchange
Size	3-month Eurodollar time deposit, $1 million principal
Months	Mar, Jun, Sep, Dec, out 10 years, plus 2 serial months and spot month
Trading ends	5 A.M. (11 A.M. London) on the second London bank business day immediately preceding the third Wednesday of the contract month.
Delivery	Cash settlement
Settlement	100 − British Banker's Association Futures Interest Settlement Rate for 3-Month Eurodollar Interbank Time Deposits. (This is a 3-month rate annualized by multiplying by 360/90.)

Thus, if LIBOR is 6% at maturity of the Eurodollar futures contract, the final futures price will be 100 -6 = 94. It is important to understand that the Eurodollar contract settles based on current LIBOR, the interest rate quoted for the *next* 3 months. Thus, for example, the price of the contract that expires in June reflects the 3-month interest rate between June and September.

Like most money-market interest rates, LIBOR is quoted assuming a 360-day year. Thus, the annualized 91-day rate, r_{91}, can be extracted from the futures price, F, by computing the 90-day rate and multiplying by 91/90. The quarterly effective rate is then computed by dividing the result by 4:

$$r_{91} = (100 - F) \times \frac{1}{100} \times \frac{1}{4} \times \frac{91}{90}$$

To use Eurodollars for hedging against borrowing rate, one shall enter into short Eurodollar contracts that will result in following payoff at expiration:

$$short\ eurodollar\ futures\ payoff = [F - (100 - r_{LIBOR})] \times 100 \times \$25$$

Where

F – Eurodollar futures price

r_{LIBOR} *– current LIBOR rate*

Similarly, for hedging the lending rate, one might enter into long Eurodollar futures contracts and have the payoff at expiration of:

$$long\ eurodollar\ futures\ payoff = [(100 - r_{LIBOR}) - F] \times 100 \times \$25$$

It is highly important to note, that the Eurodollar futures price is a construct, not the price of an asset. In this sense Eurodollar futures are different from any other futures contracts.

LIBOR is quoted in currencies other than dollars, and comparable rates are quoted in different locations. In addition to LIBOR, there are PIBOR (Paris), TIBOR (Tokyo), and Euribor (the European Banking Federation).

Finally, you might be wondering why we are discussing LIBOR rather than rates on Treasury bills. Business and bank borrowing rates move more in tandem with LIBOR than with the government's borrowing rate. Thus, these borrowers use the Eurodollar futures contract to hedge. LIBOR is also a better measure of the cost of funds for a market-maker, so LIBOR is typically used to price forward contracts.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 5, pp. 158-160

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 7, pp. 218-223

29. What is the market value of a swap? Give the general formula for swap price. Define swap's implicit loan balance

A **swap** is a contract calling for an exchange of payments over time. One party makes a payment to the other depending upon whether a price turns out to be greater or less than a reference price that is specified in the swap contract. A swap thus provides a means to hedge a stream of risky payments. By entering into an oil swap, for example, an oil buyer confronting a stream of uncertain oil payments can lock in a fixed price for oil over a period of time. The swap payments would be based on the difference between a fixed price for oil and a market price that varies over time. From this description, you can see that there is a relationship between swaps and forward contracts. In fact, a forward contract is a single-payment swap.

To illustrate the general calculations for determining the swap rate, suppose there are n swap settlements, occurring on dates t_i, $i = 1, ..., n$. The implied forward interest rate from date t_{i-1} to date t_i, known at date 0, is $r_0(t_{i-1}, t_i)$. [We will treat $r_0(t_{i-1}, t_i)$ as not having been annualized.; i.e., it is the return earned from t_{i-1} *to* t_i]. The price of a zero-coupon bond maturing on date t_i is $P(0, t_i)$.

The market-maker can hedge the floating – rate payments using forward rate agreements. The requirement that the hedged swap have zero net present value is:

$$\sum_{i=1}^{n} P(0, t_i)[R - r_0(t_{i-1}, t_i)] = 0$$

Where there are n payments on dates $t_1, t_2, \ldots, t_n$. The cash flows $R - r_0(t_{i-1}, t_i)$can also be obtained by buying a fixed-rate bond paying R and borrowing at the floating rate. After solving for R and rearranging, we get:

$$R = \frac{1 - P_0(0, t_n)}{\sum_{i=1}^{n} P_0(0, t_i)}$$

The conclusion is that the swap rate is the coupon rate on a par coupon bond. This· result is intuitive since a firm that swaps from floating-rate to fixed-rate exposure ends up with the economic equivalent of a fixed-rate bond.

When the buyer first enters the swap, its market value is zero, meaning that either party could enter or exit the swap without having to pay anything to the other party (apart from commissions and bid-ask spreads). The forward contracts and forward rate agreement have zero value, so the swap does as well. Once the swap is struck, however, its market value will generally no longer be zero, for two reasons. First, the forward prices for oil and interest rates will change over time. New swaps would no longer have a fixed price; hence, one party will owe money to the other should one party wish to exit or unwind the swap.

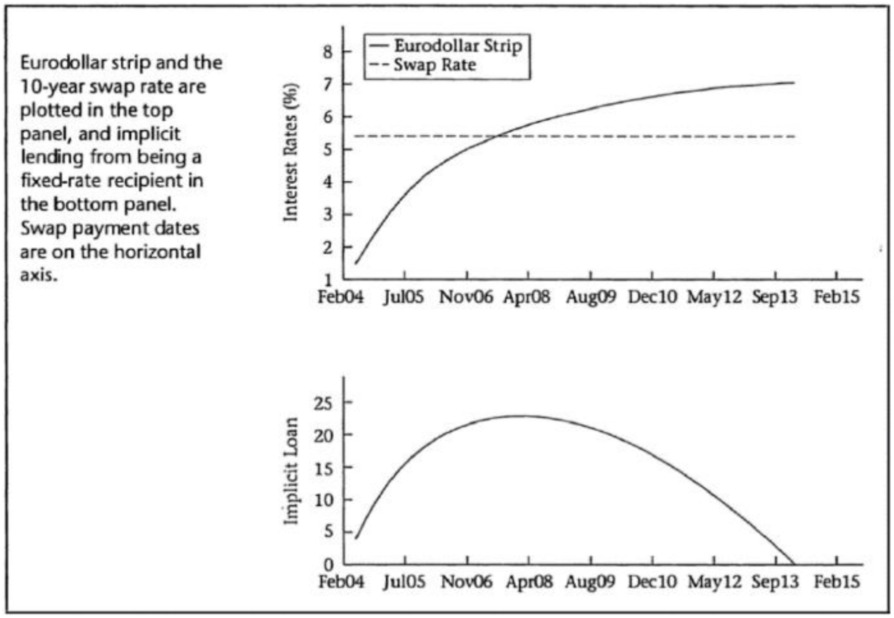
Second, even if oil and interest rate forward prices do not change, the value of the swap will remain zero only *until the first swap payment is made*. Once the first swap payment is made, the buyer has overpaid by some amount relative to the forward curve, and hence, in order to exit the swap, the counterparty would have to pay the oil buyer this amount. Thus, even if prices do not change, the market value of swaps can change over time due to the implicit borrowing and lending caused by fixed swap price structure compared to forward strip.

An interest rate swap behaves much like the oil swap. At inception, the swap has zero value to both parties. If interest rates change, the present value of the fixed payments and, hence, the swap rate will change. The market value of the swap is the difference in the present value of payments between the old swap rate and the new swap rate. For example, consider the 3-year swap in table below. If interest rates rise after the swap is entered into, the value of the existing 6.9548% swap will fall for the party receiving the fixed payment.

Cash flows faced by a floating-rate borrower who enters into a 3-year swap with a fixed rate of 6.9548%.

Year	Floating-Rate Debt Payment	Net Swap Payment	Net
1	-6%	$6\% - 6.9548\%$	-6.9548%
2	$-\tilde{r}_2$	$\tilde{r}_2 - 6.9548\%$	-6.9548%
3	$-\tilde{r}_3$	$\tilde{r}_3 - 6.9548\%$	-6.9548%

Even in the absence of interest rate changes, however, the swap in the table changes value over time. Once the first swap payment is made, the swap acquires negative value for the market-maker (relative to the use of forwards) because in the second year the market-maker will make net cash payment. Similarly, the swap will have positive value for the borrower (again relative to the use of forwards) after the first payment is made. In order to smooth payments, the borrower pays "too much" (relative to the forward curve) in the first year and receives a refund in the second year. *The swap is equivalent to entering into forward contracts and undertaking some additional borrowing and lending.* The implicit loan balance in the swap is illustrated in the figure below:



Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 8, pp. 247-261

Options Pricing

30. Explain early exercise for American options

When might we want to exercise an option prior to expiration? An important result is that an American call option on a non-dividend-paying stock should never be exercised prior to expiration. You may, however, rationally exercise an American-style put option prior to expiration.

Early exercise for calls: We can demonstrate that an American-style call option on a non-dividend-paying stock should never be exercised prior to expiration. Early exercise is not optimal if the price of an American call prior to expiration satisfies:

$$C_{\text{Amer}}(S_t, K, T-t) > S_t - K$$

If this inequality holds, you would lose money by early-exercising (receiving $S_t - K$) as opposed to selling the option and receiving more. We will use put – call parity to demonstrate that early exercise is not rational. If the option expires at T, parity implies that:

$$C_{\text{Eur}}(S_t, K, T) = \underbrace{S_t - K}_{\text{Exercise value}} + \underbrace{P_{\text{Eur}}(S_t, K, T-t)}_{\text{Insurance against } S_T < K} + \underbrace{K(1 - e^{-r(T-t)})}_{\text{Time value of money on } K} > S_t - K$$

Since the put price and the time value of money on the strike are both positive, this equation establishes that the European call option premium on a non-dividend-paying stock always is at least as great as $S_t - K$. We also know that American option always costs at least as much as European option, hence, we have:

$$C_{\text{Amer}} \geq C_{\text{Eur}} > S_t - K$$

So, we would lose money exercising an American call prior to expiration, as opposed to selling the option.

Early-exercising has three effects. First, we throw away the implicit put protection should the stock later move below the strike price. Second, we accelerate the payment of the strike price. A third effect is the possible loss from deferring receipt of the stock. However, when there are no dividends, we lose nothing by waiting to take physical possession of the stock.

Early exercise for puts: When the underlying stock pays no dividend, a call will not be early-exercised, but a put might be. To see that early exercise for a put can make economic sense, suppose a company is bankrupt and the stock price falls to zero. Then a put that would not be exercised until expiration will be worth $PV_{t.T}(K)$. If we could early-exercise, we would receive K, if the interest rate is positive, then $K>PV(K)$. Therefore, early exercise would be optimal in order to receive the strike price earlier.

We can also use a parity argument to understand this. The put will never be exercised as long as $P> K – S$. Parity for the put implies:

$$P(S_t, K, T-t) = C(S_t, K, T-t) - S_t + \text{PV}_{t.T}(K)$$

$P > K – S$

$$C(S_t, K, T-t) - S_t + \text{PV}_{t.T}(K) > K - S_t$$

Or

$$C(S_t, K, T-t) > K - \text{PV}_{t.T}(K)$$

If the call is sufficiently valueless (as in the above example of a bankrupt company), parity cannot rule out early exercise. This does not mean that we *will* early-exercise; it simply means that we cannot rule it out.

We can summarize this discussion of early exercise. When we exercise an option, we receive something (the stock with a call, the strike price with a put).

A necessary condition for early exercise is that we prefer to receive this something sooner rather than later. For calls, dividends on the stock are a reason to want to receive the stock earlier. For puts, interest on the strike is a reason to want to receive the strike price earlier. Thus, dividends and interest play similar roles in the two analyses of early exercise.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 9, pp. 294-297

31. Formulate replicating portfolio principle for one-period binomial option pricing model

The binomial option pricing model assumes that, over a period of time, the price of the underlying asset can move only up or down by a specified amount, that is, the asset price follows a binomial distribution. Given this assumption, it is possible to determine a no-arbitrage price for the option. Surprisingly, this approach, which appears at first glance to be overly simplistic, can be used to price options, and it conveys much of the intuition underlying more complex (and seemingly more realistic) option pricing models. It is hard to overstate the value of thoroughly understanding the binomial approach to pricing options.

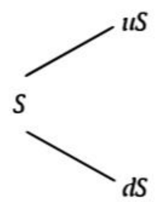
Binomial pricing achieves its simplicity by making a very strong assumption about the stock price: At any point in time, the stock price can change to either an up value or a down value. In-between, greater, or lesser values are not permitted. The restriction to two possible prices is why the method is called "binomial." The appeal of binomial pricing is that it displays the logic of option pricing in a simple setting, using only algebra to price options.

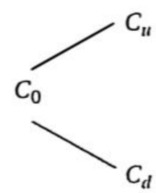
We have two instruments to use in replicating a call option: shares of stock and a position in bonds (i.e., borrowing or lending). To find the replicating portfolio, we need to find a combination of stock and bonds such that the portfolio mimics the option.

To be specific, we wish to find a portfolio consisting of Δ shares of stock and a dollar amount B in lending, such that the portfolio imitates the option whether the stock rises or falls. We will suppose that the stock has a continuous dividend yield of δ, which we reinvest in the stock. Thus, if you buy one share at time t, at time $t + h$ you will have $e^{\delta h}$ shares. The up and down movements of the stock price reflect the *ex-dividend* price.

We can write the stock price as $u\,S_0$ when the stock goes up and as $d\,S_0$ when the price goes down. We can represent the stock price tree as shown below. In this tree u is interpreted as one plus the rate of capital gain on the stock if it goes up, and d is one plus the rate of capital loss if it goes down. (If there are dividends, the total return is the capital gain or loss, plus the dividend.)

Let C_u, and C_d represent the value of the option when the stock goes up or down, respectively. The tree for the stock implies a corresponding tree for the value of the option shown below as well:





If the length of a period is h, the interest factor per period is e^{rh}. The problem is to solve for Δ and B such that our portfolio of Δ shares and B in lending duplicates the option payoff. The value of the replicating portfolio at time h, with stock price S_h, is

$$\Delta S_h + e^{rh} B$$

At the prices $S_h = dS$ and $S_h = uS$, a successful replicating portfolio will satisfy:

$$(\Delta \times dS \times e^{\delta h}) + (B \times e^{rh}) = C_d$$
$$(\Delta \times uS \times e^{\delta h}) + (B \times e^{rh}) = C_u$$

This is two equations in the two unknown's Δ and B. Solving for Δ and B gives:

$$\Delta = e^{-\delta h} \frac{C_u - C_d}{S(u - d)}$$

$$B = e^{-rh} \frac{uC_d - dC_u}{u - d}$$

Note that when there are dividends, the formula adjusts the number of shares in the replicating portfolio, Δ, to offset the dividend income.

Given the expressions for Δ and B, we can derive a simple formula for the value of the option. The cost of creating the option is the net cash required to buy the shares and bonds. Thus, the cost of the option is $\Delta S + B$. Finally, we have:

$$\Delta S + B = e^{-rh}\left(C_u \frac{e^{(r-\delta)h} - d}{u - d} + C_d \frac{u - e^{(r-\delta)h}}{u - d}\right)$$

The assumed stock price movements, u and d, should not give rise to arbitrage opportunities. In particular, we require that:

$$u > e^{(r-\delta)h} > d$$

Although probabilities are not needed for pricing the option, there is a probabilistic interpretation of equation for cost of an option. Notice that in equation the terms $(e^{(r-\delta)h} - d) / (u - d)$ and $(u - e^{(r-\delta)h})/(u - d)$ sum to 1 and are both positive (this follows from no-arbitrage condition stated above). Thus, we can interpret these terms as probabilities. Let

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

Option pricing equation then can be rewritten as:

$$C = e^{-rh}[p^* C_u + (1 - p^*)C_d]$$

This expression has the appearance of a discounted expected value. It is peculiar, though, because we are discounting at the risk-free rate, even though the risk of the option is at least as great as the risk of the stock (a call option is a leveraged position in the stock since $B< 0$). In addition, there is no reason to think that p^* is the true probability that the stock will go up; in general, it is not. We will call p^* the **risk-neutral probability** of an increase in the stock price.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 10, pp. 313-321

32. How to use binomial option pricing for American put options?

The binomial method easily accommodates put options also, as well as other derivatives. We compute put option prices using the same stock price tree and in almost the same way as call option prices; the only difference with a European put option occurs at expiration: Instead of computing the price as *max(0, S – K)*, we use *max(O, K –S)*.

In this case of the American option we should also check whether early exercise is optimal or not. The value of the option if it is left "alive" (i.e., unexercised) is given by the value of holding it for another period, equation for put option will be:

$$e^{-rh}\left[P(uS, K, t+h)p^* + P(dS, K, t+h)(1-p^*)\right]$$

The value of the option if it is exercised is given by *max(O, S – K)* if it is a call and *max(O, K – S)* if it is a put. Thus, for an American put, the value of the option at a node is given by:

$$P(S, K, t) = \max\left(K - S, e^{-rh}\left[P(uS, K, t+h)p^* + P(dS, K, t+h)(1-p^*)\right]\right)$$

Where

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

In general, the valuation of American options proceeds with checking at each node for early exercise. If the value of the option is greater when exercised, we assign that value to the node. Otherwise, we assign the value of the option unexercised. We work backward through the tree and the greater value of the option at each node ripples back through the tree resulting in at least as much value as for European style option.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 10, pp. 329-330

33. Explain the concept of risk-neutral pricing

The binomial option pricing formula can be written as:

$$C = e^{-rh}[p^*C_u + (1-p^*)C_d]$$

Where

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

We labeled p^* the risk-neutral probability that the stock will go up. The equation has the appearance of a discounted expected value, where the expected value calculation uses p^* and discounting is done at the risk – free rate.

It is common in finance to emphasize that investors are risk averse. To see what risk aversion means, suppose you are offered either (a) \$1000, or (b)

$2000 with probability 0.5, and $0 with probability 0.5. A **risk-averse** investor prefers (a), since alternative (b) is risky and has the same expected value as (a). This kind of investor will require a premium to bear risk when expected values are equal

A **risk-neutral** investor, on the other hand, is indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing. A risk-neutral investor, for example, will be equally happy with alternative (a) or (b).

Before proceeding, we need to emphasize that at no point are we assuming that investors are risk-neutral. Having said this, let's consider what an imaginary world populated by risk-neutral investors would be like. In such a world, investors care only about expected returns and not about riskiness. Assets would have no risk premium since investors would be willing to hold assets with an expected return equal to the risk-free rate.

In this hypothetical risk-neutral world, we can solve for the probability of the stock going up, p^*, such that the stock is expected to earn the risk – free rate. In the binomial model we assume that the stock can go up to uS or down to dS. If the stock is to earn the risk-free return on average, then the probability that the stock will go up, p^*, must satisfy:

$$p^* u S e^{\delta h} + (1 - p^*) d S e^{\delta h} = e^{rh} S$$

Solving for p^* gives:

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d}$$

This is exactly the definition of p^* in option pricing equation. This is why we refer top^* as the *risk-neutral probability that the stock price will group*. It is the probability that the stock price would increase in a risk –neutral world.

Not only would the risk-neutral probability be used in a risk neutral world, but also all discounting would take place at the risk-free rate. Thus, the option pricing formula can be said to price options as if investors are risk-neutral. At the risk of being repetitious, we are not assuming that investors are actually risk-neutral, and we are not assuming that risky assets are actually expected to earn the risk-free rate of return. Rather, *risk-neutral pricing* is an interpretation of the formulas above. Those formulas in turn arise from finding the cost of the portfolio that replicates the option payoff.

Interestingly, this interpretation of the option-pricing procedure has great practical importance; risk -neutral pricing can sometimes be used where other pricing methods are too difficult.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 10, pp. 320-321
Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 11, pp. 346-347

34. Write formulas for constructing Cox-Ross-Rubinstein Binomial Tree

The best – known way to construct a binomial tree is that in Cox et al. (1979), in which the tree is constructed as

$$u = e^{\sigma\sqrt{h}}$$
$$d = e^{-\sigma\sqrt{h}}$$

The Cox-Ross-Rubinstein approach is often used in practice. A problem with this approach, however, is that if $h$is large or σ is small, it is possible that $e^{rh}>e^{\sigma h}$, in which case the binomial tree violates the restriction of no – arbitrage condition. In real applications h would be small, so this problem does not occur.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 11, pp. 359

35. List properties of continuously compounded returns

Here is a summary of the important properties of continuously compounded returns:

The logarithmic function computes returns from prices Let S_t and S_{t+h} be stock prices at times t and $t + h$. The continuously compounded return between t and $t + h$, $r_{t,\,t+h}$ is then:

$$r_{t,t+h} = \ln(S_{t+h}/S_t)$$

The exponential function computes prices from returns if we know the continuously compounded return, we can obtain S_{t+h} by exponentiation of both sides of equation above. This gives:

$$S_{t+h} = S_t e^{r_{t,t+h}}$$

Continuously compounded returns are additive Suppose we have continuously compounded returns over a number of periods – for example, $r_{t,\, t+h}$, $r_{t+h,\, t+2h}$, etc. The continuously compounded return over a long period is the *sum* of continuously compounded returns over the shorter periods, i.e.

$$r_{t,t+nh} = \sum_{i=1}^{n} r_{t+(i-1)h,t+ih}$$

Continuously compounded returns can be less than -100%a continuously compounded return that is a large negative number still gives a positive stock price. The reason is that e^r is positive for any r. Thus, if the log of the stock price follows a random walk, the stock price cannot become negative.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 11, pp. 353-354

36. Explain estimation of the volatility of underlying asset

In practice we need to figure out what parameters to use in the binomial model. The most important decision is the value we assign to σ, which we cannot observe directly. One possibility is to measure σ by computing the standard deviation of continuously compounded historical returns. Volatility computed from historical stock returns is **historical volatility**.

Table below lists 3 weeks of Wednesday closing prices for the S&P 500 composite index and for IBM, along with the standard deviation of the continuously compounded returns, computed using the *StDev* function in Excel.

Over the 13-week period in the table, the weekly standard deviation was 0.0309 and 0.0365 for the S&P 500 index and IBM, respectively. These are weekly standard deviations since they are computed from weekly returns; they therefore measure the variability in weekly returns. We obtain annualized standard deviations by multiplying the weekly standard deviations by $\sqrt{52}$, giving annual standard deviations of 22.32% for the S&P 500 index and 26.32% for IBM.

Weekly prices and continuously compounded returns for the S&P 500 index and IBM, from 3/5/03 to 5/28/03.

Date	S&P 500 Price	S&P 500 $\ln(S_t/S_{t-1})$	IBM Price	IBM $\ln(S_t/S_{t-1})$
03/05/03	829.85	—	77.73	—
03/12/03	804.19	−0.0314	75.18	−0.0334
03/19/03	874.02	0.0833	82.00	0.0868
03/26/03	869.95	−0.0047	81.55	−0.0055
04/02/03	880.90	0.0125	81.46	−0.0011
04/09/03	865.99	−0.0171	78.71	−0.0343
04/16/03	879.91	0.0159	82.88	0.0516
04/23/03	919.02	0.0435	85.75	0.0340
04/30/03	916.92	−0.0023	84.90	−0.0100
05/07/03	929.62	0.0138	86.68	0.0207
05/14/03	939.28	0.0103	88.70	0.0230
05/21/03	923.42	−0.0170	86.18	−0.0288
05/28/03	953.22	0.0318	87.57	0.0160
Std. deviation		0.0309	—	0.0365
Std. deviation $\times\sqrt{52}$		0.2232	—	0.2632

The procedure outlined above is a reasonable way to estimate volatility when continuously compounded returns are independent and identically distributed. However, if returns are not independent – as with some commodities, for example – volatility estimation becomes more complicated. If a high price of oil today leads to decreased demand and increased supply, we would expect prices in the future to come down. In this case, the volatility over T years will be less than $\sigma\sqrt{T}$, reflecting the tendency of prices to revert from extreme values. Extra care is required with volatility if the random walk model is not a plausible economic model of the asset's price behavior.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 11, pp. 360-361

37. Write The Black-Scholes formula. List assumption of The Black-Scholes formula

To introduce the Black-Scholes formula, we first revise the binomial model. When computing a binomial option price, we can vary the number of binomial steps, holding fixed the time to expiration. Changing the number of steps changes the option price, but once the number of steps becomes great enough we appear to approach a limiting value for the price. We can't literally have infinity of steps in a binomial tree, but it is possible to show that as the number of steps approaches infinity, the option price is given by the Black-Scholes formula. Thus, the Black-Scholes formula is a limiting case of the binomial formula for the price of a European option.

The Black-Scholes formula for a European call and put options on a stock that pays dividends at the continuous rate δ is:

$$C(S, K, \sigma, r, T, \delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2)$$

$$P(S, K, \sigma, r, T, \delta) = Ke^{-rT} N(-d_2) - Se^{-\delta T} N(-d_1)$$

Where

$$d_1 = \frac{\ln(S/K) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

As with the binomial model, there are six inputs to the Black-Scholes formula: S, the current price of the stock; K, the strike price of the option; σ, the volatility of the stock; r the continuously compounded risk-free interest rate; T, the time to expiration; and δ, the dividend yield on the stock.

$N(x)$ in the Black-Scholes formula is the cumulative normal distribution function, which is the probability that a number randomly drawn from a standard normal distribution (i.e., a normal distribution with mean *0* and variance *1*) will be less than x. Most spreadsheets have a built-in function for computing $N(x)$. In Excel, the function is *NormSDist*.

Two of the inputs (K and T) describe characteristics of the option contract. The others describe the stock (S, σ, and δ) and the discount rate for a risk-free investment (r). All of the inputs are self-explanatory with the exception of volatility, which is the standard deviation of the rate of return on the stock – a measure of the uncertainty about the future return on the stock.

Derivations of the Black-Scholes formula make a number of assumptions that can be sorted into two groups: assumptions about how the stock price is distributed, and assumptions about the economic environment. For the version of the formula we have presented, assumptions about the distribution of the stock price include the following:

- Continuously compounded returns on the stock are normally distributed and independent over time (we assume there are no "jumps" in the stock price)

- The volatility of continuously compounded returns is known and constant
- Future dividends are known, either as a dollar amount or as a fixed dividend yield

Assumptions about the economic environment include these:

- The risk-free rate is known and constant
- There are no transaction costs or taxes
- It is possible to short-sell costlessly and to borrow at the risk-free rate

Many of these assumptions can easily be relaxed. For example, with a small change in the formula, we can permit the volatility and interest rate to vary over time in a known way.

As a practical matter, the first set of assumptions – those about the stock price distribution – are the most crucial. Most academic and practitioner research on option pricing concentrates on relaxing these assumptions. They will also be our focus when we discuss empirical evidence. You should keep in mind that almost *any* valuation procedure, including ordinary discounted cash flow, is based on assumptions that appear strong; the interesting question is how well the procedure works in practice.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 12, pp. 375-379

38. Define option Greeks and Greek measures for portfolios

Option Greeks are formulas that express the change in the option price when an input to the formula changes, taking as fixed all the other inputs. Specifically, the Greeks are mathematical derivatives of the option price formula with respect to the inputs. One important use of Greek measures is to assess risk exposure. For example, a market – making bank with a portfolio of options would want to understand its exposure to stock price changes, interest rates, volatility, etc. An options investor would like to know how interest rate changes and volatility changes affect profit and loss.

Keep in mind that the Greek measures by assumption change only one input at a time. In real life, we would expect interest rates and stock prices, for example, to change together. The Greeks answer the question, what happens when *one and only one* input changes? Here are list of some Greeks and their definitions:

- **Delta** (Δ) measures the option price change when the stock price increases by $1.
- **Gamma** (Γ) measures the change in delta when the stock price increases by $1.
- **Vega** measures the change in the option price when there is an increase in volatility of one percentage point.
- **Theta** (θ) measures the change in the option price when there is a decrease in the time to maturity of 1 day.
- **Rho** (ρ) measures the change in the option price when there is an increase in the interest rate of 1 percentage point (100 basis points).
- **Psi** (ψ) measures the change in the option price when there is an increase in the continuous dividend yield of 1 percentage point (100 basis points).

A useful mnemonic device for remembering some of these is that "vega" and "volatility" share the same first letter, as do "theta" and "time." Also "r" is often used to denote the interest rate and is the first letter in "rho."

The Greek measure of a portfolio is the sum of the Greeks of the individual portfolio components. This relationship is important because it means that the risk of complicated option positions is easy to evaluate. For a portfolio containing *n* options with a single underlying stock, where the quantity of each option is given by ω_i, we have:

$$\Delta_{\text{portfolio}} = \sum_{i=1}^{n} \omega_i \Delta_i$$

The same relation holds true for the other Greeks as well.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 12, pp. 382-389

39. Describe option elasticity

An option is an alternative to investing in the stock. Delta tells us the dollar risk of the option relative to the stock: If the stock price changes by $1, by how much does the option price change? The option elasticity, by comparison, tells us the risk of the option relative to the stock in percentage terms: If the stock price changes by 1%, what is the percentage change in the value of the option?

Dollar risk of option: If the stock price changes by ϵ, the change in the option price is:

$$\begin{aligned}\text{Change in option price} &= \text{Change in stock price} \times \text{option delta} \\ &= \epsilon \times \Delta\end{aligned}$$

Percentage risk of option: The **option elasticity** computes the percentage change in the option price relative to the percentage change in the stock price. The percentage change in the stock price is simply ϵ / S. The percentage change in the option price is the dollar change in the option price, $\epsilon\Delta$, divided by the option price, C**:**

$$\frac{\epsilon\Delta}{C}$$

The option elasticity, denoted by Ω, is the ratio of these two:

$$\Omega \equiv \frac{\text{\% change in option price}}{\text{\% change in stock price}} = \frac{\frac{\epsilon\Delta}{C}}{\frac{\epsilon}{S}} = \frac{S\Delta}{C}$$

The elasticity tells us the percentage change in the option for a 1% change in the stock. It is effectively a measure of the leverage implicit in the option. Elasticity can be used to compute option volatility and the risk premium of an option.

Option Volatility: The volatility of an option is the elasticity times the volatility of the stock:

$$\sigma_{\text{option}} = \sigma_{\text{stock}} \times |\Omega|$$

Where $|\Omega|$ is the absolute value of Ω.

The risk premium of an option: Since elasticity measures the percentage sensitivity of the option relative to the stock, it tells us how the risk premium of the option compares to that of the stock.

At a point in time, the option is equivalent to a position in the stock and in bonds; hence, the return on the option is a weighted average of the return on the stock and the risk-free rate. Let α denote the expected rate of return on the stock, γ the expected return on the option, and r the risk-free rate. We have:

$$\gamma = \frac{\Delta S}{C(S)}\alpha + \left(1 - \frac{\Delta S}{C(S)}\right) r$$

Since $\Delta S / C(S)$ is elasticity, this can be rewritten as:

$$\gamma - r = (\alpha - r) \times \Omega$$

Thus, the risk premium on the option equals the risk premium on the stock times Ω.

Using our earlier facts about elasticity, we conclude that if the stock has a positive risk premium, then a call always has an expected return at least as great as the stock and that, other things equal, the expected return on an option goes down as the stock price goes up. In terms of the capital asset pricing model, we would say that the option beta goes down as the option becomes more in-the-money. For puts, we conclude that the put always has an expected return less than that of the stock.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 12, pp. 389-395

40. Draw profit diagrams before maturity for a call option

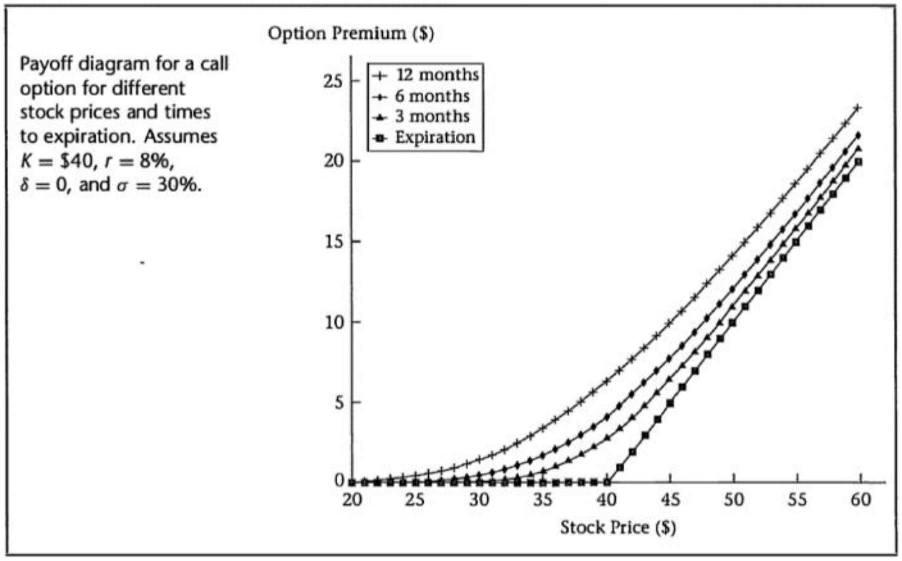
In order to evaluate investment strategies using options, we would like to be able to answer questions such as: If the stock price in 1 week is $5 greater than it is today, what will be the change in the price of a call option? What is the profit diagram for an option position in which the options have different times to expiration? To do this we need to use an option pricing formula.

Consider the purchase of a call option. Just as with expiring options, we can ask what the value of the option is at a particular point in time and for a particular stock price. The table below shows the Black-Scholes value of a call option for five different stock prices at four different times to expiration. By varying the stock price for a given time to expiration, keeping everything else the same, we are able to graph the value of the call.

Value of 40-strike call option at different stock prices and times to expiration. Assumes $r = 8\%$, $\sigma = 30\%$, $\delta = 0$.

	Time to Expiration			
Stock Price ($)	**12 Months**	**6 Months**	**3 Months**	**0 (Expiration)**
36	3.90	2.08	1.00	0
38	5.02	3.02	1.75	0
40	6.28	4.16	2.78	0
42	7.67	5.47	4.07	2
44	9.15	6.95	5.58	4

The figure below plots Black-Scholes call option prices for stock prices ranging from \$20 to \$60, including the values in the table above. Notice that the value of the option prior to expiration is a smoothed version of the value of the option at expiration.



Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 12, pp. 395-397

41. How to compute implied volatility of an underlying asset?

Volatility is unobservable, and though option prices, particularly for near-the-money options, can be quite sensitive to volatility. Thus, choosing a volatility to use in pricing an option is difficult but also quite important.

One approach to obtaining volatility is to use the history of returns to compute historical volatility. A problem with historical volatility is that history is not a reliable guide to the future: Markets have quiet and turbulent periods and predictable events such as Federal Reserve Board Open Market Committee meetings can create periods of greater than normal uncertainty. There are sophisticated statistical models designed to improve upon simple volatility estimates, but no matter what you do, you cannot count on history to provide you with a reliable estimate of future volatility.

In many cases we can observe option prices for an asset. We can then invert the question: Instead of asking what volatility we should use to price an option, we can compute an option's **implied volatility**, which is the volatility that would explain the observed option price. Assuming that we observe the stock price *S*, strike price *K*, interest rate *r*, dividend yield *δ*, and time to expiration *T*, the implied call volatility is the *ό* that solves:

$$\text{Market option price} = C(S, K, \hat{\sigma}, r, T, \delta)$$

By definition, if we use implied volatility to price an option, we obtain the market price of the option. Thus, we cannot use implied volatility to assess whether an option price is correct, but implied volatility does tell us the market's assessment of volatility.

Computing an implied volatility requires that we (1) observe a market price for an option and (2) have an option pricing model with which to infer volatility. Equation above cannot be solved directly for the implied volatility, *ό*, so it is necessary to use an iterative procedure to solve the equation. Any pricing model can be used to calculate an implied volatility, but Black-Scholes implied volatilities are frequently used as benchmarks.

Table below lists ask prices of calls and puts on the S&P 500 index, along with implied volatilities computed using the Black-Scholes formula. These S&P options are European style, so the Black-Scholes model is appropriate. Notice that, although the implied volatilities in the table are not all equal, they are all in a range between 13% and 16%. We could describe the general level of S&P option prices by saying that the options are trading at about a 15% volatility level. There are typically numerous options on a given asset; implied volatility can be used to succinctly describe the general level of option prices for a given underlying asset.

When you graph implied volatility against the strike price, the resulting line can take different shapes, often described as "smiles," "frowns," and "smirks”. This systematic change in implied volatility across strike prices occurs generally for different underlying assets and is called **volatility skew**.

When examining implied volatilities, it is helpful to keep put-call parity in mind. If options are European, then puts and calls with the same strike and time to expiration must have the same implied volatility. This is true because prices of European puts and calls must satisfy the parity relationship or else there is an arbitrage opportunity. Although call and put volatilities are not exactly equal in the table above, they are close enough that parity arbitrage would not be profitable after transaction costs are taken into account.

Implied volatilities for S&P 500 options, 10/28/2004. Option prices (ask) from **www.cboe.com**; assumes $S = \$1127.44$, $\delta = 1.85\%$, $r = 2\%$.

Strike ($)	Expiration	Call Price ($)	Implied Volatility	Put Price ($)	Implied Volatility
1100	11/20/2004	34.80	0.1630	6.80	0.1575
1125	11/20/2004	17.10	0.1434	14.70	0.1447
1150	11/20/2004	5.80	0.1284	29.20	0.1389
1100	12/18/2004	41.70	0.1559	13.80	0.1539
1125	12/18/2004	24.50	0.1396	22.50	0.1436
1150	12/18/2004	13.00	0.1336	35.50	0.1351
1100	1/22/2005	49.10	0.1567	20.40	0.1518
1125	1/22/2005	33.00	0.1463	29.40	0.1427
1150	1/22/2005	20.00	0.1363	41.50	0.1337

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 12, pp. 400-402

42. List and define types of exotic options

By altering the terms of standard contracts like options, futures, and swaps, you obtain a "nonstandard" or "exotic" option. Exotic options can provide precise tailoring of risk exposures, and they permit investment strategies difficult or costly to realize with standard options and securities. Some basic kinds of exotic options include Asian, barrier, compound, gap, and exchange options.

Asian Options: An **Asian option** has a payoff that is based on the average price over some period of time. An Asian option is an example of a **path-dependent option**, which means that the value of the option at expiration depends upon the path by which the stock arrived at its final price.

The payoff at maturity can be computed using the average stock price either as the price of the underlying asset or as the strike price. When the average is used as the asset price, the option is called an average price option. When the average is used as the strike price, the option is called an average strike option. Here are the eight variants of options based on the geometric and arithmetic average:

$$Arithmetic\ average\ price\ call = max[0, A(T) - K]$$
$$Geometric\ average\ price\ call = max[0, G(T) - K]$$
$$Arithmetic\ average\ price\ put = max[K - A(T), 0]$$
$$Geometric\ average\ price\ put = max[K - G(T)]$$
$$Arithmetic\ average\ strike\ call = max[0, S_T - A(T)]$$
$$Geometric\ average\ strike\ call = max[0, S_T - G(T)]$$
$$Arithmetic\ average\ strike\ put = max[0, A(T) - S_T]$$
$$Geometric\ average\ strike\ put = max[0, G(T) - S_T]$$

The terms "average price" and "average strike" refer to whether the average is used in place of the asset price or the strike price.

Barrier Options: A barrier option is an option with a payoff depending upon whether, over the life of the option, the price of the underlying asset reaches a specified level, called the barrier. Barrier puts and calls either come into existence or go out of existence the first time the asset price reaches the barrier. If they are in existence at expiration, they are equivalent to ordinary puts and calls.

Since barrier puts and calls never pay more than standard puts and calls, they are no more expensive than standard puts and calls. Barrier options are another example of a path-dependent option.

Barrier options are widely used in practice. One appeal of barrier options may be their lower premiums, although the lower premium of course reflects a lower average payoff at expiration. There are three basic kinds of barrier options:

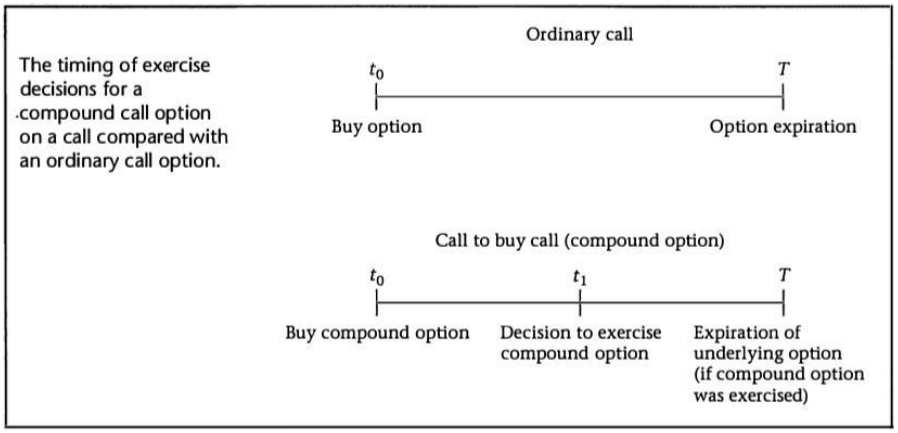
1. *Knock-out options:* These go out of existence (are "knocked-out") if the asset price reaches the barrier. If the price of the underlying asset has to fall to reach the barrier, the option is a down-and-out. If the price of the underlying asset has to rise to reach the barrier, the option is an up-and-out.
2. *Knock-in options:* These come into existence (are "knocked-in") if the barrier is touched. If the price of the underlying asset has to fall to reach the barrier, the option is a down-and-in. If the asset price has to rise to reach the barrier, it is an up-and-in.
3. *Rebate options:* These make a fixed payment if the asset price reaches the barrier. The payment can occur either at the time the barrier is reached, or at the time the option expires, in which case it is a deferred rebate. Rebate options can be either "up rebates" or "down rebates," depending on whether the barrier is above or below the current price.

The important parity relation for barrier options is:

"Knock-in" option + "Knock-out" option = Ordinary option

Compound Options: A compound option is an option to buy an option. If you think of an-ordinary option as an asset – analogous to a stock – then a compound option is similar to an ordinary option. Compound options are a little more complicated than ordinary options because there are two strikes and two expirations, one each for the underlying option and for the compound option.

Figure below compares the timing of the exercise decisions for *CallonCall* compound option with the exercise decision for an ordinary call expiring at time *T*.

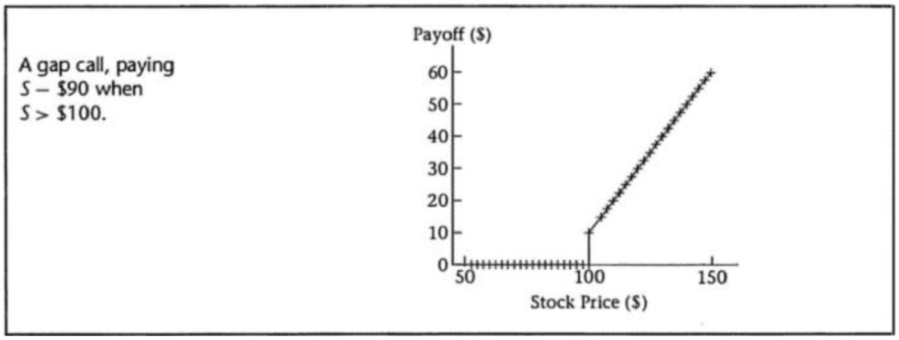


The four examples of compound options are:

- An option to buy a call (*CallOnCall*)
- An option to sell a call (*PutOnCall*)
- An option to buy a put (*CallOnPut*), and
- An option to sell a put (*PutOnPut*)

Gap Options: A call option pays $S – K$ when $S > K$. The strike price, K, here serves to determine both the range of stock prices where the option makes a payoff (when $S > K$) and also the size of the payoff ($S – K$). However, we could imagine separating these two functions of the strike price. Consider an option that pays $S – 90$ when $S > 100$. Note that there is a difference between the prices that govern when there is a payoff ($100) and the price used to determine the size of the payoff ($90). This difference creates a discontinuity – or gap – in the payoff diagram, which is why the option is called a **gap option.**

Figure below shows a gap call option with payoff $S – 90$ when $S > 100$. The gap in the payoff occurs when the option payoff jumps from $0 to $10 as a result of the stock price changing from $99.99 to $100.01.



Exchange Options: Exchange options are one kind of executives' compensation. Executive stock options are sometimes constructed so that the strike price of the option is the price of an index, rather than a fixed cash amount. The idea is to have an option that pays off only when the company outperforms competitors, rather than one that pays off simply because all stock prices have gone up. As a hypothetical example of this, suppose Bill Gates, chairman of Microsoft, is given compensation options that pay off only if Microsoft outperforms Google. He will exercise these options if and only if the share price of Microsoft, S_{MSFT}, exceeds the share price of Google, S_{GOOG}, i.e., $S_{MSFT}>S_{GOOG}$. From Gates's perspective, this is a call option, with the payoff:

$$max(0, S_{MSFT} - S_{GOOG})$$

Now consider the compensation option for Eric Schmidt, CEO of Google. He will receive a compensation option that pays off only if Google outperforms Microsoft, i.e.,

$$max(0, S_{GOOG} - S_{MSFT})$$

This is a call from Schmidt's perspective. Here is the interesting twist: Schmidt's Google call looks to Gates like a Microsoft put! And Gates's Microsoft call looks to Schmidt like a Google put. Either option can be viewed as a put or call; it is simply a matter of perspective. *The distinction between a put and a call in this example depends upon what we label the underlying asset and what we label as the strike asset.*

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 9, pp. 288-289
Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 14, pp. 443-461

Derivatives and Corporate Finance

43. Define Modigliani-Miller Theorem

The starting point for any discussion of modern financial engineering is the analysis of Franco Modigliani and Merton Miller (Modigliani and Miller, 1958). Before their work, financial analysts would puzzle over how to compare the values of firms with similar operating characteristics but different financial characteristics. Modigliani and Miller realized that different financing decisions (for example, the choice of the firm's debt-to-equity ratio) may carve up the firm's cash flows in different ways, but if the total cash flows paid to all claimants is unchanged, the total value of all claims would remain the same. They showed that if firms differing only in financial policy differed in market value, profitable arbitrage would exist. Using their famous analogy, the price of whole milk should equal the total prices of the skim milk and butterfat that can be derived from that milk.

The Modigliani-Miller analysis requires numerous assumptions: For example, there are no taxes, no transaction costs, no bankruptcy costs, and no private information. Nevertheless, the basic Modigliani-Miller result provided clarity for a confusing issue, and it created a starting point for thinking about the effects of taxes, transaction costs, and the like, revolutionizing finance.

All of the no – arbitrage pricing arguments embody the Modigliani-Miller spirit. For example, we could synthetically create a forward contract using options, a call option using a forward contract, bonds, and a put, and so forth. An option could also be synthetically created from a position in the stock and borrowing or lending. If prices of actual claims differ from their synthetic equivalents, arbitrage is possible.

Financial engineering is an application of the Modigliani-Miller idea. We can combine claims such as stocks, bonds, forwards, and options and assemble them to create new claims. The price for this new security is the sum of the pieces combined to create it. When we create a new instrument in this fashion, as in the Modigliani Miller analysis, value is neither created nor destroyed. Thus, financial engineering has no value in a pure Modigliani-Miller world. However, in real life, the new instrument may have different tax, regulatory, or accounting characteristics, or may provide a way for the issuer or buyer to obtain a particular payoff at lower transaction costs than the alternatives. Financial engineering thus provides a way to create instruments that meet specific needs of investors and issuers.

As a starting point, you can ask the following questions when you confront new financial instruments:

- What is the payoff of the instrument?
- Is it possible to synthetically create the same payoffs using some combination of assets, bonds, and options?
- Who might issue or buy such an instrument?
- What problem does the instrument solve?

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 15, pp. 473-474

44. How debt and equity can be viewed as options?

Firms often issue securities that have derivative components. For example, firms issue options to employees for financing, and convertible debt is a bond coupled with a call option. However, even simple securities, such as ordinary debt and equity, can be viewed as derivatives.

Consider a firm with the following very simple capital structure. The firm has non – dividend – paying equity outstanding, along with a single zero – coupon debt issue. Represent the time *t* values of the assets of the firm, the debt, and the equity as A_t, B_t, and E_t. The debt matures at time *T* and has maturity value *B*.

The value of the debt and equity at time *T* will depend upon the value of the firm's assets. Equity – holders are the legal owners of the firm; in order for them to have unambiguous possession of the firm's assets, they must pay the debt – holders *B* at time *T*. Therefore, the value of the equity will be:

$$E_T = \begin{cases} A_T - B, & if\ A_T > B \\ 0, & if\ A_T < B \end{cases}$$

This is analogous to the payoff to a call option with the assets of the firm as the underlying asset and *B* as the strike price:

$$E_T = max(0, A_T - B)$$

Because equity-holders control the firm, bondholders receive the smallest payment to which they are legally entitled. If the firm is bankrupt – i.e., if $A_T<B$ – the bondholders receive A_T. If the firm is solvent – i.e., if $A_T>$ B – the bondholders receive *B*. Thus, the value of the debt is:

$$B_T = min(A_T, B)$$

This can be rewritten as:

$$B_T = A_T + \min(0, \overline{B} - A_T)$$
$$= A_T - \max(0, A_T - \overline{B})$$

Equation above says that the bondholders own the firm, but have written a call option to the equity – holders. A different way to rewrite original equation for the debt value of the firm is:

$$B_T = \overline{B} + \min(0, A_T - \overline{B})$$
$$= \overline{B} - \max(0, \overline{B} - A_T)$$

The interpretation of above equation is that the bondholders own risk – free debt with a payoff equal to *B*, but have written a put option on the assets with strike price *B*.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 16, pp. 503-506

45. How to calculate expected return on debt and equity

We can compute the expected return on both debt and equity using the concept of option elasticity. Recall that the elasticity of an option tells us the relationship between the expected return on the underlying asset and that on the option. So, we can compute the expected return on equity as:

$$r_E = r + (r_A - r) \times \Omega_E$$

Where

r_A is the expected return1 on assets,

r is the risk-free rate, and

Ω_E is the elasticity of the equity

With

$$\Omega_E = \frac{A_t \Delta_E}{E_t}$$

Where

Δ_E is the option delta

We can compute the expected return on debt using the debt elasticity, Ω_B:

$$r_B = r + (r_A - r) \times \Omega_B$$

The elasticity calculation is slightly more involved for debt than for equity. Since we compute debt value as $B_t = A_t - E_t$. The elasticity of debt is a weighted average of the asset and equity elasticity:

$$\Omega_B = \frac{A_t}{A_t - E_t}\Omega_A - \frac{E_t}{A_t - E_t}\Omega_E$$

Using above equations you can verify that if you owned a proportional interest in the debt and equity of the firm, the expected return on your portfolio would be the expected return on the assets of the firm:

$$(\%\text{Equity} \times r_E) + (\%\text{Debt} \times r_B) = r_A$$

It bears emphasizing that this relationship requires that r_B represent the expected return on debt, not the yield to maturity.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 16, pp. 503-512

46. Write a lognormal model of stock prices

A random variable, y, is said to be lognormally distributed if *ln(y)* is normally distributed. Put another way, if x is normally distributed, y is lognormal if it can be written in either of two equivalent ways:

$$\ln(y) = x \quad or \quad y = e^x$$

How do we implement lognormality as a model for the stock price? If the stock price S_t is lognormal, we can write:

$$\frac{S_t}{S_0} = e^x$$

Where x is the continuously compounded return from 0 to t, is normally distributed. We want to find a specification for x that provides a useful way to think about stock prices.

Let the continuously compounded return from time t to some later time s be $R_{(t,\ s)}$. Suppose we have times $t_0<t_1<t_2$. By the definition of the continuously compounded return, we have:

$$S_{t_1} = S_{t_0}e^{R(t_0,t_1)}$$
$$S_{t_2} = S_{t_1}e^{R(t_1,t_2)}$$

The stock price at t_2 can therefore be expressed as

$$\begin{aligned} S_{t_2} &= S_{t_1} e^{R(t_1,t_2)} \\ &= S_{t_0} e^{R(t_0,t_1)} e^{R(t_1,t_2)} \\ &= S_{t_0} e^{R(t_0,t_1)+R(t_1,t_2)} \end{aligned}$$

Thus, the continuously compounded return from t_0 to t_1, $R(t_0, t_2)$, is the sum of the continuously compounded returns over the shorter periods:

$$R(t_0, t_2) = R(t_0, t_1) + R(t_1, t_2)$$

As we saw together with the assumption that returns are independent and identically distributed over time, implies that the mean and variance of returns over different horizons are proportional to the length of the horizon. Take the period of time from *0* to *T* and carve it up into 11 intervals of length *h*, where *h* = *T / n*. We can then write the continuously compounded return from *0* to *T* as the sum of the n returns over the shorter periods:

$$\begin{aligned} R(0, T) &= R(0, h) + R(h, 2h) + \cdots + R[(n-1)h, T] \\ &= \sum_{i=1}^{n} R[(i-1)h, ih] \end{aligned}$$

Let

$$E(R[(i-1)h, ih]) = \alpha_h \text{ and } \mathrm{Var}(R[(i-1)h, ih]) = \sigma_h^2$$

Then over the entire period, the mean and variance are:

$$\begin{aligned} E[R(0, T)] &= n\alpha_h \\ \mathrm{Var}[R(0, T)] &= n\sigma_h^2 \end{aligned}$$

Thus, if returns are independent and identically distributed, the mean and variance of the continuously compounded returns are proportional to time.

Now we have enough background to present an explicit lognormal model of the stock price. Generally, let *t* be denominated in years and *α* and *σ* be the annual mean and standard deviation, with *δ* the annual dividend yield on the stock. We will assume that the continuously compounded capital gain from *0* to*t*, $ln\ (S_t / S_0)$, is normally distributed with mean $(\alpha - \delta - 0.5\ \sigma^2)t$ and variance $\sigma^2 t$.

This gives us two equivalent ways to write an expression for the stock price. First, recall that we can convert a standard normal random variable, *z*, into one with an arbitrary mean or variance by multiplying by the standard deviation and adding the mean. We can write:

$$\ln(S_t/S_0) = (\alpha - \delta - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}z$$

Second, we can exponentiate above to obtain an expression for the stock price:

$$S_t = S_0 e^{(\alpha-\delta-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}z}$$

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 18, pp. 587-598

47. Write stock price probabilities under lognormal distribution

If S_t is lognormally distributed, we can use this fact to compute a number of probabilities and expectations. For example, we can compute the probability that an option will expire in the money, and, given that it expires in the money, the expected stock price.

If the stock price today is S_0, what is the probability that $S_t < K$, where K is some arbitrary number? Note that $S_t < K$ exactly when $ln(S_t) < ln(K)$. Since $ln(S)$ is normally distributed, we have:

$$\ln(S_t/S_0) \sim \mathcal{N}[(\alpha - \delta - 0.5\sigma^2)t, \sigma^2 t]$$

Or, equivalently,

$$\ln(S_t) \sim \mathcal{N}\left[\ln(S_0) + (\alpha - \delta - 0.5\sigma^2)t, \sigma^2 t\right]$$

We can create a standard normal number random variable, z, by subtracting the mean and dividing by the standard deviation:

$$z = \frac{\ln(S_t) - \ln(S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}$$

We have Prob ($S_t<K$) = Prob[$ln(S_t)<ln(K)$]. Subtracting the mean from both $ln(S_t)$ and $ln(K)$and dividing by standard deviation, we obtain:

$$\text{Prob}(S_t < K) =$$
$$\text{Prob}\left[\frac{\ln(S_t) - \ln(S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}} < \frac{\ln(K) - \ln(S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right]$$

Since the left-hand side is a standard normal random variable, the probability that $S_t < K$ is:

$$\text{Prob}(S_t < K) = \text{Prob}\left[z < \frac{\ln(K) - \ln(S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right]$$

Since $z \sim N(0, 1)$, Prob ($S_t<K$) is:

$$\text{Prob}(S_t < K) = N\left[\frac{\ln(K) - \ln(S_0) - (\alpha - \delta - 0.5\sigma^2)t}{\sigma\sqrt{t}}\right]$$

This can also be written as:

$$\text{Prob}(S_t < K) = N(-\hat{d}_2)$$

Where $\hat{d}_2$ is the standard Black – Scholes argument with the risk-free rate, r, replaced with the actual expected return on the stock, α. We can also perform the complementary calculation. We have Prob($S_t>K$) = 1– Prob($S_t<K$), so:

$$\text{Prob}(S_t > K) = N(\hat{d}_2)$$

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 18, pp. 599-600

48. Write conditional expected stock prices under lognormal stock prices distribution

Given that an option expires in the money, what is the expected stock price? The answer to this question is the conditional expected stock price. For a put with strike price K, we want to calculate $E\ (S_t \mid S_t<K)$, the expected stock price conditional on $S_t<K$. To compute this expectation, we need to take into account only the portion of the probability density representing stock prices below K.

The partial expectation of S_t, conditional on $S_t<K$, is:

$$\begin{aligned}\int_0^K S_t g(S_t; S_0)\,dS_t &= S_0 e^{(\alpha-\delta)t} N\left(\frac{\ln(K) - [\ln(S_0) + (\alpha - \delta + 0.5\sigma^2)t]}{\sigma\sqrt{t}}\right)\\ &= S_0 e^{(\alpha-\delta)t} N(-\hat{d}_1)\end{aligned}$$

Where $g\ (S_t; S_0)$ is the probability density of S_t conditional on S_0, and $\hat{d}_1$ is the Black – Scholes $d1$ equation with α replacing r.

The probability that $S_t<K$ is $N(-d2)$. Thus, the expectation of S_t conditional on $S_t<K$ is:

$$E(S_t|S_t < K) = Se^{(\alpha-\delta)t}\frac{N(-\hat{d}_1)}{N(-\hat{d}_2)}$$

For a call, we are interested in the expected price conditional on $S > K$. The partial expectation of S_t conditional on $S_t>K$ is:

$$\begin{aligned}\int_K^\infty S_t g(S_t;S_0)dS_t &= Se^{(\alpha-\delta)t}N\left(\frac{\ln(S_0)-\ln(K)+(\alpha-\delta+0.5\sigma^2)t}{\sigma\sqrt{t}}\right)\\ &= S_0e^{(\alpha-\delta)t}N(\hat{d}_1)\end{aligned}$$

As before, except for the fact that it contains the expected rate of return on the stock, α, instead of the risk – free rate, the second term is just the Black-Scholes expression, $N(d_1)$. The conditional expectation is:

$$E(S_t|S_t > K) = Se^{(\alpha-\delta)t}\frac{N(\hat{d}_1)}{N(\hat{d}_2)}$$

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 18, pp. 602-604

49. Derive the Black-Scholes formula

Using equations for calculating Prob ($S_t<K$), Prob ($S_t>K$), the expectation of S_t conditional on $S_t<K$ and expectation of S_t conditional on $S_t>K$, we can heuristically derive the Black – Scholes formula. Recall that the Black-Scholes formula can be derived by assuming risk-neutrality. In this case, the expected return on stocks, α, will equal r, the risk – free rate. If we let g^* denote the risk – neutral lognormal probability density, E^* denote the expectation taken with respect to risk-neutral probabilities, and Prob* denote these probabilities, the price of a European call option on a non – dividend paying stock will be:

$$\begin{aligned}C(S,K,\sigma,r,t,\delta) &= e^{-rt}\int_K^\infty (S_t-K)g^*(S_t;S_0)dS_t\\ &= e^{-rt}E^*(S-K|S>K)\times \text{Prob}^*(S>K)\end{aligned}$$

This can be rewritten as:

$$\begin{aligned}C(S,K,\sigma,r,t,\delta) = e^{-rt}E^*(S|S>K)\times \text{Prob}^*(S>K)\\ - e^{-rt}E^*(K|S>K)\times \text{Prob}^*(S>K)\end{aligned}$$

Substituting α with r, this becomes:

$$C(S, K, \sigma, r, t, \delta) = e^{-\delta t} SN(d_1) - Ke^{-rt} N(d_2)$$

This is the Black – Scholes formula.

Similarly, the formula for a European put option on a non – dividend paying stock is derived by computing:

$$P(S, K, \sigma, r, t, \delta) = e^{-rt} E^*(K - S|K > S) \times \text{Prob}^*(K > S)$$

This can be rewritten as:

$$P(S, K, \sigma, r, t, \delta) = e^{-rt} E^*(K|K > S) \times \text{Prob}^*(K > S) \\ - e^{-rt} E^*(S|K > S) \times \text{Prob}^*(K > S)$$

Substituting α with r, this becomes the Black – Scholes formula:

$$P(S, K, \sigma, r, t, \delta) = Ke^{-rt} N(-d_2) - e^{-\delta t} SN(-d_1)$$

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 18, pp. 604-605

50. Explain Monte Carlo Valuation of a European Call

A lognormal stock price can be written as:

$$S_T = S_0 e^{(\alpha - \delta - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z}$$

Suppose we wish to draw random stock prices for 2 years from today. From lognormal stock price equation, the stock price is driven by the normally distributed random variable Z. Set $T = 2$, $\alpha = 0.10$, $\delta = 0$, and $\sigma = 0.30$. If we then randomly draw a set of standard normal Z's and substitute the results into lognormal stock price, the result is a random set of lognormally distributed S_2's. The continuously compounded mean return will be 20% (10% per year) and the continuously compounded standard deviation of $ln\ (S_2)$ will be 0.3 x $\sqrt{2}$ = 42.43%.

In Monte Carlo valuation, we perform similar calculation to that above for lognormal stock price. The option payoff at time T is a function of the stock price, S_t. Representing this payoff as $V(S_T,T)$. The time – 0 Monte Carlo prices, $V\ (S_0, 0)$ is then:

$$V(S_0, 0) = \frac{1}{n} e^{-rT} \sum_{i=1}^{n} V(S_T^i, T)$$

Where $S_t^1,\ldots,S_T^n$ are n randomly drawn time – T stock prices. For the case of a call option, for example, $V(S_T^i, T) = max(0, S_T^i - T)$.

As an illustration of Monte Carlo techniques, let's consider pricing of a European call option. We assume that the underlying stock follows lognormal distribution with $\alpha = r$. We generate random standard normal variables, Z, substitute them into equation for lognormal stock price and generate many random future stock prices. Each Z creates one trial. Suppose we compute N trials. For each trial, i, we compute the value of a call as:

$$\max(0, S_T^i - K) = \max\left(0, S_0 e^{(r-\delta-0.5\sigma^2)T+\sigma\sqrt{T}Z_i} - K\right); \qquad i = 1, \ldots, N$$

Average the resulting values:

$$\frac{1}{N}\sum_{i=1}^{N}\max(0, S_T^i - K)$$

This expression gives us an estimate of the expected option payoff at time T. We discount the average payoff back at the risk-free rate in order to get an estimate of the option value, getting:

$$\overline{C} = e^{-rT}\frac{1}{N}\sum_{i=1}^{N}\max(0, S_T^i - K)$$

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 19, pp. 623-626

51. Explain Control Variate Method for Monte Carlo Valuation

"Naive" Monte Carlo is often used term for describing simplest simulation without making an attempt to reduce the variance of the simulated answer for a given number of trials. There are a number of methods to achieve faster Monte Carlo valuations.

Control Variate Method is use to increase Monte Carlo accuracy. The idea underlying this method is to estimate the error on each trial by using the price of a related option that does have a pricing formula. The error estimate obtained from this control price can be used to improve the accuracy of the Monte Carlo price on each trial.

To be specific, we use simulation to estimate the arithmetic price, A', and the geometric price, G'. Let G and A represent the true geometric and arithmetic prices. The error for the Monte Carlo estimate of the geometric price is (G – G’). We want to use this error to improve our estimate of the arithmetic price. Consider calculating:

$$A^* = \overline{A} + \left(G - \overline{G}\right)$$

This is a control variate estimate. Since Monte Carlo provides an unbiased estimate, *E (G’) = G*. Hence, *E (A*) = E(A;) =A*. Moreover, the variance of *A** is:

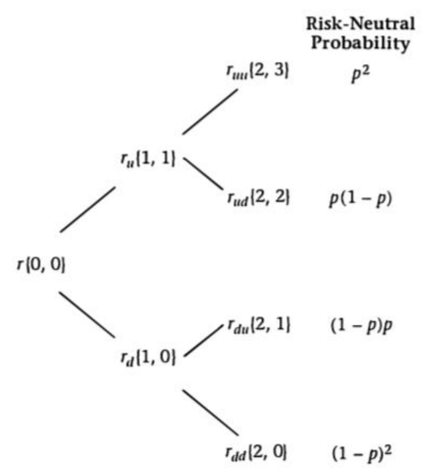
$$\text{Var}(A^*) = \text{Var}(\overline{A}) + \text{Var}(\overline{G}) - 2\text{Cov}(\overline{A}, \overline{G})$$

As long as the estimate *G’* is highly correlated with the estimate *A’*, the variance of the estimate *A** can be less than the variance of A’. In practice, the variance reduction from the control variate method can be dramatic.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 19, pp. 624-626

52. Construct a binomial interest rate model

Binomial interest rate models permit the interest rate to move randomly over time. One approach is to model the short-term rate, where the definition of short-term is, *h*, the length of the binomial period. Let’s assume *h*= 1. To construct a binomial tree of the 1-year rate, note that we can observe today's 1 – year rate. We assume the 1 – year rate moves up or down the second year, and again the third year. This behavior gives us the tree in Figure below, which is drawn so that it need not recombine.



Three-period interest rate tree showing the evolution of the continuously compounded 1-year rate. The state at each node (represented by time index *i* and level index *j*) are in braces at each node. The risk-neutral probability of an up move is *p*.

The notation used for interest rate trees is $r_{t0}(t, T)$ as the forward interest rate at time t_0 for time t to time T. This notation accounts for the fact that at a point in time, there is a set of forward interest rates at different future times (t) and covering different times to maturity ($T - t$). When $t_0 = t$, $r_t\,(t, T)$ is the set of current spot interest rates for different times to maturity.

At time 0 we can determine a bond price on the binomial tree in much the same way we determined option prices in a binomial stock – price tree. The one – period bond price at any time is determined by discounting at the current one – period rate, which is given at each node:

$$P_i(i, i+1; j) = e^{-r_i(i,i+1;j)h}$$

We can value a two – period bond by discounting the expected one – period bond price, one period hence. At any node we can value an n – period zero – coupon bond by proceeding in this way recursively. Beginning in period $i + n$, we value one – period bonds, then in period $i + n - 1$ we have two – period bond values, and so forth.

For the one-period bond we have:

$$P_i(0,1;0) = e^{-rh}$$

The two – year bond is priced by working backward along the tree. In the second period, the price of the bond is $1. One year from today, the bond will have the price e^{-ru} with probability p or e^{-rd} with probability $1 - p$. The price of the bond is therefore:

$$\begin{aligned} P_i(0,2;0) &= e^{-rh}[pe^{-r_u h} + (1-p)e^{-r_d h}] \\ &= e^{-rh}[pP_1(1,2;1) + (1-p)P_1(1,2;0)] \end{aligned}$$

Thus, we can price the 2 – year bond using either the interest rate tree or the implied bond prices. Finally, the 3 – year bond is again priced by traversing the entire tree:

$$\begin{aligned} P_i(0,3;0) = e^{-r}[&pe^{-r_u}(pe^{-r_{uu}} + (1-p)e^{-r_{ud}}) \\ &+ (1-p)e^{-r_d}(pe^{-r_{du}} + (1-p)e^{-r_{dd}})] \end{aligned}$$

The 3 – year bond calculation can be written differently. By collecting terms in above equation, we can rewrite it as:

$$\begin{aligned} P_0(0,3;0) = {} & p^2 e^{-(r+r_u+r_{uu})} + p(1-p)e^{-(r+r_u+r_{ud})} \\ & + (1-p)pe^{-(r+r_d+r_{du})} + (1-p)^2 e^{-(r+r_d+r_{dd})} \end{aligned}$$

This version of the equation makes clear that we can value the bond by considering separately each *path* the interest rate can take. Each path implies a realized discount factor. We then compute the expected discount factor, using risk – neutral probabilities. Denoting this expectation as E^*, the value of the zero – coupon bond is:

$$E^*\left(e^{-(r_0+r_1+r_2)h}\right)$$

More generally, letting r_i represent the time – i rate, we have:

$$E^*\left(e^{-\sum_{i=0}^{n} r_i h}\right)$$

All bond valuation models implicitly calculate this equation.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 24, pp. 793-798

53. What is optimal hedge ratio for cross-hedging?

Suppose that we can produce one widget with λ ounces of gold. If we produce N_w widgets at a price of P_w and do not hedge, profit is:

$$Profit = N_w P_w - \lambda N_w P_g$$

where P_g is the future spot price of gold.

Now suppose that we hedge by going long H gold futures contracts, with each contract covering q oz of gold. If F is the gold forward price, the profit on the hedged position is

$$\begin{aligned} Hedged\ Profit &= N_w P_w - \lambda N_w P_g + Hq\left(P_g - F\right) \\ &= N_w P_w + (Hq - \lambda N_w)P_g - H_q F \end{aligned}$$

The variance of hedged profit is

$$\sigma_{hedged}^2 = N_w^2\sigma_w^2 + (Hq - \lambda N_w)^2\sigma_g^2 + 2N_w(Hq - \lambda N_w)\rho\sigma_w\sigma_g$$

where σ_w is the standard deviation of the widget price, σ_g is the standard deviation of the gold price, and ρ is the price correlation between widgets and gold.

By differentiating with respect to H and setting equation to 0, we can solve for H that minimizes the variance of hedged profit. The variance-minimizing hedge position, H^*, is

$$qH^* = \lambda N_w - N_w\frac{\rho\sigma_w}{\sigma_g}$$

When we hedge with H^* futures, the variance of hedged position becomes

$$\sigma^2_{hedged} = N_w^2 \sigma_w^2 (1 - \rho^2)$$

The uncertainty remaining in the hedged position is due to basis risk, which is risk due to the hedging instrument (gold) and the hedged price (widgets) not moving as predicted. The variance of profits ultimately depends upon the ability to hedge the price of widgets, which, since we are using gold to hedge, depends on the correlation, ρ, between widgets and gold. The larger that ρ is, the less is basis risk.

Derivatives Markets, 3rd ed., Robert L. McDonald; Pearson, 2013; Chapter 4, pp. 112-114

54. What is optimal hedge ratio for quantity uncertainty?

The quantity a firm produces and sells may vary with the prices of inputs or outputs. When this happens, using the "obvious" hedge ratio (for example, by hedging the expected quantity of an input) can increase rather than decrease risk.

Agricultural producers commonly face quantity uncertainty because crop size is affected by factors such as weather and disease. Moreover, we expect there to be a correlation between quantity and price, because good weather gives rise to bountiful harvests. What quantity of forward contracts should a corn producer enter to minimize the variability of revenue?

Let S and Q denote the price and quantity in 1 year. Revenue is SQ without hedging. In general, if the producer enters into forward contracts on H units, hedged revenue, $R(H)$, will be

$$Hedged\ Revenue = R(H) = SQ + H(S - F)$$

When there is uncertainty, the variability of hedged revenue, $\sigma^2_{R(H)}$, is

$$\sigma^2_{R(H)} = \sigma^2_{SQ} + H^2\sigma^2_S + 2H\rho_{SQ,S}\sigma_S\sigma_{SQ}$$

The standard deviation of total revenue, SQ, is σ_{SQ}, and the correlation of total revenue with price is ρ_{SQ}, S. As in the preceding questions' discussion of cross-hedging, the H that minimizes the variance of hedged revenue will be

$$H = -\frac{\rho_{SQ,S}\sigma_{SQ}}{\sigma_S}$$

This formula for the variance-minimizing hedge ratio is the negative of the coefficient from a regression of unhedged revenue on price. We can therefore determine the variance-minimizing hedge ratios either by using above equation directly or by running a regression of revenue on price.

When correlation is positive, the optimal hedge quantity exceeds expected quantity. The fact that quantity goes up when price goes up makes revenue that much more variable than when price alone varies, and a correspondingly larger hedge position is required.

Derivatives Markets, 3rd ed., Robert L. McDonald; Pearson, 2013; Chapter 4, pp. 114-117

55. Explain the concept of cheapest-to-deliver arising in Treasury-bond and Treasury-note futures contracts

The Treasury-note and Treasury-bond futures contracts are important instruments for hedging interest rate risk. The specifications for the T-note contract are listed in the figure below. The bond contract is similar except that the deliverable bond has a maturity of at least 15 years, or if the bond is callable, has 15 years to first call.

Description	8-Year 7% Coupon, 6.4% Yield	7-Year 5% Coupon, 6.3% Yield
Market price	103.71	92.73
Price at 6% (conversion factor)	106.28	94.35
Invoice price (futures × conversion factor)	103.71	92.09
Invoice − market	0	−0.66

The basic idea of the T-note contract is that a long position is an obligation to buy a 6% bond with between 6.5 and 10 years to maturity. To a first approximation, we can think of the underlying as being like a stock with a dividend yield of 6%. The futures price would then be computed as with a stock index: the future value of the current bond price, less the future value of coupons payable over the life of the futures contract. This description masks a complication that may already have occurred to you. The delivery procedure permits the short to deliver any note maturing in 6.5 to 10 years. Hence, the delivered note can be one of many outstanding notes, with a range of coupons and maturities. Which bond does the futures price represent?

Of all bonds that could be delivered, there will generally be one that is the most advantageous for the short to deliver. This bond is called the cheapest to deliver. A description of the delivery procedure will demonstrate the importance of the cheapest-to-deliver bond.

In fulfilling the note futures contract, the short delivers the bond in exchange for payment. The payment to the short – the invoice price for the delivered bond – is the futures price times the conversion factor. The conversion factor is the price of the bond if it were priced to yield 6%. Thus, the short delivering a bond is paid

$$Invoice\ Price = (futures\ price\ \times conversation\ factor) \\ + accrued\ interest$$

***Example**: Consider two bonds making semiannual coupon payments. Bond A is a 7% coupon bond with exactly 8 years to maturity, a price of 103.71, and a yield of 6.4%. This bond would have a price of 106.28 if its yield were 6%. Thus its conversion factor is 1.0628. Bond B has 7 years to maturity and a 5% coupon. Its current price and yield are 92.73 and 6.3%. It would have a conversion factor of 0.9435, since that is its price at a 6% yield.*

Now suppose that the futures contract is close to expiration, the observed futures price is 97.583, and the only two deliverable bonds are Bonds A and B. The short can decide which bond to deliver by comparing the market value of the bond to its invoice price if delivered. For Bond A we have

$$\text{Invoice price} - \text{market price} = (97.583 \times 1.0628) - 103.71 = 0.00$$

For Bond B we have

$$\text{Invoice price} - \text{market price} = (97.583 \times 0.9435) - 92.73 = -0.66$$

Based on the yields for the two bonds, the short breaks even delivering the 8-year 7% bond and would lose money delivering the 7-year 5% coupon bond (the invoice price is less than the market price). In this example, the 8-year 7% bond is thus the cheapest to deliver.

In general, there will be a single cheapest-to-deliver bond. You might be wondering why both bonds are not equally cheap to deliver. The reason is that the conversion factor is set by a mechanical procedure (the price at which the bond yields 6%), taking no account of the current relative market prices of bonds. Except by coincidence, two bonds will not be equally cheap to deliver.

Also, all but one of the bonds must have a negative delivery value. If two bonds had a positive delivery value, then arbitrage would be possible. The only no-arbitrage configuration in general has one bond worth zero to deliver (Bond A in example above) and the rest lose money if delivered. To avoid arbitrage, the futures price is

$$Futures\ price = \frac{price\ of\ cheapest\ to\ deliver}{conversion\ factor\ of\ cheapest\ to\ deliver}$$

This discussion glosses over subtleties involving transaction costs (whether you already own a bond may affect your delivery profit calculation) and uncertainty before the delivery period about which bond will be cheapest to deliver. Also, the T-note is deliverable at any time during the expiration month, but trading ceases with 7 business days remaining. Consequently, if there are any remaining open contracts during the last week of the month, the short has the option to deliver any bond at a price that might be a week out of date. This provides a delivery option for the short that is also priced into the contract.

Derivatives Markets, 3rd ed., Robert L. McDonald; Pearson, 2013; Chapter 4, pp. 217-220

APPLICATIONS

Types and importance of correlations

56. List statistical correlation measure and discuss how they can be applied to finance?

Financial models always deal with uncertainty and are, therefore, only approximations of a very complex pricing system that is influenced by numerous dynamic factors. Almost all financial models require market valuations as inputs. Unfortunately, these values are often determined by investors who do not always behave rationally. Therefore, asset values are sometimes random and may exhibit unexpected changes. Financial models also require assumptions regarding the underlying distribution of the asset returns. The BSM option pricing model assumes strike prices have constant volatility. However, numerous empirical studies find higher volatility for out-of-the money options and a volatility skew in equity markets. Thus, option traders and risk managers often use a volatility smile with higher volatilities for out-of-the money call and put options. Financial models at times may fail to accurately measure risk due to mathematical inconsistencies. For example, regarding barrier options, when applying the BSM option pricing model to up-and-out calls and puts and down-and-out calls and puts, there are rare cases where the inputs make the model insensitive to changes in implied volatility and option maturity.

The choice of time period used to calibrate the parameter inputs for the model can have a big impact on the results. All financial models should be tested using scenarios of extreme economic conditions. This process is referred to as stress testing. The copula correlation models failed for two reasons. First, the copula correlation models assumed a negative correlation between the equity and senior tranches of CDOs. However, during the crisis, the correlations for both tranches significantly increased causing losses for both. Second, the copula correlation models were calibrated using volatility and correlation estimates with data from time periods that had low risk, and correlations changed significantly during the crisis (risk managers used volatility and correlation estimates from pre-crisis periods).

When applying the Pearson correlation coefficient in financial models, risk managers and investors need to be aware of the following five limitations:

1. The Pearson correlation coefficient measures the linear relationship between two variables, but financial relationships are often nonlinear.
2. A Pearson correlation of zero does not imply independence between the two variables. It simply means there is not a linear relationship between the variables.
3. When the joint distribution between variables is not elliptical, linear correlation measures do not have meaningful interpretations. Examples of common elliptical joint distributions are the multivariate normal distribution and the multivariate Student's *t* – distribution.
4. The Pearson correlation coefficient requires that the variance calculations of the variables X and Y are finite. In cases where kurtosis is very high, such as the Student's *t* – distribution, the variance could be infinite, so the Pearson correlation coefficient would be undefined.
5. The Pearson correlation coefficient is not meaningful if the data is transformed. For example, the correlation coefficient between two variables X and Y will be different than the correlation coefficient between ln(X) and ln(Y).

Ordinal measures are based on the order of elements in data sets. Two examples of ordinal correlation measures are the Spearman rank correlation and the Kendall τ. The Spearman rank correlation is a nonparametric approach because no knowledge of the joint distribution of the variables is necessary.

The Spearman rank correlation coefficient is determined in three steps:

1. Order the set pairs of variables X and Y with respect to the set X.
2. Determine the ranks X_i and Y_i for each time period i.
3. Calculate the difference of the variable rankings and square the difference.

Kendall's τ is another ordinal correlation measure that is becoming more widely applied in financial models for ordinal variables such as credit ratings. Kendall's τ is also a nonparametric measure that does not require any assumptions regarding the joint probability distributions of variables.

Ordinal correlation measures based on ranking are implemented in copula correlation models to analyze the dependence of market prices and counterparty risk. Because ordinal numbers simply show the rank of observations, problems arise when ordinal measures are used for cardinal observations, which show the quantity, number, or value of observations. Another limitation of Kendall's τ occurs when there are a large number of pairs that are neither concordant nor discordant. In other words, the Kendall τ calculation can be distorted when there are only a few concordant and discordant pairs.

Example: Calculate the Spearman rank correlation for the returns of stocks X and Y provided below.

Year	*X*	*Y*
2010	25.0%	-20.0%
2011	60.0%	40.0%
2012	-20.0%	10.0%
2013	40.0%	20.0%
2014	-10.0%	30.0%
Average	19.0%	16.0%

The calculations for determining the Spearman rank correlation coefficient are shown below. The first step involves ranking the returns for stock X from lowest to highest in the second column. The first column denotes the respective year for each return. The returns for stock Y are then listed for each respective year. The fourth and fifth columns rank the returns for variables X and Y. The differences between the rankings for each year are listed in column six. Lastly, the sum of squared differences in rankings is determined in column 7.

Year	*X*	*Y*	*X Rank*	*Y Rank*	d_i	d_i^2
2012	-20.0%	10.0%	1	2	-1	1
2014	-10.0%	30.0%	2	4	-2	4
2010	25.0%	-20.0%	3	1	2	4
2013	40.0%	20.0%	4	3	1	1
2011	60.0%	40.0%	5	5	0	0
					Sum	10

The Spearman rank correlation coefficient can then be determined as 0.5:

$$\rho_S = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n(n^2-1)} = 1 - \frac{6 \times 10}{5(25-1)} = .5$$

Example: Calculate the Kendall τ correlation coefficient for the stock returns of X and Y listed below.

Year	*X*	*Y*	*X Rank*	*Y Rank*
2012	-20.0%	10.0%	1	2
2014	-10.0%	30.0%	2	4
2010	25.0%	-20.0%	3	1
2013	40.0%	20.0%	4	3
2011	60.0%	40.0%	5	5

Begin by comparing the rankings of X and Y stock returns in columns four and five of Figure 3. There are five pairs of observations, so there will be ten combinations. Below are summarized the pairs of rankings based on the stock returns for X and Y. There are two concordant pairs, four discordant pairs, and four pairs that are neither concordant nor discordant.

Concordant Pairs	*Discordant Pairs*	*Neither*
{(1,2), (2,4)}	{(1,2), (3,1)}	{(1,2), (5,5)}
{(3,1), (4,3)}	{(1,2), (4,3)}	{(2,4), (5,5)}
	{(2,4), (3,1)}	{(3,1), (5,5)}
	{(2,4), (4,3)}	{(4,3), (5,5)}

Kendall's τ can then be determined as: $\tau = \frac{n_c - n_d}{n(n-1)/2} = \frac{2-4}{(5\times4)/2} = -0.2$.

Market Risk Measurement and Management, 2019 SchweserNotes, FRM exam prep Part II, Book 1; Reading 8, pp. 96-106

57. Describe some correlation basics: Properties, Motivation, Terminology

Correlation risk measures the risk of financial loss resulting from adverse changes in correlations between financial or nonfinancial assets. Nonfinancial assets can also be impacted by correlation risk. For example, the correlation of sovereign debt levels and currency values can result in financial losses for exporters.

Static financial correlations do not change and measure the relationship between assets for a specific time period. Examples of static correlation measures are value at risk (VaR), correlation copulas for collateralized debt obligations (CDOs), and the binomial default correlation model. Dynamic financial correlations measure the co-movement of assets over time. Examples of dynamic financial correlations are pairs trading, deterministic correlation approaches, and stochastic correlation processes.

A CDS transfers credit risk from the investor (CDS buyer) to a counterparty (CDS seller). The fixed CDS spread is valued based on the default probability of the reference asset (French Bond) and the joint default correlation of Deutsche Bank and France. If there is positive correlation risk between Deutsche Bank and France, the investor has wrong-way risk (WWR). The higher the correlation risk, the lower the CDS spread, s. The increasing correlation risk increases the probability that both the French bond (reference asset) and Deutsche Bank (counterparty) default.

Five common finance areas where correlations play an important role are (1) investments, (2) trading, (3) risk management, (4) global markets, and (3) regulation.

Correlation trading strategies involve trading assets that have prices determined by the co-movement of one or more assets over time. Correlation options have prices that are very sensitive to the correlation between two assets and are often referred to as multi-asset options.

Correlation strategies	**Payoff**
Option on higher of two stocks	$max(S_1, S_2)$
Call option on maximum of two stocks	$\max(0, max(S_1, S_2) - K)$
Exchange Option	$max(0, S_2 - S_1)$
Spread call option	$max(0, S_2 - S_1 - K)$
Dual-strike call option	$max(0, S_1 - K_1, S_2 - K_2)$
Portfolio of basket options	$\left\lvert \sum_{i=1}^{n} n_i \times S_i - K, 0 \right\rvert, where\ n_i = weight\ of\ asset\ i$
Option on the worse of two stocks	$min(S_1, S_2)$

The minimum of the two stock prices is the only correlation option where a lower correlation is not desirable because it reduces the correlation option price. In contrast, the price of the exchange option is close to zero when the correlation is close to 1 because the two asset prices move together, and the spread between them does not change.

The quanto option is another investment strategy using correlation options. It protects a domestic investor from foreign currency risk.

A correlation swap is used to trade a fixed correlation between two or more assets with the correlation that actually occurs. The correlation that will actually occur is unknown and is referred to as the realized or stochastic correlation. Another example of buying correlation is to buy call options on a stock index (such as the Standard & Poor's 300 Index) and sell call options on individual stocks held within the index. If correlation increases between stocks within the index, this causes the implied volatility of call options to increase. The increase in price for the index call options is expected to be greater than the increase in price for individual stocks that have a short call position. An investor can also buy correlation by paying fixed in a variance swap on an index and receiving fixed on individual securities within the index.

The primary goal of risk management is to mitigate financial risk in the form of market risk, credit risk, and operational risk. A common risk management tool used to measure market risk is value at risk (VaR). VaR for a portfolio measures the potential loss in value for a specific time period for a given confidence level. The Basel Committee on Banking Supervision (BCBS) requires banks to hold capital based on the VaR for their portfolios. The BCBS requires banks to hold capital for assets in the trading book of at least three times greater than 10-day VaR. The trading book includes assets that are marked-to-market, such as stocks, futures, options, and swaps.

The correlations of assets within and across different sectors and geographical regions were a major contributing factor for the financial crisis of 2007 to 2009. The economic environment, risk attitude, new derivative products, and new copula correlation models all contributed to the crisis. Risk managers, financial institutions, and investors did not understand how to properly measure correlation. Risk managers used the newly developed copula correlation model for measuring correlation in structured products. It is common for CDOs to contain up to 125 assets. The copula correlation model was designed to measure [n x (n — 1) / 2] assets in structured products. Thus, risk managers of CDOs needed to estimate and manage 7,750 correlations (i.e., (125 x 124) / 2). The copula correlation model was trusted to monitor the default correlations across different tranches. A number of large hedge funds were short the CDO equity tranche and long the CDO mezzanine tranche. In other words, potential losses from the equity tranche were thought to be hedged with gains from the mezzanine tranche. Unfortunately, huge losses lead to bankruptcy filings by several large hedge funds because the correlation properties across tranches were not correctly understood.

Bonds within specific credit quality levels typically are more highly correlated. Bonds across credit quality levels typically have lower correlations.

The CDO market, comprised primarily of residential mortgages, increased from $64 billion in 2003 to $455 billion in 2006. The CDO equity tranche spread typically decreases when default correlations increase. A lower equity tranche spread typically leads to an increase in value of the equity tranche. Unfortunately, the probability of default in the subprime market increased so dramatically in 2007 that it lowered the value of all CDO tranches. Thus, the default correlations across CDO tranches increased. The default rates also increased dramatically for all residential mortgages. Even the highest quality CDO tranches with AAA ratings lost 20% of their value as they were no longer protected from the lower tranches. In addition to the rapid growth in the CDO market, the credit default swap (CDS) market grew from $8 trillion to $60 trillion during the 2004 to 2007-time period. The recent financial crisis revealed that American International Group (AIG) was overextended, selling $500 billion in CDSs with little reinsurance. Also, Lehman Brothers had leverage 30.7 times greater than equity in September 2008 leading to its bankruptcy. However, the leverage was much higher considering the large number of derivatives transactions that were also held with 8,000 different counterparties.

New correlation models are being developed and implemented such as the Gaussian copula, credit value adjustment (CVA) for correlations in derivatives transactions, and wrong-way risk (WWR) correlation. These new models hope to address correlated defaults in multi-asset portfolios.

A major concern for risk managers is the relationship between correlation risk and other types of risk such as market, credit, systemic, and concentration risk. Given that correlation risk refers to the risk that the correlation between assets changes over time, the concern is how the covariance matrix used for calculating VaR or ES changes over time due to changes in market risk. Risk managers are also concerned with measuring credit risk with respect to migration risk and default risk. Migration risk is the risk that the quality of a debtor decreases following the lowering of quality ratings. Default correlation is of critical importance to financial institutions in quantifying the degree that defaults occur at the same time. A lower default correlation is associated with greater diversification of credit risk. Most default correlations across industries are positive with the exception of the energy sector. The energy sector has little or no correlation with other sectors and is, therefore, more resistant to recessions. Systematic factors impacting the overall market and credit risk have much more influence in defaults than individual or company-specific factors. For example, if Chrysler defaults, then Ford and General Motors are more likely to default and have losses rather than benefit from increased market share. The default term structure increases slightly with time to maturity for most investment grade bonds. Conversely, for non-investment grade bonds, the probability of default is higher in the immediate time horizon. If the company survives the near-term distressed situation, the probability of default decreases over time.

The consumer staples and pharmaceutical sector are often recession resistant as individuals continue to need basic necessities such as food, household supplies, and medications. The educational sector is also resilient as more unemployed workers go back to school for education and career changes. The severity of correlation risk is even greater during a systemic crisis when one considers the higher correlations of U.S. equities with bonds and international equities. Concentration risk is the financial loss that arises from the exposure to multiple counterparties for a specific group. Concentration risk is measured by the concentration ratio. A lower (higher) concentration ratio reflects that the creditor has more (less) diversified default risk.

Market Risk Measurement and Management, 2019 SchweserNotes, FRM exam prep Part II, Book 1; Reading 6, pp. 67-88

58. How do correlations behave in the real world?

The state of the economy was defined as an expansionary period when GDP was greater than 3.3%, a normal economic period when GDP was between 0% and 3.5%, and a recession when there were two consecutive quarters of negative growth rates. Correlation levels during a recession, normal period, and expansionary period were 37.0%, 32.7%, and 27.5%, respectively. Thus, as expected, correlations were highest during recessions when common stocks in equity markets tend to go down together. The low correlation levels during an expansionary period suggest common stock valuations are determined more on industry and company-specific information rather than macroeconomic factors. The correlation volatilities during a recession, normal period, and expansionary period were 80.5%, 83.4%, and 71.2%, respectively. Investors expect stocks to go down during a recession and up during an expansionary period, but they are less certain of direction during normal times, which results in higher correlation volatility.

Mean reversion implies that over time, variables or returns regress back to the mean or average return. Empirical studies reveal evidence that bond values, interest rates, credit spreads, stock returns, volatility, and other variables are mean reverting. Mean reversion is statistically defined as a negative relationship between the change in a variable over time, $S_t - S_{t-1}$, and the variable in the previous period, S_{t-1}. For example, if S_{t-1} increases and is high at time period $t - 1$, then mean reversion causes the next value at *S*t to reverse and decrease toward the long-run average or mean value. The mean reversion rate is the degree of the attraction back to the mean and is also referred to as the speed or gravity of mean reversion. Standard regression analysis is one method used to estimate the mean reversion rate, α. A regression is run where $S_t - S_{t-1}$ (i.e., the Y variable) is regressed with respect to S_{t-1} (i.e., the X variable). Thus, the β coefficient of the regression is equal to the negative of the mean reversion rate, α.

Autocorrelation measures the degree that a current variable value is correlated to past values. Autocorrelation is often calculated using an autoregressive conditional heteroskedasticity (ARCH) model or a generalized autoregressive condition al heteroskedasticity (GARCH) model. An alternative approach to measuring autocorrelation is running a regression equation. In fact, autocorrelation has the exact opposite properties of mean reversion. Mean reversion measures the tendency to pull away from the current value back to the long-run mean. Autocorrelation instead measures the persistence to pull toward more recent historical values. If the mean reversion rate was 78% for Dow stocks. Thus, the autocorrelation for a one-period lag is 22% for the same sample. The sum of the mean reversion rate and the one-period autocorrelation rate will always equal one (i.e., 78% + 22% = 100%). This autocorrelation equation was used to calculate the one-period lag autocorrelation of Dow stocks for the 1972 to 2012-time period, and the result was 22%, which is identical to subtracting the mean reversion rate from one. The autocorrelation for longer lags decreased gradually to approximately 10% using a 10-day lag. It is common for autocorrelations to decay with longer time period lags.

Based on the results of the Kolmogorov-Smirnov, Anderson-Darling, and chi-squared distribution fitting tests, the Johnson SB distribution (which has two shape parameters, one location parameter, and one scale parameter) provided the best fit for equity correlations. The Johnson SB distribution best fit was also robust with respect to testing different economic states for the time period in question. The normal, lognormal, and beta distributions provided a poor fit for equity correlations. In summary:

Correlation Type	*Average rho*	*Rho volatility*	*Reversion Rate*	*Best Fit Distribution*
Equity	35%	80%	78%	Johnson SB
Bond	42%	64%	26%	Generalized Extreme Value
Default Probability	30%	88%	30%	Johnson SB, also Normal

Market Risk Measurement and Management, 2019 SchweserNotes, FRM exam prep Part II, Book 1; Reading 7, pp. 88-95

59. Describe financial correlation copulas

A correlation copula is created by converting two or more unknown distributions that may have unique shapes and mapping them to a known distribution with well – defined properties, such as the normal distribution. A copula creates a joint probability distribution between two or more variables while maintaining their individual marginal distributions. This is accomplished by mapping multiple distributions to a single multivariate distribution.

A Gaussian copula maps the marginal distribution of each variable to the standard normal distribution which, by definition, has a mean of zero and a standard deviation of one. The key property of a copula correlation model is preserving the original marginal distributions while defining a correlation between them. The mapping of each variable to the new distribution is done on percentile-to-percentile basis. For example, the 5th percentile observation for marginal distribution A is mapped to the 5^{th} percentile point on the univariate standard normal distribution. When the 5^{th} percentile is mapped, it will have a value of – 1.645. This is repeated for each observation on a percentile – to –

percentile basis. Likewise, every observation on the marginal distribution of Y is mapped to the corresponding percentile on the univariate standard normal distribution. The new joint distribution is now a multivariate standard normal distribution. A copula is a way to indirectly define a correlation relationship between two variables when it is not possible to directly define a correlation.

When a Gaussian copula is used to derive the default time relationship for more than two assets, a Cholesky decomposition is used to derive a sample M_n (•) from a multivariate copula M_n (•) E [0, 1]. The default correlations of the sample are determined by the default correlation matrix p_M for the n – variate standard normal distribution, M_n.

Example: Suppose a risk manager owns two non-investment grade assets. Below are listed the default probabilities for the next five years for companies B and C that have B and C credit ratings, respectively. How can a Gaussian copula be constructed to estimate the joint default probability, Q, of these two companies in the next year, assuming a one-year Gaussian default correlation of 0.4?

Time, t	*B default probability*	*C default probability*
1	0.065	0.238
2	0.081	0.152
3	0.072	0.113
4	0.064	0.092
5	0.059	0.072

In this example, there are only two companies, B and C. Thus, a bivariate standard normal distribution, M_2, with a default correlation coefficient of p can be applied. With two companies, only a single correlation coefficient is required, and not a correlation matrix of p_M.

$$C_{GD}[Q_B(t), Q_C(t)] = M_2\left[N^{-1}\left(Q_B(t)\right), N^{-1}\left(Q_C(t)\right); \rho\right]$$

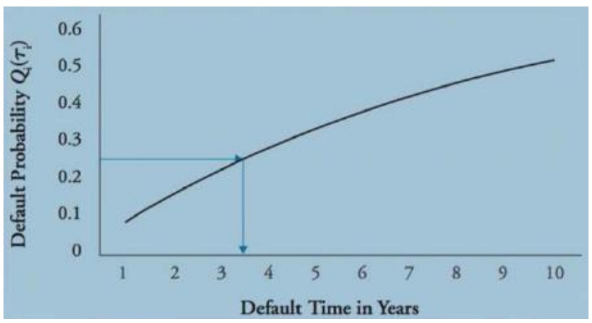
Time, t	*B default*	*$Q_B(t)$*	*$N^{-1}(Q_B(t))$*	*C default*	*$Q_C(t)$*	*$N^{-1}(Q_C(t))$*
1	0.065	0.065	-1.513	0.238	0.238	-0.712
2	0.081	0.146	-1.053	0.152	0.390	-0.279
3	0.072	0.218	-0.779	0.113	0.503	0.008
4	0.064	0.282	-0.577	0.092	0.595	0.241
5	0.059	0.341	-0.409	0.072	0.667	0.432

The joint probability of both Company B and Company C defaulting within one year is calculated as:

$$Q(t_B \leq 1 \cap t_C \leq 1) \equiv M(X_B \leq -1.513 \cap X_C \leq -0.712, \rho = 0.4) = 3.4\%$$

Example: Illustrate how a risk manager estimates the expected default time of asset *i* using an n-variate Gaussian copula.

Suppose a risk manager draws a 25% cumulative default probability for asset *i* from a random 72-variate standard normal distribution, M_n (•). The n – variate standard normal distribution includes a default correlation matrix, p_M, that has the default correlations of asset *i* with all n assets. Below is illustrated how to equate this 25% with the market determined cumulative individual default probability Q_i (τ_i). Suppose the first random sample equates to a default time τ of 3.5 years. This process is then repeated 100,000 times to estimate the default time of asset *i*.



Market Risk Measurement and Management, 2019 SchweserNotes, FRM exam prep Part II, Book 1; Reading 9, pp. 107-112

Bonds hedging strategies

60. Describe DV01 neutral hedge and regression hedge

A standard DV01 – neutral hedge assumes that the yield on a bond and the yield on a hedging instrument rise and fall by the same number of basis points. However, a one-to-one relationship does not always exist in practice. In general, more dispersion surrounds the change in the nominal yield for a given change in the real yield. Empirically, the nominal yield adjusts by more than one basis point for every basis point adjustment in the real yield. To improve this DV01 – neutral hedge approach, we can apply regression analysis techniques. Using a regression hedge examines the volatility of historical rate differences and adjusts the DV01 hedge accordingly, based on historical volatility. A regression hedge takes DV01 – style hedges and adjusts them for projected nominal yield changes compared to projected real yield changes. The advantage of a regression framework is that it provides an estimate of a hedged portfolio's volatility. An investor can gauge the expected gain in advance and compare it to historical volatility to determine whether the hedged portfolio is an attractive investment. If least squares estimation determines the yield beta to be 1.0198, then this means that over the sample period, the nominal yield increases by 1.0198 basis points for every basis point increase in real yields. The regression hedge approach assumes that the hedge coefficient, β, is constant over time. This of course is not always the case, so it is best to estimate the coefficient over different time periods and make comparisons. Two other factors should be also considered in our analysis: (1) the R-squared (i.e., the coefficient of determination), and (2) the standard error of the regression (SER). The R-squared gives the percentage of variation in nominal yields that is explained by real yields. The standard error of the regression is the standard deviation of the realized error terms in the regression.

Regression hedging can also be conducted with two independent variables. For example, assume a trader in euro interest rate swaps buys/receives the fixed rate in a relatively illiquid 20-year swap and wishes to hedge this interest rate exposure. In this case, a regression hedge with swaps of different maturities would be appropriate. Since it may be impractical to hedge this position by immediately selling 20-year swaps, the trader may choose to sell a combination of 10- and 30-year swaps. The following regression equation describes this relationship:

$$\Delta y_t^{20} = \alpha + \beta^{10}\Delta y_t^{10} + \beta^{30}\Delta y_t^{30} + \varepsilon_t$$

Similar to the single-variable regression hedge, this hedge of the 20-year euro swap can be expressed in terms of risk weights, which are the beta coefficients in the above equation:

$$\frac{(-F^{10} \times DV01^{10})}{(F^{20} \times DV01^{20})} = change\ in\ 10 - year\ swap\ rate, \beta^{10} = 0.2221$$

$$\frac{(-F^{30} \times DV01^{30})}{(F^{20} \times DV01^{20})} = change\ in\ 30 - year\ swap\ rate, \beta^{30} = 0.7765$$

Given these regression results and an illiquid 20-year swap, the trader would hedge 22.21% of the 20-year swap DV01 with a 10-year swap and 77.65% of the 20-year swap DV01 with a 30-year swap. Because these weights sum to approximately one, the regression hedge DV01 will be very close to the 20-year swap DV01. The two-variable approach will provide a better hedge (in terms of R-squared) compared to a single-variable approach. However, regression hedging is not an exact science. There are several cases in which simply doing a one-security DV01 hedge, or a two-variable hedge with arbitrary risk weights, is not appropriate (e.g., hedging during a financial crisis).

When setting up and establishing regression-based hedges, there are two schools of thought. Some regress changes in yields on changes in yields, as demonstrated previously, but an alternative approach is to regress yields on yields. With both approaches, the estimated *regression coefficients are unbiased and consistent*; however, the error terms are unlikely to be independent of each other. Thus, since the error terms are correlated over time (i.e., serially correlated), the estimated *regression coefficients are not efficient*.

Empirical approaches, such as principal components analysis (PCA), take a different approach by providing a single empirical description of term structure behavior, which can be applied across all bonds. For example, if we consider the set of swap rates from 1 to 30 years, at annual maturities, the PCA sets up the 30 factors with the following properties:

- The sum of the variances of the 30 principal components (PCs) equals the sum of the variances of the individual rates. The PCs thus capture the volatility of the set of rates.
- The PCs are not correlated with each other.
- Each PC is chosen to contain the highest possible variance, given the earlier PCs.

The advantage of this approach is that we only really need to describe the volatility and structure of the first three (small number, 3 is not a must) PCs since the sum of the variances of the first three PCs is a good approximation of the sum of the variances of all rates. Thus, the PCA approach creates three factors that capture similar data as a comprehensive matrix containing variances and covariances of all interest rate factors.

Example 20 and Answer 20: Assume a relative value trade is established whereby a trader sells a U.S. Treasury bond and buys a U.S. TIPS (which makes inflation-adjusted payments) to hedge the T-bond. Assume the following data for yields and DV01s of a TIPS and a T-bond. Also assume that the trader is selling 100 million of the T-bond.

Bond	*Yield (%)*	*DV01*
TIPS	1.325	0.084
T-Bond	3.475	0.068

The calculation for the amount of TIPS to purchase to hedge the short nominal bond is as follows where F^R is face amount of the real yield bond:

$$F^R \times \frac{0.084}{100} = 100M \times \frac{0.068}{100}, \rightarrow F^R = 100M \times \frac{0.068}{0.084} = \$80.95\, million.$$

Now considering the variability between the nominal and real yields, the hedge can be adjusted by the hedge adjustment factor of 1.0198:

$$F^R = F^N \times \left(\frac{DV01^N}{DV01^R}\right) \times \beta = 100M \times \left(\frac{0.068}{0.084}\right) \times 1.0198 = \$82.55\, million.$$

Market Risk Measurement and Management, 2019 SchweserNotes, FRM exam prep Part II, Book 1; Reading 10, pp. 115-122

61. Describe immunization strategies for bond portfolios

A bond portfolio's value in the future depends on the interest-rate structure prevailing up to and including the date at which the portfolio is liquidated. If a portfolio has the same payoff at some specific future date, no matter what interest-rate structure prevails, then it is said to be immunized. Here we will discuss immunization strategies, which are closely related to the concept of duration discussed earlier in the text. Immunization strategies have been discussed for many concepts of duration, but this material is restricted to the simplest duration concept, that of Macauley.

Consider the following situation: A firm has a known future obligation, Q. the discounted value of this obligation is:

$$V_0 = \frac{Q}{(1+r)^N}$$

Suppose that this future obligation is currently hedged by a bond held by the firm. That is, the firm currently holds a bond whose value V_B is equal to the discounted value of the future obligation V_0. If P_1, $P_2 \ldots P_M$ is the stream of anticipated payments made by the bond, then the bond's present value is given by

$$V_B = \sum_{t=1}^{M} \frac{P_t}{(1+r)^t}$$

Now suppose that the underlying interest rate, r, changes to $r + \Delta r$. Using a first-order linear approximation, we find that the new value of the future obligation is given by

$$V_0 + \Delta V_0 \approx V_0 + \frac{dV_0}{dr}\Delta r = V_0 + \Delta r\left[\frac{-NQ}{(1+r)^{N+1}}\right]$$

However, the new value of the bond is given by

$$V_B + \Delta V_B \approx V_B + \frac{dV_B}{dr}\Delta r = V_B + \Delta r \sum_{t=1}^{N} \frac{-tP}{(1+r)^{t+1}}$$

If these two expressions are equal, a change in *r* will not affect the hedging properties of the company's portfolio. Setting the expressions equal gives us the condition

$$V_B + \Delta r \sum_{t=1}^{N} \frac{-tP}{(1+r)^{t+1}} = V_0 + \Delta r \left[\frac{-NQ}{(1+r)^{N+1}}\right]$$

Recalling that

$$V_B = V_0 = \frac{Q}{(1+r)^N}$$

We can simplify this expression to get

$$\frac{1}{V_B}\sum_{t=1}^{M} \frac{tP_t}{(1+r)^t} = N$$

That is $D_B = N$, or as duration of single future obligation (i.e., zero-coupon bond) is equal to its maturity we can rewrite it as:

$$D_B = D_Q$$

Where D_B is the duration of the bond, D_Q is the duration of the obligation.

This statement is worth restating as a formal proposition: Suppose that the term structure of interest rates is always flat (that is, the discount rate for cash flows occurring at all future times is the same) or that the term structure moves up or down in parallel movements. Then a necessary and sufficient condition that the market value of an asset be equal under all changes of the discount rate *r* to the market value of a future obligation *Q* is that the duration of the asset equal the duration of the obligation. Here we understand the word "equal" to mean equal in the sense of a first-order approximation. An obligation against which an asset of this type is held is said to be immunized. The preceding statement has two critical limitations:

- The immunization discussed applies only to first-order approximations. When we get to a numerical example in the succeeding calculations below, we shall see that there is a big difference between first-order equality and "true" equality

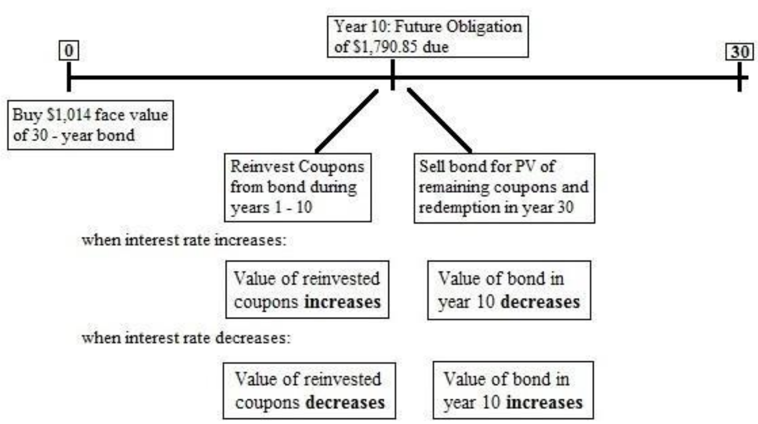
- We have assumed either that the term structure is flat or that the term structure moves up or down in parallel movements. At best, this assumption might be considered to be a poor approximation to reality

In this example we consider a basic numerical immunization. Suppose you are trying to immunize a 10 – year obligation whose present value is $1,000 (for example, at a current interest rate of 6 percent, its future value would be $1,000 × 1.0610 = $1,790.85). You intend to immunize the obligation by purchasing $1,000 worth of a bond or a combination of bonds. You consider three bonds:

1. Bond 1 has 10 years remaining until maturity, a coupon rate of 6.7 percent, and a face value of $1,000.
2. Bond 2 has 15 years until maturity, a coupon rate of 6.988 percent, and a face value of $1,000.
3. Bond 3 has 30 years until maturity, a coupon rate of 5.9 percent, and a face value of $1,000.

At the existing yield to maturity of 6 percent, the prices of the bonds differ. Bond 1, for example is worth $1,051.52 = $\sum_{t}^{10} \frac{67}{(1.06)^t} + \frac{1,000}{(1.06)^{10}}$; thus, in order to purchase $1,000 worth of this bond, you have to purchase $951 = $1,000/$1,051.52 of *face value* of the bond. However, Bond 3 is currently worth $986.24, so that in order to buy $1,000 of market value of this bond, you will have to buy $1,013.96 of face value. If you intend to use this bond to finance a $1,790.85 obligation 10 years from now, following is a schematic of the problem you face.

The Immunization Problem
Illustrated for the 30 - Year



As we will see, the 30-year bond will exactly finance the future obligation of $1,790.85 only for the case in which the current market interest rate of 6 percent remains unchanged. Here is a summary of price and duration information for the three bonds:

2				
3	***YTM***	6%		
4				
5		**Bond 1**	**Bond 2**	**Bond 3**
6	***Coupon Rate***	6.70%	6.988%	5.90%
7	***Maturity***	10	15	30
8	***Face Value***	1000	1000	1000
9				
10	***Bond Price***	$1,051.52	$1,095.96	$986.24
11	***Face Value equal to $1,000 of market value***	$951.00	$912.44	$1,013.96
12				
13	***Duration***	7.6655	10.0000	14.6361
14				

If the yield to maturity does not change, then you will be able to reinvest each coupon at 6 percent. Thus, bond 2, for example, will give a terminal wealth at the end of 10 years of

$$\sum_{t=1}^{9} 69.88 \times (1.06)^t + \left[\sum_{t=1}^{5} \frac{69.88}{(1.06)^t} + \frac{1{,}000}{(1.06)^5}\right] = 921.07 + 1{,}041.62$$
$$= 1{,}962.69$$

The first term in this expression is the sum of the reinvested coupons. The second term represents the market value of the bond maturing in year 10, when the bond has five more years until maturity. Since we will be buying only $912.44 of face value of this bond, we have, at the end of 10 years, 0.91244 × $1,962.69 = $1,790.85. This is exactly the amount we wanted to have at this date. The results of this calculation for all three bonds, provided there is no change in the yield to maturity, are given in the following table:

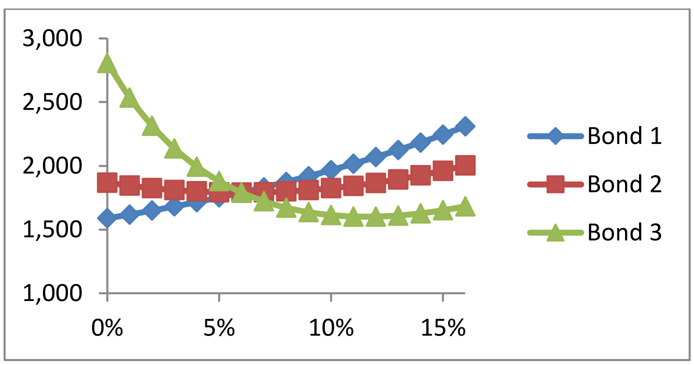
2					
3	***YTM***	6%			
4		**Bond 1**	**Bond 2**	**Bond 3**	
5	***Coupon Rate***	6.70%	6.988%	5.90%	
6	***Maturity***	10	15	30	
7	***Face Value***	1000	1000	1000	
8					
9	***Bond Price***	$1,051.52	$1,095.96	$986.24	
10	***Face Value equal to $1,000 of market value***	$951.00	$912.44	$1,013.96	
11					
12	***Duration***	7.6655	10.0000	14.6361	
13					
14		**Bond 1**	**Bond 2**	**Bond 3**	
15	*Bond Price*	$1,000.00	$1,041.62	$988.53	<= PV($B$3,D7-10,-D6*D8)+D8/(1+$B$3)^(D7-10)
16	*Reinvested Coupons*	$883.11	$921.07	$777.67	<= FV($B$3,10,-D6*D8)
17	***Total***	$1,883.11	$1,962.69	$1,766.20	
18					
19	*Percent of face value bought*	95.10%	91.24%	101.40%	<= 1000/D9
20	***Terminal Wealth***	$1,790.85	$1,790.85	$1,790.85	

The upshot of this table is that purchasing $1,000 of any of the three bonds will provide – 10 years from now – funding for your future obligation of $1,790.85, provided the market interest rate of 6 percent doesn't change.

Now suppose that, immediately after you purchase the bonds, the yield to maturity changes to some new value and stays there. This change will obviously affect the calculation we already did. For example, if the yield falls to 5 percent, the table will now look as follows:

	A	B	C	D
22	***New YTM***	5%		
23		**Bond 1**	**Bond 2**	**Bond 3**
24	*Bond Price*	$1,000.00	$1,086.07	$1,112.16
25	*Reinvested Coupons*	$842.72	$878.94	$742.10
26	***Total***	$1,842.72	$1,965.01	$1,854.26
27				
28	*Percent of face value bought*	95.10%	91.24%	101.40%
29	***Terminal Wealth***	$1,752.43	$1,792.97	$1,880.14

Thus, if the yield falls, bond 1 will no longer fund our obligation, whereas bond 3 will overfund it. Bond 2's ability to fund the obligation – not surprisingly, in view of the fact that its duration is exactly 10 years – hardly changes. We can repeat this calculation for any new yield to maturity. The results are shown in the following figure:

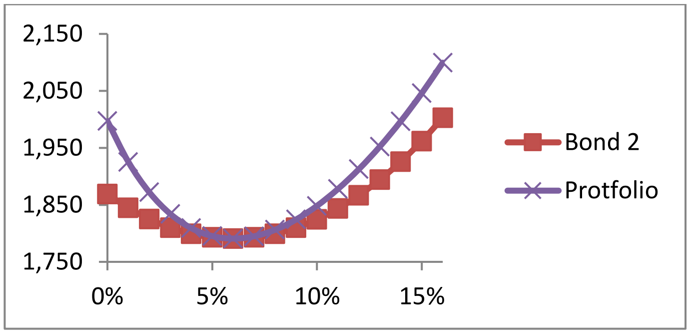


Clearly, if you want an immunized strategy, you should buy bond 2! As its duration is equal to the duration of the obligation we want to hedge.

The duration of a portfolio is the weighted average duration of the assets in the portfolio. As a result, there is another way to get a bond investment with duration of 10: If we invest $665.09 in bond 1 and $ 344.91 in bond 3, the resulting portfolio also has duration of 10. These weights are calculated as follows:

$$\lambda \times D_{bond\ 1} + (1-\lambda) \times D_{bond\ 3} = 7.665\lambda + 14.636(1-\lambda) = 10 \rightarrow \lambda = 0.66509$$

Suppose we repeat our experiment with this portfolio of bonds used previously (varying the YTM), but add in the portfolio of bond 1 and bond 3. Building a data table based on this experiment and graphing the results shows that the portfolio's performance is better than that of bond 2 by itself.



Look again at the graph: Notice that, while the terminal value is somewhat convex in the yield to maturity for both bond 2 and the bond portfolio, the terminal value of the portfolio is *more convex* than that of the single bond. Redington (1952), one of the influential propagators of the concept of duration and immunization, thought this convexity very desirable, and we can see why: No matter what the change in the yield to maturity, the portfolio of bonds provides *more overfunding* of the future obligation than the single bond. This is obviously a desirable property for an immunized portfolio, and it leads us to formulate the following rule:

In a comparison between two immunized portfolios, both of which are to fund a known future obligation, the portfolio whose terminal value is more convex with respect to changes in the yield to maturity is preferable.

Despite what was said in the preceding above, there is some interest in deriving the characteristics of a bond portfolio whose terminal value is as insensitive to changes in the yield as possible. One way of improving the performance (when so defined) of the bond portfolio is not only to match the first derivatives of the change in value (which leads to the duration concept), but also to match the second derivatives.

A direct extension of this analysis leads us to the conclusion that matching the second derivatives requires:

$$N(N+1) = \frac{1}{V_B}\sum_{t=1}^{M}\frac{t(t+1)P_t}{(1+r)^t}$$

The following example illustrates the kind of improvement that can be made in a portfolio where the second derivatives are also matched. Consider 4 bonds, one of which, bond 2 is from the previous example, whose duration is exactly 10. The bonds are described in the following table:

	A	B	C	D	E	F
1	**BOND CONVEXITY**					
2	Yield to maturity	6%				
3						
4		**Bond 1**	**Bond 2**	**Bond 3**	**Bond 4**	
5	Coupon rate	4.50%	6.988%	3.50%	11.00%	
6	Maturity	20	15	14	10	
7	Face value	1,000	1,000	1,000	1,000	
8						
9	Bond price	$827.95	$1,095.96	$767.63	$1,368.00	<-- =-PV($B$2,E6,E5*E7)+E7/(1+$B$2)^E6
10	Face value equal to $1,000 of market value	$ 1,207.80	$ 912.44	$ 1,302.72	$ 730.99	<-- =E7/E9*E7
11						
12	Duration	12.8964	10.0000	10.8484	7.0539	<-- =dduration(E6,E5,$B$2,1)
13	Second derivative of duration	229.0873	136.4996	148.7023	67.5980	<-- =secondDur(E6,E5,$B$2)/bondprice(E6,E5,$B$2)

Here secondDur(numberPayments, couponRate, YTM) is a VBA function we have defined to calculate the second derivative of the duration:

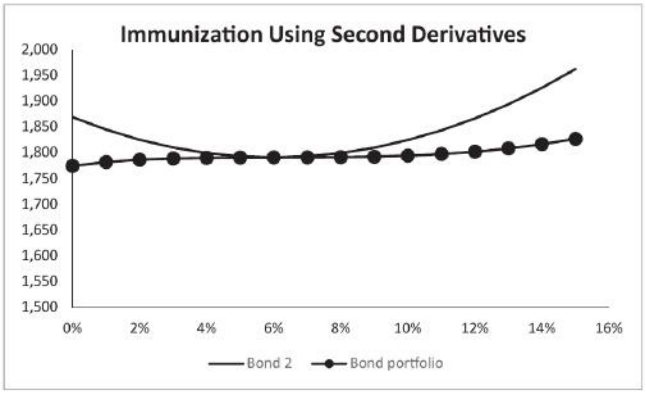
```
Function secondDur(numberPayments, couponRate,
YTM)

    For Index = 1 To numberPayments
        If Index < numberPayments Then
            secondDur = couponRate * Index * _
            (Index + 1) / (1 + YTM) ^ Index + _
                 secondDur
        Else
            secondDur = (couponRate + 1) * _
            Index * (Index + 1) _
                 / (1 + YTM) ^ Index + _
                 secondDur
        End If

        secondDur = secondDur
    Next Index

End Function
```

We need three bonds in order to calculate a portfolio of bonds whose duration and whose second duration derivative are exactly equal to those of the liability. The proportions of a portfolio which sets both the duration and its second derivative equal to those of the liability are bond 1 = −0.5619, bond 3 = 1.6415, bond 4 = −0.0797.[2] As the following figure shows, this portfolio provides a better hedge against the terminal value than even bond 2:



Computing the Bond Portfolio

We want to invest proportions x1, x3, x4 in bonds 1, 3, and 4 so that:

- The portfolio is totally invested: $x_1 + x_3 + x_4 = 1$
- The portfolio duration is matched to that of bond 2: $x_1 D_1 + x_3 D_3 + x_4 D_4 = D_2$, where D_i is the duration of bond i
- The second derivative of the portfolio duration is matched to that of bond 2: $x_1 D_1^2 + x_3 D_3^2 + x_4 D_4^2 = D_2^2$, where D_i^2 is the duration derivative

Writing this in matrix form, we get:

$$\begin{bmatrix} 1 & 1 & 1 \\ D_1 & D_3 & D_4 \\ D_1^2 & D_3^2 & D_4^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ D_2 \\ D_2^2 \end{bmatrix}$$

whose solution is given by:

$$\begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ D_1 & D_3 & D_4 \\ D_1^2 & D_3^2 & D_4^2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ D_2 \\ D_2^2 \end{bmatrix}$$

This can easily be set up in Excel:

	I	J	K	L	M	N
15	**Calculating the bond portfolio:**					
16					Vector of	
17	Matrix of coefficients				constants	
18	1	1	1		1	
19	12.8964	10.8484	7.0539		10.0000	
20	229.0873	148.7023	67.5980		110.0000	
21						
22	Solution					
23	-0.5619					
24	1.6415	<-- {=MMULT(MINVERSE(I18:K20),M18:M20)}				
25	-0.0797					
26						
27						
28	Explanation of the above: We want to invest proportions					
29	x_1, x_3, and x_4 in bonds 1, 3 and 4 respectively, in order					
30	that: a) The total investment is $1000; this means $x_1+x_2+x_4=1$					
31	b) Portfolio duration is matched to that of bond 2; this means					
32	that $x_1*D_1+x_3*D_3+x_4*D_4 = D_2$, where D_i is the duration					
33	of bond I.					
34	c) The weighted average duration derivatives are equal					
35	to that of bond 2.					
36						
37	These three conditions give us the matrix system in					
38	cells I18:K20 and the corresponding solution in					
39	cells I23:I25 .					

Given this solution, the last chart is produced by the following data table:

B	C	D	E	F
	Bond 2	**Bond portfolio**		
			<-- =I23*B23+I24*D23+I25*E23 , data table header (hidden)	
0%	$ 1,868.87	$ 1,774.63		
1%	$ 1,844.71	$ 1,781.79		
2%	$ 1,825.14	$ 1,786.37		
3%	$ 1,810.05	$ 1,789.02		
4%	$ 1,799.35	$ 1,790.32		
5%	$ 1,792.97	$ 1,790.78		
6%	$ 1,790.85	$ 1,790.85		
7%	$ 1,792.95	$ 1,790.91		
8%	$ 1,799.26	$ 1,791.31		
9%	$ 1,809.76	$ 1,792.38		
10%	$ 1,824.46	$ 1,794.38		
11%	$ 1,843.37	$ 1,797.58		
12%	$ 1,866.53	$ 1,802.21		
13%	$ 1,893.98	$ 1,808.46		
14%	$ 1,925.77	$ 1,816.55		
15%	$ 1,961.98	$ 1,826.65		

Consider a situation when a fund manager owns n_1 units of bond B_1. The value of the holding will obviously be changing with the change in the level of the yield curve. If the yield goes up the value of the holding will fall. The fund manager wishes to protect the portfolio against this unfavorable movement.

One of the ways of doing so is to add to the holding n_2 units of bond B_2 so that the any change in the value of bond B_1 will be exactly offset by the change in the value of bond B_2.

Let the value of the portfolio be $V = n_1 B_1(y_1) + n_2 B_2(y_2)$. Here y_1 and y_2 are the yields of the bonds respectively. If the yields are shifted by the same amount Δy the new value of the portfolio will be $V_\Delta = n_1 B_1(y_1 + \Delta y) + n_2 B_2(y_2 + \Delta y)$ and the change in the value of the portfolio will be:

$$\Delta V = V_\Delta - V = n_1\big(B_1(y_1 + \Delta y) - B_1(y_1)\big) + n_2\big(B_2(y_2 + \Delta y) - B_2(y_2)\big)$$
$$\approx n_1 D_1(y_1)\Delta y + n_2 D_2(y_2)\Delta y$$

Where D_1 and D_2 are respective bond durations; for an immunized portfolio the value change should be zero, so:

$$n_1 D_1(y_1)\Delta y + n_2 D_2(y_2)\Delta y = 0 \to n_2 = -n_1 \frac{D_1}{D_2}$$

Several things are to be noted here.

1. This type of portfolio immunization may not be allowed, as many funds are under restriction of not being able to hold ‘short’ bonds that is bonds that are nominally sold. There sometimes are ways around this restriction. Funds may, for example, ‘borrow’ bonds from other parties.
2. Even if bonds can be sold short this solution may be quite expensive by the way of losing the coupon income, transaction costs and liquidity issues.
3. This approach will immunize the portfolio only against parallel shocks, but not changes in the slope and/or curvature of the curve.
4. By immunizing the portfolio we at the same time forgo the potential benefits, for example, the increase in the value of the portfolio if the yields go down.
5. As durations (and yield) change with time, portfolio needs to be periodically rebalanced.

The method above will protect against small changes of the yields. If the market gaps (yield drop or go up unexpectedly) the price convexity will generate profit or loss on the portfolio. To immunize against large movements of the yield the convexity must be considered. Using Taylor’s decomposition of our portfolio to the second order we obtain:

$$\Delta V = V_\Delta - V = n_1\big(B_1(y_1+\Delta y) - B_1(y_1)\big) + n_2\big(B_2(y_2+\Delta y) - B_2(y_2)\big)$$
$$\approx n_1 D_1(y_1)\Delta y + \frac{1}{2}n_1 C_1(y_1)\Delta y^2 + n_2 D_2(y_2)\Delta y$$
$$+\frac{1}{2}n_2 C_2(y_2)\Delta y^2$$

Where C_i are the convexities of respective bonds; by equating the value increment to zero and solving for n_2 the portfolio composition can be easily obtained.

Consider a fund manager who needs to meet a liability of $1,000,000 in 4 years. He has two bonds available to him to hedge his liability. The information is summarized in the table.

	FV	Maturity	Coupon	Yield	Price	Modified Duration
Bond	$100,000	3	5%	6.1%	$97,025.48	2.74
Bond	$100,000	7	6%	6.4%	$97,771.28	5.62
Liability	$1,000,000	4		6.3%	$780,272.21	4.00

The fund manager needs to buy n_1 units of the first bond and n_2 of the second bond so that:

1. Match the value of the liability
2. Match the duration of the liability

This leads to the following simultaneous equations:

$$97025.48n_1 + 97771.28n_2 = 780272.21$$
$$2.74 \times 97025.48 \times n_1 + 5.62 \times 97771.28 \times n_2 = 4 \times 780272.21$$

The second equation requires some explanation. Remember, that modified duration measures the change in the value for a unit price. So, to measure the actual change (in $values) the modified duration needs to be multiplied by the value. Solving these equations, we obtain $n_1 = 4.5$ and $n_2 = 3.5$.

Financial Modeling, Second Edition, Simon Benninga; MIT Press 2000, pp. 317-326

Financial Modeling, Fourth Edition, Simon Benninga; MIT Press 2014, pp. 547-551

Fundamentals of Financial Markets, Volf Frishling; National Australia Bank 2007, pp. 15-17

Value-at-Risk (VAR)

62. Describe Value-at-Risk

Value-at-Risk (VaR) measures the worst expected loss under normal market conditions over a specific time interval at a given confidence level. VaR answers the question: How much can I lose with x percent probability over a preset horizon? Another way of expressing this idea is that VaR is the lowest quantile of the potential losses that can occur within a given portfolio during a specified time period. The basic time period T and the confidence level (the quantile) q are the two major parameters that should be chosen in a way appropriate to the overall goal of risk measurement. The time horizon can differ from a few hours for an active trading desk to a year for a pension fund. When the primary goal is to satisfy external regulatory requirements, such as bank capital requirements, the quantile is typically very small (for example, 1 percent of worst outcomes). However, for an internal risk management model used by a company to control the risk exposure, the typical number is around 5 percent.

In the jargon of VaR, suppose that a portfolio manager has a daily VaR equal to $1 million at 1 percent. This statement means that there is only one chance in 100 that a daily loss bigger than $1 million occurs under normal market conditions.

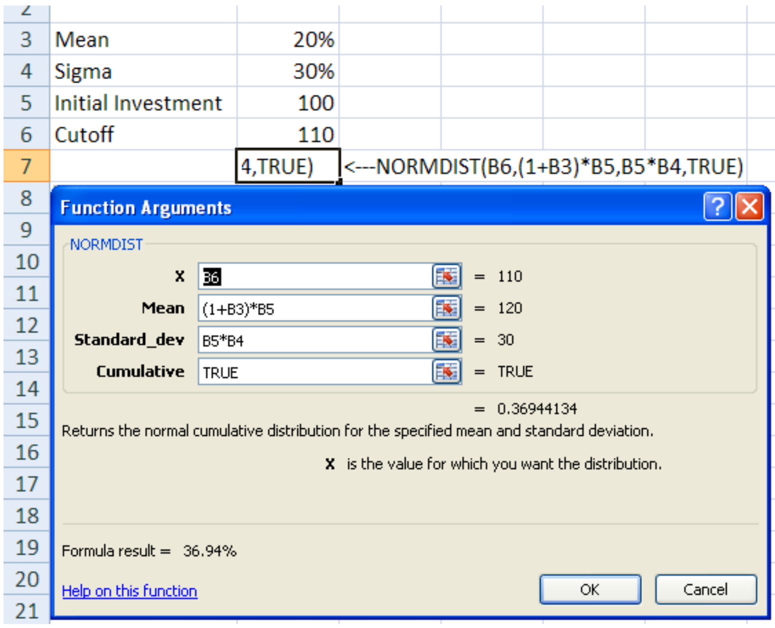
Suppose a manager has a portfolio that consists of a single asset. The return of the asset is normally distributed with mean return 20 percent and standard deviation 30 percent. The value of the portfolio today is $100 million. We want to answer various simple questions about the end-of-year distribution of portfolio value:

- What is the distribution of the end-of-year portfolio value?
- What is the probability of a loss of more than $20 million dollars by year-end (i.e., what is the probability that the end-of-year value is less than $80 million)?
- With 1 percent probability what is the maximum loss at the end of the year? This is the VaR at 1 percent.

The probability that the end-of-year portfolio value is less than $80 million is about 37 percent. ("Million" is omitted in the example.)

	A	B	C	D	E	F
1						
2						
3	Mean	20%				
4	Sigma	30%				
5	Initial Investment	100				
6	Cutoff	110				
7		36.94%	<---NORMDIST(B6,(1+B3)*B5,B5*B4,TRUE)			

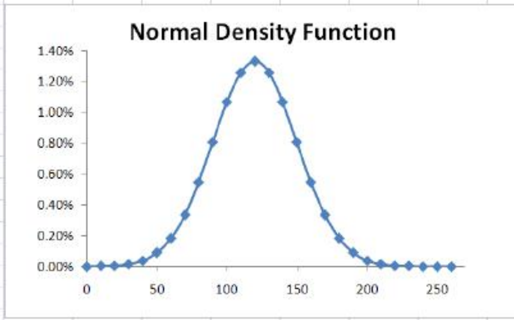
Here's how we apply the **NormDist** function:



This picture shows that the Excel function **Normdist** can give both the cumulative normal distribution and the probability mass function. Using the latter option and a data table gives the standard bell-shaped graph:

	A	B	C	D
1				
2				
3	Mean	20%		
4	Sigma	30%		
5	Initial Investment	100		
6	Cutoff	110		
7		36.94%	<---NORMDIST(B6,(1+B3)*B5,B5*B4,TRUE)	
8				
9				
10				
11			1.26%	
12		0	4.5E-06	
13		10	1.6E-05	
14		20	5.1E-05	
15		30	0.00015	
16		40	0.00038	
17		50	0.00087	
18		60	0.0018	
19		70	0.00332	
20		80	0.00547	
21		90	0.00807	
22		100	0.01065	
23		110	0.01258	
24		120	0.0133	

Data Table Header
=NORMDIST(B6,(1+B3)*B5,B5*B4,FALSE)

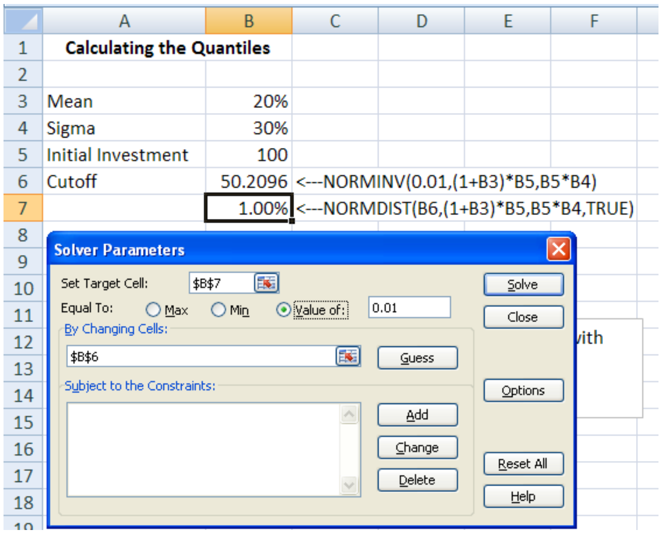


With a probability of 1 percent the end-of-year portfolio value will be less than 50.20865; thus the VaR of the distribution is 100 − 50.20865 = 49.79135.

	A	B	C	D	E	F	G
1	**Calculating the Quantiles**						
2							
3	Mean	20%					
4	Sigma	30%					
5	Initial Investment	100					
6	Cutoff	50.2096	<---NORMINV(0.01,(1+B3)*B5,B5*B4)				
7		1.00%	<---NORMDIST(B6,(1+B3)*B5,B5*B4,TRUE)				
8							
9	Var at 1.00% level	49.7904					
10							
11							
12							
13							
14							
15							

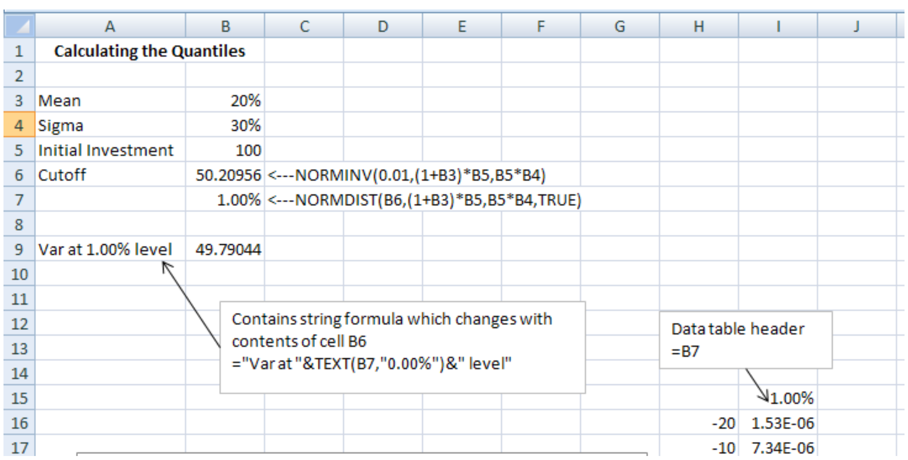
Contains string formula which changes with contents of cell B6
="Var at "&TEXT(B7,"0.00%")&" level"

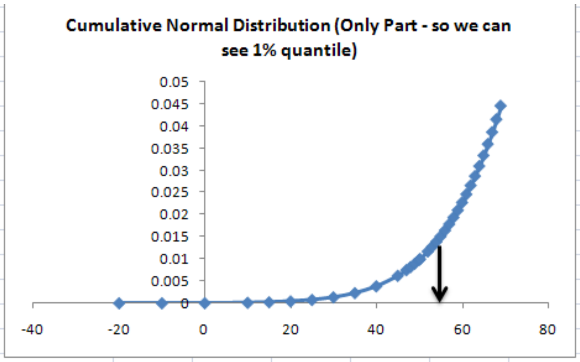
The cutoff is known as the quantile of the distribution. We found this solution by using **Excel's Solver**:



We can use Solver to find the quantiles for any distribution. For Normal Distribution Excel has built in functions that find the quantile. These functions—Norminv and Normsinv—find the inverse for the normal and standard normal.

Here's an example for the numbers that we've been using; this time we have written the function Norminv(0.01(1 +B3)*B5,B4) in cell B6. This function finds the cutoff point for which the normal distribution with a mean of 120 and a standard deviation of 30 has probability of 1 percent. You can see this point on the following graph, which shows part of the cumulative distribution:





18	0	3.17E-05
19	10	0.000123
20	15	0.000233
21	20	0.000429
22	25	0.000771
23	30	0.00135
24	35	0.002303
25	40	0.00383
26	45	0.00621
27	47	0.00748
28	48	0.008198
29	49	0.008975
30	50	0.009815
31	50.20974	0.01

The lognormal distribution is a more reasonable distribution for many asset prices (which cannot become negative) than the normal distribution. Suppose that the return on the portfolio is normally distributed with annual mean μ and annual standard deviation σ. Furthermore, suppose that the current value of the portfolio is given by V_0. Then it follows that the logarithm of the portfolio value at time T, V_T, is normally distributed:

$$\ln\,(V_T) \sim Normal\left[\ln(V_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

Suppose, for example, that $V_0 = 100$, $\mu = 10$ percent, and $\sigma = 30$ percent. Thus the end-of-year log of the portfolio value is distributed normally:

$$\ln\,(V_T) \sim \mathrm{Normal}\left[\ln(100) + \left(0.10 - \frac{0.3^2}{2}\right), 0.3\right]$$

Thus a portfolio whose initial value is \$100 million and whose annual returns are lognormally distributed with parameters μ = 10 percent and σ = 30 percent, has an annual VaR equal to \$47.42 million at 1 percent:

	A	B	C	D	E
2					
3	Initial Value, V_0	100			
4	Mean, μ	10%			
5	Sigma, σ	30%			
6	Time Period, T	1	<-- In years		
7					
8	Parameters of Normal Distributions of ln(V_T)				
9	Mean	4.66			
10	Sigma	0.3			
11					
12	Cutoff	52.57631976	<---LOGINV(0.01,B9,B10)		
13	VaR at 1% level	47.42368024	<---B3-B12		
14					

Most VaR calculations are not concerned with annual value at risk. The main regulatory and management concern is with loss of portfolio value over a much shorter time period (typically several days or perhaps weeks). It is clear that the distribution formula:

$$\ln\,(V_T) \sim \text{Normal}\left[\ln(V_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$$

can be used to calculate the VaR over any horizon. Recall that T is measured in annual terms; if there are 250business days in a year, then the daily VaR corresponds to T = 1/250 (for many fixed-income instruments one should use 1/360, 1/365, or 1/365.25 depending on the market convention).

As can be seen from the preceding examples, VaR is not—in principle, at least—a very complicated concept. In the implementation of VaR, however, there are two big practical problems:

1. The first problem is the estimation of the parameters of asset return distributions. In "real-world" applications of VaR, it is necessary to estimate means, variances, and correlations of returns. This is a not-inconsiderable problem! We illustrate the importance of the correlations between asset returns. Next, we give a highly simplified example of the estimation of return distributions from market data. For example, you can imagine that a long position in euros and a short position in U.S. dollars is less risky than a position in only one of the currencies, because of a high probability that profits of one position will be mainly offset by losses of another.

2. The second problem is the actual calculation of position sizes. A large financial institution may have thousands of loans outstanding. The database of these loans may not classify them by their riskiness, nor even by their term to maturity. Or-to give a second example-a bank may have offsetting positions in foreign currencies at different branches in different locations. A long position in deutschemarks in New York may be offset by a short position in deutschemarks in Geneva; the bank's risk-which we intend to measure by VaR-is based on the net position. We start with the problem of correlations between asset returns.

We continue the previous example but assume that there are three risky assets. As before, the parameters of the distributions of the asset returns are known: all the means, μ1, μ2, μ3, as well as the variance-covariance matrix of the returns:

$$S = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

The matrix S is of course symmetric, with μ_i the variance of the ith asset's return and σ_{ij} the covariance of the returns of assets i and j (if i = j, σij is the variance of asset i's return).

Suppose that the total portfolio value today is $100 million, with $30 million invested in asset 1, $25 million in asset 2, and $45 million in asset 3. Then the return distribution of the portfolio is given by

$$Mean\ Return = \{x_1 \quad x_2 \quad x_3\} \begin{Bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{Bmatrix} = x_1\mu_1 + x_2\mu_2 + x_3\mu_3$$

$$Variance\ of\ Return = \{x_1 \quad x_2 \quad x_3\} S \{x_1 \quad x_2 \quad x_3\}^T$$

Where x = {x1, x2, x3} = {0.3, 0.25, 0.45} is the vector of proportions invested in each of the three assets. Assuming that the returns are normally distributed (meaning that prices are lognormally distributed), we may calculate the VaR as in the following spreadsheet fragment:

	A	B	C	D	E	F	G	H	I
1				**VaR for Three Asset Portfolio**					
2									
3		Mean		Variance-Covariance				Portfolio	
4		Returns		Matrix				Proportions	
5	Asset 1	10%		0.1	0.04	0.03		0.3	
6	Asset 2	12%		0.04	0.2	-0.04		0.25	
7	Asset 3	13%		0.03	-0.04	0.6		0.45	
8									
9	Initial Investment	100							
10	Mean Return	0.1185	<---MMULT(TRANSPOSE(B5:B7),H5:H7)						
11	Portfolio Sigma	0.38484	<---SQRT(MMULT(MMULT(TRANSPOSE(H5:H7),D5:F7),H5:H7))						
12									
13	Mean Investment Value	11.85							
14	Sigma of Invesvestment Value	38.4838							
15									
16	Cutoff	22.3234	<---NORMINV(0.01,(1+B10)*B9,B11*B9)						
17	Cumulative PDF	0.01	<---NORMDIST(B16,B13,B14,0)						
18	VaR at 1,00% Level	77.6766	<---B9-B16						

Sometimes it helps to simulate data. Suppose that the current date is February10, 1997, and consider a firm that has an investment in two assets:

- It is long two units of an index fund. The fund's current market price is 293, so that the investment in the index fund is worth 2 * 293 = 586.
- It is short a foreign bond denominated in rubles. The bond is a zero-coupon bond (i.e., pays no interest), has face value of 100 rubles and

maturity of May 8, 2000. If the current ruble interest rate is 5.30 percent, then the February 10, 1997, ruble value of the bond is −100 * exp [−5.30 percent * (May 8, 2000 − Feb. 10, 1997)/365] = −84.2166

In dollars, the value of the bond is −84.2166 * 3.40 = −286.3365, so that the net portfolio value is 586 − 286.3365 =299.66.

This example is illustrated in the following display:

	A	B	C	D	E	F	G	H	I	J	K
1											
2											
3	Units of Index Held	2									
4	Bond Maturity	5/8/2000									
5											
6				Ruble	Ruble		Total	Ruble	Dollar		
7			Index	Interest	Exchange		Index	Bond	Bond	Portfolio	
8		Date	Value	Rate	Rate		Value	Value	Value	Value	
9		2/10/1997	293	5.30%	3.40		586	-84.21660441	-286.336455	299.66355	
10											
11							B3*C9		H9*E9		G9+I9
12											
13							-100*EXP(-5.3%*(B4-B9)/365)				
14											

Now suppose we have exchange-rate and index data.

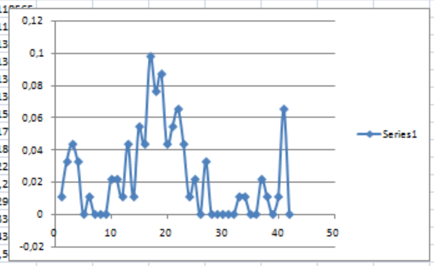
	A	B	C	D	E	F	G
5							
6			Foreign	Exchange		Portfolio	
7	Day	Index	Interest R	Rate		Value	
8	1/2/1997	462.71	5.28%	3.5		632.13	
9	1/3/1997	514.71	526.60%	3.47		738.41	
10	1/4/1997	456.5	5.23%	3.46		622.46	
11	1/5/1997	487.39	5.24%	3.45		685.17	
12	1/6/1997	470.42	5.25%	3.45		651.28	
13	2/8/1997	467.14	5.31%	3.44		644.75	
14	2/9/1997	562.06	5.32%	3.41		637.17	
15	2/10/1997	481.61	5.30%	3.4		676.88	
16							
17							

We want to use these data as a basis for generating "random" return data. We illustrate one technique for doing so, called bootstrapping. This term refers to random reshufflings of the data. For each iteration, we reorder the series of index prices, interest rates, and exchange rates and calculate the return on the portfolio. We need to iterate random numbers for each variable and lookup value in first table.

	A	B	C	D	E	F	G	H	I	J
1	Bootstraping Return Distribution									
2										
3			Units	2						
4			Maturity	07.05.00						
5										
6					Foreign	Exchange		Portfolio		
7			Day	Index	Interest Rate	Rate		Value		
8		1	02.01.97	462,71	5,28%	3,5		632,13	<---B8*$B$1-100*EXP(-C8*(($B$2-A8)/365))*D8	
9		2	03.01.97	514,71	5,26%	3,47		738,41		
10		3	04.01.97	456,5	5,23%	3,46		622,49		
11		4	05.01.97	487,39	5,24%	3,45		685,17		
12		5	06.01.97	470,42	5,25%	3,45		651,28		
13		6	08.02.97	467,14	5,31%	3,44		644,75		
14		7	09.02.97	562,06	5,32%	3,41		837,17		
15		8	10.02.97	481,61	5,30%	3,4		676,88		
16										
17				Foreign Intrest	Foreign	Exchange Rate			Portfolio	
18		Index Rand	Index	Rate Rand	Interest Rate	Rand	Exchange Rate		Value	
19	07.01.97	4	487,39	2	0,0526	8	3,4		689,47	
20	08.01.97	4	487,39	2	0,0526	1	3,5		681,04	-0,012232
21	09.01.97	6	467,14	6	0,0531	7	3,41		648,53	-0,047739
22	10.01.97	8	481,61	2	0,0526	1	3,5		669,39	0,0321763
23	11.01.97	3	456,5	8	0,053	6	3,44		624,55	-0,066987
24	12.01.97	4	487,39	6	0,0531	2	3,47		683,87	0,0949779

The bootstrapped return data look like this:

	M	N	O	P	Q	R	S	T
16								
17		Portfolio						
18		Value						
19		686,02			-0,25818	1	0,01087	0,01087
20		684,04	-0,002884		-0,24297	3	0,032609	0,043478
21		685,58	0,0022475		-0,22775	4	0,043478	0,086957
22		654,57	-0,04523		-0,21254	3	0,032609	0,11
23		652,49	-0,003179		-0,19733	0	0	0,11
24		684,71	0,0493822		-0,18212	1	0,01087	0,13
25		677,30	-0,010822		-0,16691	0	0	0,13
26		624,62	-0,077782		-0,1517	0	0	0,13
27		619,45	-0,008277		-0,13649	0	0	0,13
28		832,27	0,3435665		-0,12128	2	0,021739	0,15
29		645,39	-0,224542		-0,10606	2	0,021739	0,17
30		649,58	0,0064937		-0,09085	1	0,01087	0,18
31		647,54	-0,003138		-0,07564	4	0,043478	0,22
32		838,18	0,2944042		-0,06043	1	0,01087	0,2
33		621,78	-0,258177		-0,04522	5	0,054348	0,29
34		626,05	0,006854		-0,03001	4	0,043478	0,33
35		829,55	0,3250626		-0,0148	9	0,097826	0,43
36		739,40	-0,108676		0,000415	7	0,076087	0,5
37		648,70	-0,122663		0,015626	8	0,086957	0,597826
38		676,01	0,0420921		0,030838	4	0,043478	0,641304
39		671,68	-0,006405		0,046049	5	0,054348	0,695652



The graph on the right indicates the return distribution, which is far from normal. From columns Q, R, and S, you can tell that the 5 percent VaR is about −47 percent, meaning that with a probability of 5 percent, the firm could lose 24percent of its investment.

Financial Modeling, 2nd ed., Simon Benninga, 2000; Chapter 12, pp.175-184

Portfolio Optimization Methods

63. Define necessary conditions and solution to Pareto multi-objective optimization (MOO) problem for optimal portfolio construction

We can define volatility and return of portfolio, p, as follows:

$$\sigma_p^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + \cdots + x_n^2\sigma_n^2 + 2x_1x_2Cov(r_1,r_2) + \cdots + 2x_1x_nCov(r_1,r_n) + \cdots + 2x_2x_nCov(r_2,r_n)$$

$$r_p = x_1r_1 + x_2r_2 + \cdots + x_nr_n + x_fr_f$$

Necessary condition for optimization problem

$$x_1 + x_2 + \cdots + x_n + x_f = 1$$

Portfolio optimization problem - MOO (Pareto multi-objective optimization) is formulated as: if $P = f\left(\sigma_p^2, r_p\right)$ we can find the optimal weights (x) by simultaneously minimizing volatility and maximizing return of the portfolio by the following way

$$\min P = \mu_1\sigma_p^2 - \mu_2 r_p$$

$$\text{Subject to: } x_1 + x_2 + \cdots + x_n + x_f - 1 = 0$$

with $\mu_1 = \frac{1}{2}$ and $\mu_2 = 1$ we get

$$\min \frac{1}{2}\left(x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + \cdots + x_n^2\sigma_n^2 + 2x_1x_2Cov(r_1,r_2) + \cdots + 2x_1x_nCov(r_1,r_n) + \cdots + 2x_2x_nCov(r_2,r_n)\right) - \left(x_1r_1 + x_2r_2 + \cdots + x_nr_n + x_fr_f\right) + \lambda\left(x_1 + x_2 + \cdots + x_n + x_f - 1\right)$$

Taking partial derivatives we obtain the following

$$\frac{\partial P}{\partial x_1} = x_1\sigma_1^2 + x_2Cov(r_1,r_2) + \cdots + x_nCov(r_1,r_n) - r_1 + \lambda = 0$$

$$\frac{\partial P}{\partial x_2} = x_1Cov(r_1,r_2) + x_2\sigma_2^2 + \cdots + x_nCov(r_2,r_n) - r_2 + \lambda = 0$$

$$\cdots$$

$$\frac{\partial P}{\partial x_n} = x_1Cov(r_1,r_n) + x_2Cov(r_2,r_n) + \cdots + x_n\sigma_n^2 - r_n + \lambda = 0$$

$$\frac{\partial P}{\partial x_f} = -r_f + \lambda = 0$$

$$\frac{\partial P}{\partial \lambda} = x_1 + x_2 + \cdots + x_n + x_f - 1 = 0$$

Finding that $\lambda = r_f$ we can obtain the following system of equations

$$
\begin{gathered}
x_1\sigma_1^2 + x_2Cov(r_1, r_2) + \cdots + x_nCov(r_1, r_n) = r_1 - r_f \\
x_1Cov(r_1, r_2) + x_2\sigma_2^2 + \cdots + x_nCov(r_2, r_n) = r_2 - r_f \\
x_1Cov(r_1, r_n) + x_2Cov(r_2, r_n) + \cdots + x_n\sigma_n^2 = r_n - r_f
\end{gathered}
$$

In matrix form

$$
\begin{bmatrix} \sigma_1^2 & Cov(r_1, r_2) & Cov(r_1, r_n) \\ Cov(r_1, r_2) & \sigma_2^2 & Cov(r_2, r_n) \\ Cov(r_1, r_n) & Cov(r_2, r_n) & \sigma_n^2 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = \begin{bmatrix} r_1 - r_f \\ r_2 - r_f \\ r_n - r_f \end{bmatrix}
$$

Defining the matrices and vectors as

$$
C = \begin{bmatrix} \sigma_1^2 & Cov(r_1, r_2) & Cov(r_1, r_n) \\ Cov(r_1, r_2) & \sigma_2^2 & Cov(r_2, r_n) \\ Cov(r_1, r_n) & Cov(r_2, r_n) & \sigma_n^2 \end{bmatrix}
$$

$$
X = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}
$$

$$
R = \begin{bmatrix} r_1 \\ r_2 \\ r_n \end{bmatrix}
$$

We have the following equation for optimal weights

$$
X = C^{-1}(R - r_f)
$$

By normalizing the weights, we obtain

$$
X = \frac{C^{-1}(R - r_f)}{\sum C^{-1}(R - r_f)}
$$

K. Deb, Multi-Objective Optimization using Evolutionary Algorithms, John Wiley & Sons, Inc., 2001

64. Construct efficient portfolios with short sales

We begin with some preliminary definitions and notation. We then state the major results. We implement these results, showing

- How to calculate efficient portfolios.
- How to calculate the efficient frontier.

Throughout this example we use the following notation: There are N risky assets, each of which has expected return $E(r_i)$. The variable R is the column vector of expected returns of these assets:

$$R = \begin{bmatrix} E(r_1) = \overline{r_1} \\ E(r_2) = \overline{r_2} \\ \vdots \\ E(r_N) = \overline{r_N} \end{bmatrix}$$

and S is the N × N variance-covariance matrix:

$$S = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \vdots & \sigma_{2N} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{3N} \\ & \cdots & & \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \sigma_{NN} \end{bmatrix}$$

A portfolio of risky assets (when our intention is clear, we shall just use the word portfolio) is a column vector *x* whose coordinates sum to 1:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_N \end{bmatrix} \sum_{i=1}^{N} x_i = 1$$

Each coordinate x_i represents the proportion of the portfolio invested in risky asset i.

The expected portfolio return E(rx) of a portfolio x is given by the product of x and R:

$$E(r_x) = x^T R = \sum_{i=1}^{N} x_i E(r_i)$$

The *variance of portfolio x's return*, $\sigma^2_x \equiv \sigma_{xx}$ is given by the product

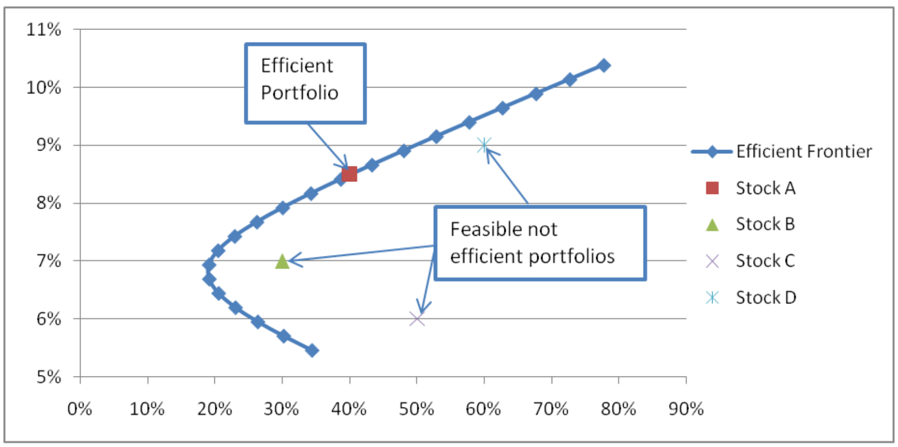
$$x^T S x = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}$$

The covariance between the return of two portfolios x and y, $\mathrm{Cov}(r_x, r_y)$, is defined by the product

$$\sigma_{xy} = x^T S y = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i y_j \sigma_{ij}$$

Note that $\sigma_{xy} = \sigma_{yx}$.

The following graph illustrates four concepts. A feasible portfolio is any portfolio whose proportions sum to one. The feasible set is the set of portfolio means and standard deviations generated by the feasible portfolios; this feasible set is the area inside and to the right of the curved line. A feasible portfolio is on the envelope of the feasible set if for a given mean return it has minimum variance. Finally, a portfolio x is an efficient portfolio if it maximizes the return given the portfolio variance (or standard deviation). That is, x is efficient if there is no other portfolio y such that $E(R_y) > E(R_x)$ and $\sigma_y \leq \sigma_x$. The set of all efficient portfolios is called the efficient frontier; this frontier is the heavier line in the graph.



To construct *Efficient Frontier*, we must find any two efficient portfolios. Any two efficient portfolios are enough to establish the whole frontier. Given any two envelope portfolios $x = \{x_1, \ldots, x_N\}$ and $y = \{y_1, \ldots, y_N\}$, all *efficient portfolios* are convex combinations of x and y.

Let's calculate the efficient frontier using Excel. We consider a world with four risky assets having the following expected returns and variance-covariance matrix:

	A	B	C	D	E	F	G
4							
5		Variance-Covariance matrix				Mean Returns	
6	0,4	0,03	0,02	0		0,06	
7	0,03	0,2	0,001	-0,06		0,05	
8	0,02	0,001	0,3	0,03		0,07	
9	0	-0,06	0,03	0,1		0,08	
10							

We separate our calculations into two parts: First we calculate two efficient portfolios of the feasible set. Then we calculate the efficient frontier.

We must solve the system R − c = Sz for z, where we use two different values for c. This follows from portfolio optimization problem derived in previous question as well as various proofs shown in the mathematical appendix that can be found in the reference to this question and that are based on the following propositions:

Proposition 0: The set of all feasible portfolios of risky assets is convex.

Proposition 1: Let c be a constant and denote by R the vector of mean returns. A portfolio x is on the envelope relative to the sample set of N assets if and only if it is the normalized solution of the system:

$$R - c = Sz$$
$$x_i = \frac{z_i}{\sum_h z_h}$$

Proposition 2: The convex combination of any two envelope portfolios is on the envelope of the feasible set.

Proposition 3: Let y be any envelope portfolio of the set of N assets. Then for any other portfolio x (including, possibly, a portfolio composed of a single asset) there exists a constant c such that the following relation holds between the expected return on x and the expected return on portfolio y:

$$E\,(r_x) = c + \beta\, x\,[\,E\,(r_y) - c]$$

where

$$\beta_x = \frac{Cov(x, y)}{\sigma_y^2}$$

Furthermore, $c = E(r_z)$, where z is any portfolio for which $Cov(z, y) = 0$.

Proposition 4: If in addition to the N risky assets, there exists a risk-free asset with return r_f, then the standard security market line holds:

$$E\,(r_x) = r_f + \beta_x\left[E\,(r_M) - r_f\right], \ \ where\ \beta_x = \frac{Cov(x, M)}{\sigma_M^2}$$

Proposition 5: Suppose that there exists a portfolio y such that for any portfolio x the following relation holds:

$$E(r_x) = c + \beta_x\left[E\left(r_y\right) - c\right], \ \ where\ \beta_x = \frac{Cov(x, y)}{\sigma_y^2}$$

Then the portfolio y is on the envelope.

Following above example to solve the system R − c = Sz for z, the c's we solve for are somewhat arbitrary, but to make life easy, we first solve this system for c = 0. This procedure gives the following results:

	A	B	C
12		Z	X
13		0,1019	0,0540
14		0,5657	0,2998
15		0,1141	0,0605
16		1,1052	0,5857
17		1,88689	
18			

The formulas in the cells are as follows:

- For z: =**MMult(MInverse(A6:D9), F6:F9)**. The range A6:D9 contains the variance-covariance matrix, and the cells F6:F9 contain the mean returns of the assets.
- For x: Each cell contains the associated value of z divided by the sum of all the z's. Thus, for example, cell C13 contains the formula =**B13/B14**

We now solve this system for some other constant c. This solution involves a few extra definitions, as the following picture from the spreadsheet shows:

	A	B	C	D	E	F	G	H
4								
5		Variance-Covariance matrix				Mean Returns	Mean Returns - Constant	
6	0,4	0,03	0,02	0		0,06	-0,005	
7	0,03	0,2	0,001	-0,06		0,05	-0,015	
8	0,02	0,001	0,3	0,03		0,07	0,005	
9	0	-0,06	0,03	0,1		0,08	0,015	
10								
11				Constant	0,065			
12		Z	X		Z	Y		
13		0,1019	0,0540		-0,0101	-0,1163		
14		0,5657	0,2998		-0,0353	-0,4067		
15		0,1141	0,0605		0,0047	0,0544		
16		1,1052	0,5857		0,1274	1,4687		
17		1,886888			0,0867538			

Each cell of the column vector labeled Mean minus constant contains the mean return of the given asset minus the value of the constant c (in this case c = 0.065). The second set of z's and its associated envelope portfolio y is given by

11	Constant	0,065		
12		Z	Y	
13		-0,0101	-0,1163	
14		-0,0353	-0,4067	
15		0,0047	0,0544	
16		0,1274	1,4687	
17		0,086754		

This vector z is calculated in a manner similar to that of the fir vector, except that the array function in the cells is **MMult(MInverse(A6:D9), G6:G9).**

To complete the basic calculations, we compute the means, standard deviations, and covariance of returns for the portfolios *x and y*.

	A	B	C	D	E	F
18						
19	Transpose X					
20	0,0540	0,2998	0,0605	0,5857		
21						
22	Transpose Y					
23	-0,1163	-0,4067	0,0544	1,4687		
24						
25	Mean X	0,0693		Mean Y	0,0940	
26	Var X	0,0367		Var Y	0,3341	
27	Sigma X	0,1917		Sigma Y	0,5780	
28						
29	Cov XY	0,049809				
30	Corr XY	0,449585				

The transpose vectors of x and of y are inserted using the array function Transpose. Now we calculate the **mean, variance,** and **covariance** as follows:

Mean(x) uses the formula **MMult(transpose_x, means)**.
Var(x) uses the formula **MMult(MMult(transpose_x, var_cov), x)**.
Sigma(x) uses the formula **Sqrt(var_x)**.
Cov(x, y) uses the formula **MMult(MMult(transpose_x, var_cov), y)**.
Corr(x, y) uses the formula **cov(x, y)/(sigma_x * sigma_y)**.

The following spreadsheet illustrates everything that has been done so far.

	A	B	C	D	E	F	G	H
5		Variance-Covariance matrix				Mean Returns	Mean Returns - Constan	
6	0,4	0,03	0,02	0		0,06	-0,005	
7	0,03	0,2	0,001	-0,06		0,05	-0,015	
8	0,02	0,001	0,3	0,03		0,07	0,005	
9	0	-0,06	0,03	0,1		0,08	0,015	
10								
11				Constant	0,065			
12		Z	X		Z	Y		
13		0,1019	0,0540		-0,0101	-0,1163		
14		0,5657	0,2998		-0,0353	-0,4067		
15		0,1141	0,0605		0,0047	0,0544		
16		1,1052	0,5857		0,1274	1,4687		
17		1,88689			0,08675			
19	Transpose X							
20	0,0540	0,2998	0,0605	0,5857				
21								
22	Transpose Y							
23	-0,1163	-0,4067	0,0544	1,4687				
24								
25	Mean X	0,0693		Mean Y	0,0940			
26	Var X	0,0367		Var Y	0,3341			
27	Sigma X	0,1917		Sigma Y	0,5780			
28								
29	Cov XY	0,04981						
30	Corr XY	0,44958						

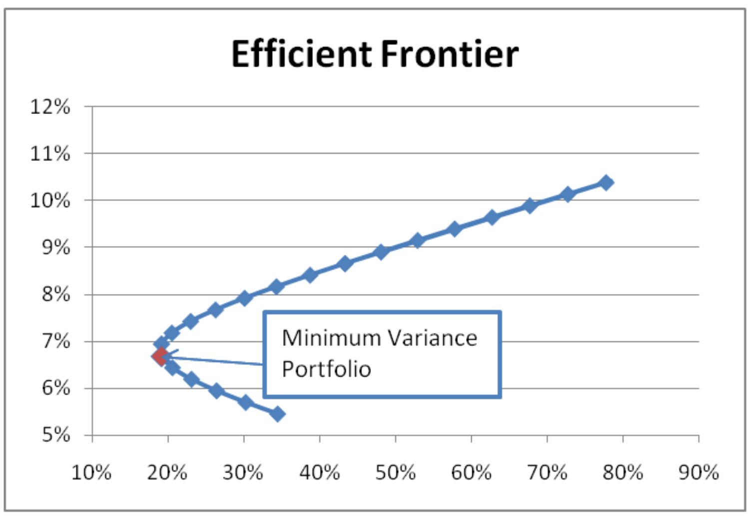
Convex combinations of the two portfolios calculated before allow us to calculate the whole frontier of the feasible set. Suppose we let p be a portfolio that has proportion α invested in portfolio x and proportion $(1-\alpha)$ invested in y. Then—as discussed before—the mean and standard deviation of p's return are

$$E(r_P) = \alpha(r_x) + (1-\alpha)(r_y)$$
$$\sigma_p^2 = \alpha^2\sigma_x^2 + (1-\alpha)^2\sigma_y^2 + 2\alpha(1-\alpha)\sigma_{xy}$$

Here's a sample calculation for our two portfolios:

	A	B	
31			
32	**Single Portfolio Calculation**		
33	Proportion of X	0,3	
34	p's Mean Return	0,086586	
35	p's Sigma	0,433516	
36			

We can turn this calculation into a **data table**. The data table itself has been outlined in black. The five data points in the fourth column give the expected return of the portfolio in the cell to the left; these data points are graphed as a separate data series in the following figure and data table below.



	A	B	C	D	E	F	G
38			Data Table for Efficient				
39			Sigma	Return			
40			0,43352	0,08659	<-- Data Table Header		
41		-0,4	0,77777	0,10385			
42		-0,3	0,72738	0,10138			
43		-0,2	0,67725	0,09892			
44		-0,1	0,62743	0,09645			
45		0	0,57801	0,09398			
46		0,1	0,52911	0,09152			
47		0,2	0,48087	0,08905			
48		0,3	0,43352	0,08659			
49		0,4	0,38738	0,08412			
50		0,5	0,34295	0,08165			
51		0,6	0,30098	0,07919			
52		0,7	0,26266	0,07672			
53		0,8	0,22982	0,07425			
54		0,9	0,20510	0,07179			
55		1	0,19167	0,06932	<-- Minimum Variance Portfolio		
56		1,1	0,19193	0,06685			
57		1,2	0,20581	0,06439			
58		1,3	0,23088	0,06192			
59		1,4	0,26396	0,05946			
60		1,5	0,30244	0,05699			
61		1,6	0,34452	0,05452			

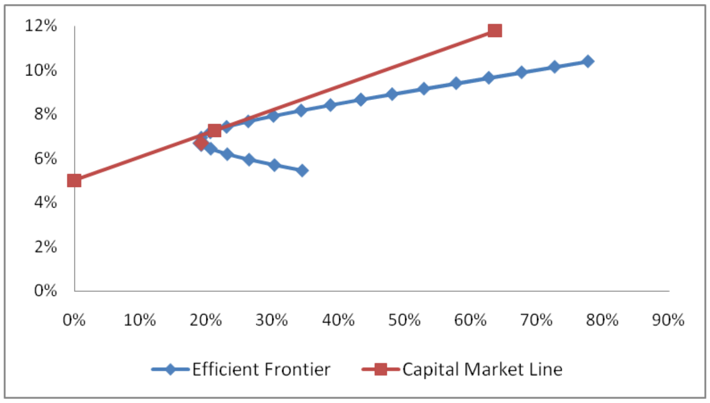
Suppose a risk-free asset exists, and suppose that this asset has expected return r_f. Let M be the efficient portfolio which is constructed using weights solved for this equation:

$$R - r_f = Sz$$

Now consider a convex combination of the portfolio M and the risk-free asset r_f; for example, and suppose that the weight of the risk-free asset in such a portfolio is α. It follows from the standard equations for portfolio return and σ that

$$\mathrm{E}(\mathrm{r_p}) = \alpha \mathrm{r_f} + (1-\alpha)\mathrm{E}(\mathrm{r_M})$$
$$\sigma_P = (1-\alpha)\sigma_m$$

The locus of all such combinations for a≥0 is known as the capital market line. It is graphed along with the efficient frontier as follows:



The portfolio M is called the market portfolio for several reasons:

- Suppose investors agree about the statistical portfolio information (i.e., the vector of expected returns R and the variance-covariance matrix S). Suppose furthermore that investors are interested only in maximizing expected portfolio return given portfolio standard deviation σ. Then it follows that all optimal portfolios will lie on the CML.
- In the case above, it further follows that *the portfolio M is the only portfolio of risky assets included in any optimal portfolio. It must therefore include all the risky assets, with each asset weighted in proportion to its market value.*

It is not difficult to find M when we know r_f: We merely have to solve for the efficient portfolio given that the constant $c = r_f$. When r_f changes, we get a different "market" portfolio—this is just the efficient portfolio given a constant of r_f. For example, in our numerical example, suppose that the risk-free rate is $r_f = 5$ percent. Then solving the system $R - r_f = Sz$ gives

	A	B	C	D	E
72					
73		Constant	5%		
74			0,015746359	0,0314	
75			0,103403879	0,2059	
76			0,029967007	0,0597	
77			0,353052225	0,7031	
78			0,50216947		
79					
80			Mean Y	0,0726	
81			Var Y	0,045	
82			Sigma Y	0,21214	
83					

When there is a risk-free asset, the following linear relationship (known as the SML-the **security market line**) holds:

$$E(R_x) = r_f + \beta_x(E(R_M) - r_f)$$

Where

$$\beta_x = \frac{Cov(x, M)}{\sigma_M^2}$$

Later we explore some statistical techniques for finding the SML that parallel those used by finance researchers.

Financial Modeling, 2nd ed., Simon Benninga, 2000; Chapter 9, pp.132-146
Investments, 8th ed., Zvi Bodie, Alex Kane, Alan J. Marcus, 2009; Chapter 7, pp. 204-218

65. Estimate Beta and construct Security Market Line

We look at some typical capital-market data and replicate a simple test of the CAPM. We have to calculate the betas for a set of assets, and we then have to determine the equation of the security market line (SML). The test herein is the simplest possible test of the CAPM. There is an enormous literature in which the possible statistical and methodological pitfalls of CAPM tests are discussed.

We illustrate the tests of the CAPM with a simple numerical example that uses the same data used in previous question. This example starts with rates of return on six securities and the S&P 500 portfolio. As a first step in analyzing these data and testing the CAPM, we calculate the mean return and the beta of each security's return, where we use the following formulas:

$$Mean\ return\ for\ Security\ i = Average(Security\ i'sReturns, 1972 - 1981)$$

$$\beta_i = \frac{Covar(Security\ i'sreturn, S\&P500\ returns)}{Varp(S\&P500\ Returns)}$$

Here **Average, Covar, and Varp** are Excel functions on the column vectors of returns. Calculating these statistics gives the results in the following spreadsheet. Note that the $\beta_{sp500} = 1$, which is the way it should be if the S&P 500 is the market portfolio. Also note that instead of calculating the β using the **Covar( )** and **Varp( )** functions, we can also use Excel's **Slope( )** function.

	A	B	C	D	E	F	G	H	I
3		AMR	BS	GE	HR	MO	UK	SP500	
4	1974	-0,3505	-0,1154	-0,4246	-0,2107	-0,0758	0,2331	-0,2647	
5	1975	0,7083	0,2472	0,3719	0,2227	0,0213	0,3569	0,372	
6	1976	0,7329	0,3665	0,255	0,5815	0,1276	0,0781	0,2384	
7	1977	-0,2034	-0,4271	-0,049	-0,0938	0,0712	-0,2721	-0,0716	
8	1978	0,1663	-0,0452	-0,0573	0,2751	0,1372	-0,1346	0,0656	
9	1979	-0,2659	0,0158	0,0896	0,0793	0,0215	0,2254	0,1844	
10	1980	0,0124	0,4751	0,335	-0,1894	0,2002	0,3657	0,3242	
11	1981	-0,0264	-0,2042	-0,0275	-0,7427	0,0913	0,0479	-0,0491	
12	1982	1,0642	-0,1493	0,6968	-0,2615	0,2243	0,0456	0,2141	
13	1983	0,1942	0,368	0,311	1,8682	0,2066	0,254	0,2251	
14									
15	Mean	0,20321	0,05314	0,15009	0,15287	0,10254	0,12	0,12384	
16	Beta	1,48214	1,08388	1,31081	1,29925	0,26223	0,49095	1	
17									
18									
19									
20									
21									

=SLOPE(B4:B13,$H$4:$H$13)
=COVAR(B4:B13,H4:H13)/VARP(H4:H13)

The CAPM's security market line postulates that the mean return of each security should be linearly related to its beta. Assuming that the historic data provide an accurate description of the distribution of future returns, we postulate that $E\ (R_i) = \alpha + \beta_i \Pi + \varepsilon_i$. In the second step of our test of the CAPM, we examine this hypothesis by regressing the mean returns on the βs.

Excel offers us several ways of producing regression output. A simple way is to use the functions **Intercept( )**, **Slope( )**, and **Rsq( )** to produce the basic ordinary least-squares results:

	A	B	C	D	E	F
22						
23	**Regressing the Means on Betas**					
24	Intercept	0,07519	<---INTERCEPT(B15:H15;B16:H16)			
25	Slope	0,05474	<---SLOPE(B15:H15;B16:H16)			
26	R-Square	0,28137	<---RSQ(B16:H16;B15:H15)			
27						

These results suggest that the SML is given by $E(R_i) = \alpha + \beta_i P$, where $\alpha = 0.0766$ and $P = 0.0545$. The R^2 of the regression (the percentage of the variability in the means explained by the betas) is 28 percent. We can also use **Tools|Data Analysis|Regression** to produce a new worksheet that has much more output.

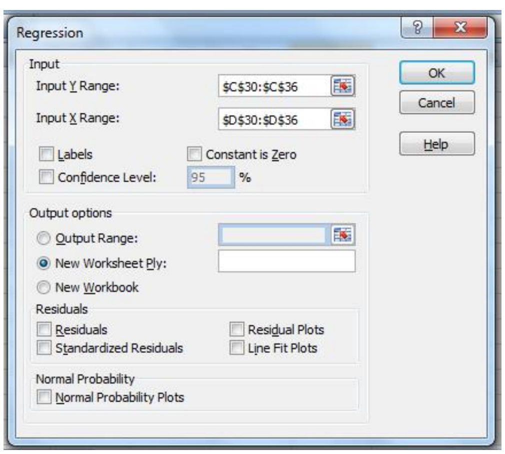
However, this tool will only work if the data are in columns, so we first rewrite the data as

	B	C	D	E
28				
29		Mean	Beta	
30		0,20321	1,482136	
31		0,05314	1,083878	
32		0,15009	1,310811	
33		0,15287	1,299254	
34		0,10254	0,262228	
35		0,12	0,490954	
36		0,12384	1	
37				

Here is some sample output:

	A	B	C	D	E	F	G	H	I
2									
3	*Regression Statistics*								
4	Multiple R	0,5304							
5	R Square	0,2814							
6	Adjusted R Square	0,1376							
7	Standard Error	0,0434							
8	Observations	7							
9									
10	ANOVA								
11		*df*	*SS*	*MS*	*F*	*Significance F*			
12	Regression	1	0,00368	0,00368	1,95766	0,22064			
13	Residual	5	0,00940	0,00188					
14	Total	6	0,01308						
15									
16		*Coefficients*	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*	*Lower 95,0%*	*Upper 95,0%*
17	Intercept	0,0752	0,04206	1,78799	0,13381	-0,03291	0,18330	-0,03291	0,18330
18	X Variable 1	0,0547	0,03913	1,39916	0,22064	-0,04583	0,15532	-0,04583	0,15532
19									

Both the standard error figures and the t-statistics show that neither α nor Π is significantly different from zero. The command that produced this output looks like this:



As discussed, Excel has two functions that give the variance: **Var(array)** gives the sample variance and **Varp (array)** gives the population variance. We use the latter function here.

Above we showed a specific numerical example in which we used some data to test the CAPM. Here we summarize what we did previously. Tests of the CAPM start with return data on a set of assets. The steps in the test are as follows:

- Determine a candidate for the market portfolio M. In the preceding example, we used the Standard & Poor's 500 Index (SP500) as a candidate for M. This is a critical step: In principle, the "true" market portfolio should contain all the market's risky assets in proportion to their value. It is clearly impossible to calculate this theoretical market portfolio, and we must therefore make do with a surrogate.
- For each of the assets in question, determine the asset beta (β).
- Regress the mean returns of the assets on their respective betas; this step should give the security market line (SML).

Our "test" yielded the following SML:

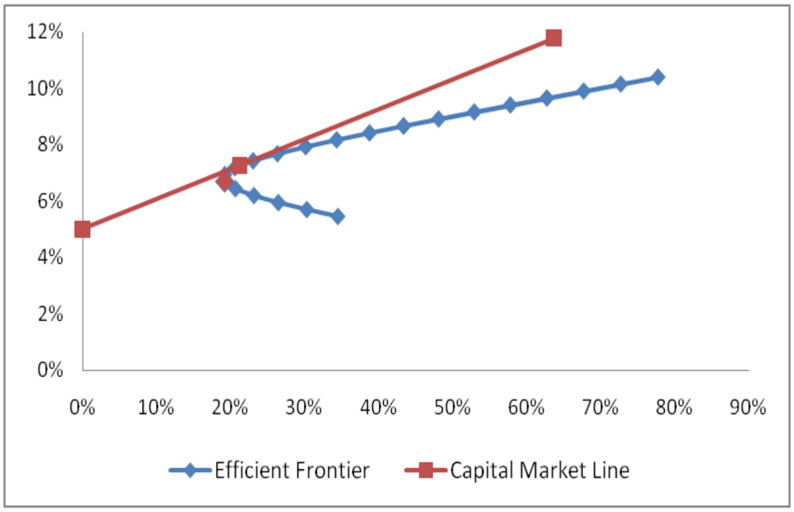
$$E(r_i) = 7.66\% + 5.45\%\beta_i$$

If the intercept is the risk-free rate (or the return on the zero-beta portfolio), then the expected return on the market is $E(r_m)$ =7.66% + 5.45% = 13.11%.

Financial Modeling, 2nd ed., Simon Benninga, 2000; Chapter 10, pp. 151-163
Investments, 8th ed., Zvi Bodie, Alex Kane, Alan J. Marcus, 2009; Chapter 8 pp. 245-259; Chapter 9 pp. 283-295

66. Construct efficient portfolio without short sales

Earlier questions discussed the problem of finding an efficient portfolio. This problem can be redefined as finding a tangent portfolio on the envelope of the feasible set of portfolios:



Our conditions for solving for such an efficient portfolio involved finding the solution to the following problem:

$$Max\Theta = \frac{E(r_x) - c}{\sigma_p}$$

such that

$$\sum_{i=1}^{N} x_i = 1$$

Where

$$E(r_x) = \sum_{i=1}^{N} x_i E(r_i) = x^T R$$

$$\sigma_p = \sqrt{x^T S x} = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij}}$$

Solutions to the maximization problem allow negative portfolio proportions; when xi < 0, this is equivalent to the following assumptions:

- The i-th security is sold short by the investor.
- The proceeds from this short sale become immediately available to the investor.

Reality is, of course, considerably more complicated than this academic model of short sales. In particular, it is rare for all of the short-sale proceeds to become available to the investor at the time of investment, since brokerage houses typically escrow some or even all of the proceeds. It may also be that the investor is completely prohibited from making any short sales (indeed, most small investors seem to proceed on the assumption that short sales are impossible).

In this chapter we investigate these problems. We show how to use Excel's Solver to find efficient portfolios of assets when we restrict short sales.[1] Although the solutions are not perfect (in particular, they take too much time), they are instructive and easy to follow.

We start with the problem of finding an optimal portfolio when there are no short sales allowed. The problem we solve is similar to the maximization problem stated previously, with the addition of the short sales constraint:

$$Max\Theta = \frac{E(r_x) - c}{\sigma_p}$$

such that

$$x_i \geq 0, i = 1,2, \ldots N$$

$$\sum_{i=1}^{N} x_i = 1$$

Where

$$E(r_x) = \sum_{i=1}^{N} x_i E(r_i) = x^T R$$

$$\sigma_p = \sqrt{x^T S x} = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j \sigma_{ij}}$$

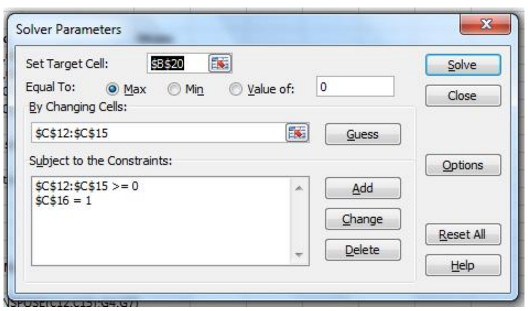
Our problem can be solved in Excel using **Tools|Solver**. We illustrate with the following numerical example, in which there are only four risky assets:

	B	C	D	E	F	G	H
1							
2							
3	**Variance-Covariance Matrix**					**Means**	
4	0,1	0,03	-0,08	0,05		8%	
5	0,03	0,2	0,02	0,03		9%	
6	-0,08	0,02	0,3	0,2		10%	
7	0,05	0,03	0,2	0,9		11%	

In order to solve the portfolio problem with no short sales, we set up the following spreadsheet (which also illustrates a solution to the problem for the c = 9 percent):

	A	B	C	D	E	F	G	H	I
1				No Short-Sales					
2									
3		Variance-Covariance Matrix					Means		
4		0,1	0,03	-0,08	0,05		8%		
5		0,03	0,2	0,02	0,03		9%		
6		-0,08	0,02	0,3	0,2		10%		
7		0,05	0,03	0,2	0,9		11%		
8									
9		c	9%	<- Constant					
10									
11		Optimal Portfolio proportions							
12		x1	0						
13		x2	0						
14		x3	0,55556						
15		x4	0,44444						
16		Total	1	<---SUM(C12:C15)					
17									
18	Portfolio Mean	10,44%	<---MMULT(TRANSPOSE(C12:C15);G4:G7)						
19	Portfolio Sigma	60,76%	<---SQRT(MMULT(TRANSPOSE(C12:C15);MMULT(B4:E7;C12:C15)))						
20	Theta	2,38%	<---(B18-C9)/B19						
21									

The solution was achieved by use the **Tools|Solver** feature of Excel. The first time we bring up the Solver, we create the following dialogue box:



The nonnegativity constraints can be added by clicking on the **Add** button in the preceding dialogue box to bring up the following window (shown here filled in):

The second constraint (which constrains the portfolio proportions to sum to 1) is added in a similar fashion. By changing the value of c in the spreadsheet we can compute other portfolios; in the following example, we have set the constant c equal to 8.50 percent.

	A	B	C	D	E	F	G	H	I
1				No Short-Sales					
2									
3		Variance-Covariance Matrix					Means		
4		0,1	0,03	-0,08	0,05		8%		
5		0,03	0,2	0,02	0,03		9%		
6		-0,08	0,02	0,3	0,2		10%		
7		0,05	0,03	0,2	0,9		11%		
8									
9		c	8,5%	<- Constant					
10									
11		Optimal Portfolio proportions							
12		x1	0						
13		x2	0,25145						
14		x3	0,48847						
15		x4	0,26008						
16		Total	1	<---SUM(C12:C15)					
17									
18	Portfolio Mean	10,01%	<---MMULT(TRANSPOSE(C12:C15);G4:G7)						
19	Portfolio Sigma	45,25%	<---SQRT(MMULT(TRANSPOSE(C12:C15);MMULT(B4:E7;C12:C15)))						
20	Theta	3,33%	<---(B18-C9)/B19						
21									

In both examples, the short-sale restriction is effective: In the first example both x_1 and x_2 are equal to zero, whereas in the second example x_1 equals zero. However, not all values of c give portfolios in which the short-sale constraint is effective. For example, if the constant is 8 percent, we get

	A	B	C	D	E	F	G	H	I
1				No Short-Sales					
2									
3		Variance-Covariance Matrix					Means		
4		0,1	0,03	-0,08	0,05		8%		
5		0,03	0,2	0,02	0,03		9%		
6		-0,08	0,02	0,3	0,2		10%		
7		0,05	0,03	0,2	0,9		11%		
8									
9		c	8,0%	<- Constant					
10									
11		Optimal Portfolio proportions							
12		x1	0,20041						
13		x2	0,25868						
14		x3	0,4219						
15		x4	0,11901						
16		Total	1	<---SUM(C12:C15)					
17									
18	Portfolio Mean	9,46%	<---MMULT(TRANSPOSE(C12:C15);G4:G7)						
19	Portfolio Sigma	31,91%	<---SQRT(MMULT(TRANSPOSE(C12:C15);MMULT(B4:E7;C12:C15)))						
20	Theta	4,57%	<---(B18-C9)/B19						
21									

As c gets lower, the short-sale constraint begins to be effective with respect to asset 4. For example, when c = 3 percent,

	A	B	C	D	E	F	G	H	I
8									
9		c	3,0%	<- Constant					
10									
11		Optimal Portfolio proportions							
12		x1	0,5856						
13		x2	0,09654						
14		x3	0,31786						
15		x4	0						
16		Total	1	<---SUM(C12:C15)					
17									
18	Portfolio Mean	8,73%	<---MMULT(TRANSPOSE(C12:C15);G4:G7)						
19	Portfolio Sigma	20,32%	<---SQRT(MMULT(TRANSPOSE(C12:C15);MMULT(B4:E7;C12:C15)))						
20	Theta	28,21%	<---(B18-C9)/B19						
21									

For very high cs (the next case illustrates c = 11 percent) only asset 4 is included in the maximizing portfolio:

	A	B	C	D	E	F	G	H	I
8									
9		c	11,0%	<- Constant					
10									
11		Optimal Portfolio proportions							
12		x1	0						
13		x2	0						
14		x3	0						
15		x4	1						
16		Total	1	<---SUM(C12:C15)					
17									
18	Portfolio Mean	11,00%	<---MMULT(TRANSPOSE(C12:C15);G4:G7)						
19	Portfolio Sigma	94,87%	<---SQRT(MMULT(TRANSPOSE(C12:C15);MMULT(B4:E7;C12:C15)))						
20	Theta	0,00%	<---(B18-C9)/B19						
21									

If **Tools|Solver** doesn't work, you may not have loaded the Solver add-in. To do so, go to **Tools|Add**-ins and click next to the **Solver Add-in**.

Financial Modeling, 2nd ed., Simon Benninga, 2000; Chapter 11, pp.164-174
Investments, 8th ed., Zvi Bodie, Alex Kane, Alan J. Marcus, 2009; Chapter 7

The Black-Scholes formula applications and analysis

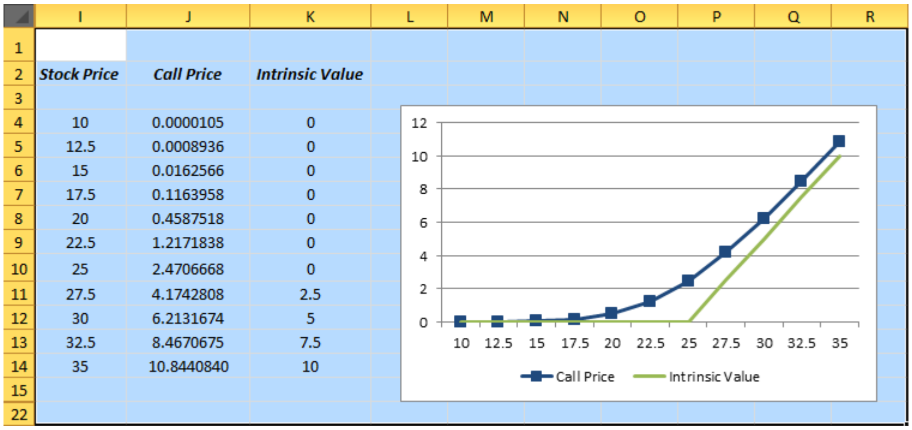
67. Calculate the Black-Scholes formula and options Greeks

The Black-Scholes formulas for call and put pricing are easily implemented in a spreadsheet. The following example shows how to calculate the price of a call option written on a stock whose current $S = 25$, when the exercise price $K = 25$, the annualized interest rate $r = 6$ percent, and $\sigma = 30$ percent. The option has $T = 0.5$ years to exercise. Note that all three of the parameters, T, r, and σ are assumed to be in annual terms.

	A	B	C	D
1		**Black-Scholes Option-Pricing Formula**		
2				
3	*S*	25		current stock price
4	*K*	25		exercise price
5	*r*	0.06		risk-free rate of interest
6	*T*	0.5		time to maturity of option (in years)
7	*σ*	0.3		stock volatility
8				
9	*d1*	0.2475	<=	(LN(S/K) + (r + 0.5*σ^2)*T)/(σ*SQRT(T))
10	*d2*	0.0354	<=	d1 - σ*SQRT(T)
11				
12	*N (d1)*	0.597734	<=	normsdist(d1)
13	*N (d2)*	0.514102	<=	normsdist(d2)
14				
15	***Call Price***	2.47	<=	S*N(d1)-K*EXP(-r*T)*N(d2)
16	***Put Price***	1.73	<=	C - S + K*EXP(-r*T); by Put-Call Parity
17		1.73	<=	K*EXP(-r*T)*N(-d2)-S*N(-d1)

Note that we have calculated the put price twice: Once by using put-call parity, the second time by the direct Black – Scholes formula.

We can use this spreadsheet to do the usual sensitivity analysis. For example, the following Data/Table gives — as the stock price S varies — the Black-Scholes value of the call compared to its intrinsic value [max (S − K, 0)].



	I	J	K
2	*Stock Price*	*Call Price*	*Intrinsic Value*
4	10	0.0000105	0
5	12.5	0.0008936	0
6	15	0.0162566	0
7	17.5	0.1163958	0
8	20	0.4587518	0
9	22.5	1.2171838	0
10	25	2.4706668	0
11	27.5	4.1742808	2.5
12	30	6.2131674	5
13	32.5	8.4670675	7.5
14	35	10.8440840	10

Option Greeks are formulas that express the change in the option price when an input to the formula changes, taking as fixed all the other inputs. Specifically, the Greeks are mathematical derivatives of the option price formula with respect to the inputs.

The Black–Scholes formula has as its inputs the current share price S, the interest rate r, the option life, and the volatility σ amongst other factors. One way to quantify the impact of changes in the inputs on the option value is to calculate the so-called option 'greeks' or hedge parameters. The most commonly calculated hedge parameters are the first-order derivatives: delta (Δ) (for change in share price), rho (ρ) (for change in interest rate), theta (Θ) (for change in option life) and vega (for change in volatility). The second-order derivative with respect to share price, called gamma (Γ), is also calculated. Apart from theta, the hedge parameters are represented by straightforward formulas. The Black–Scholes partial differential equation links theta with the option value, its delta and its gamma.

One important use of Greek measures is to assess risk exposure. For example, a market-making bank with a portfolio of options would want to understand its exposure to stock price changes, interest rates, volatility, etc. A portfolio manager wants to know what happens to the value of a portfolio of stock index options if there is a change in the level of the stock index. An options investor would like to know how interest rate changes and volatility changes affect profit and loss.

Keep in mind that the Greek measures by assumption change only one input at a time. In real life, we would expect interest rates and stock prices, for example, to change together. The Greeks answer the question, what happens when one and only one input changes.

Delta (*Δ*)

Delta measures the change in the option price for a $1 change in the stock price:

$$Call\ delta\ =\ \frac{\partial C(S,K,\sigma,r,T-t,\delta)}{\partial S} = e^{-\delta(T-t)}N(d_1)$$

$$Put\ delta\ =\ \frac{\partial P(S,K,\sigma,r,T-t,\delta)}{\partial S} = -e^{-\delta(T-t)}N(-d_1)$$

Gamma (*Γ*)

Gamma measures the change in delta when the stock price changes:

$$Call\ gamma = \frac{\partial^2 C(S,K,\sigma,r,T-t,\delta)}{\partial^2 S^2} = \frac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}}$$

$$Put\ gamma = \frac{\partial^2 P(S,K,\sigma,r,T-t,\delta)}{\partial^2 S^2} = Call\ Gamma$$

The second equation follows from put-call parity.

Theta (*Θ*)

Theta measures the change in the option price with respect to calendar time (*t*), holding fixed time to expiration (*T*):

$$Call\ theta = \frac{\partial C(S,K,\sigma,r,T-t,\delta)}{\partial t}$$
$$= \delta S e^{-\delta(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2) - \frac{Ke^{-r(T-t)}N'(d_2)\sigma}{2\sqrt{T-t}}$$

$$Call\ theta = \frac{\partial P(S,K,\sigma,r,T-t,\delta)}{\partial t}$$
$$= Call\ theta + rKe^{-r(T-t)} - \delta S e^{-\delta(T-t)}$$

If time to expiration is measured in years, theta will be the annualized change in the option value. To obtain a per-day theta, divide by 365.

Vega

Vega measures the change in the option price when volatility changes. Some writers also use the terms lambda or kappa to refer to this measure:

$$Call\ vega = \frac{\partial C(S,K,\sigma,r,T-t,\delta)}{\partial \sigma} = Se^{-\delta(T-t)}N'(d_1)\sqrt{T-t}$$

$$Put\ vega = \frac{\partial P(S,K,\sigma,r,T-t,\delta)}{\partial \sigma} = Call\ Vega$$

It is common to report Vega as the change in the option price per percentage point change in the volatility. This requires dividing the Vega formula above by 100.

Rho (ρ)

Rho is the partial derivative of the option price with respect to the interest rate:

$$Call\ rho = \frac{\partial C(S,K,\sigma,r,T-t,\delta)}{\partial r} = (T-t)Ke^{-r(T-t)}N(d_2)$$

$$Put\ rho = \frac{\partial P(S,K,\sigma,r,T-t,\delta)}{\partial r} = -(T-t)Ke^{-r(T-t)}N(-d_2)$$

These expressions for rho assume a change in *r* of 1.0. We are typically interested in evaluating the effect of a change of 0.01 (100 basis points) or 0.0001 (1 basis point). To report rho as a change per percentage point in the interest rate, divide this measure by 100. To interpret it as a change per basis point, divide by 10,000.

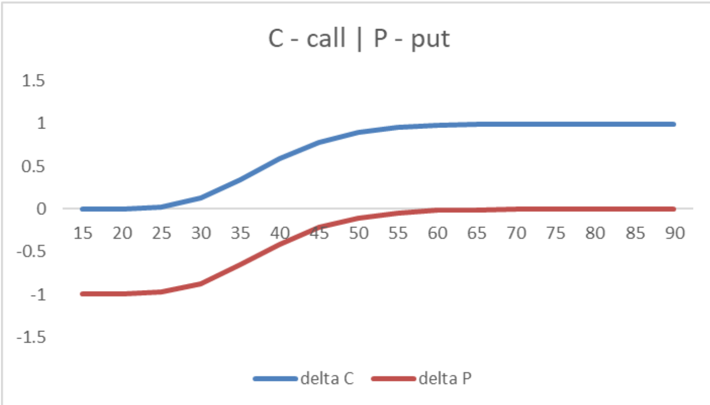
In above example, where S = 25, K = 25, r = 0.06, and σ = 0.30 and T = 0.5, we get the following greeks:

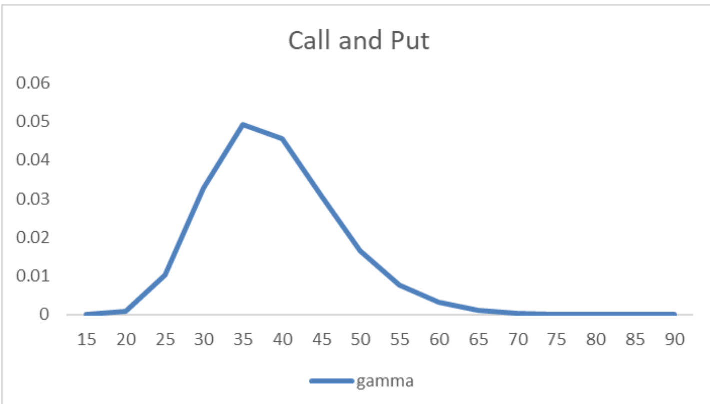
	I	J	K
16		***Call***	***Put***
17	delta	0.5977345	-0.4022655
18	gamma	0.0729564	0.0729564
19	rho	0.0623635	-0.0589422
20	theta	-0.0076719	-0.0036838
21	vega	0.0683966	0.0683966

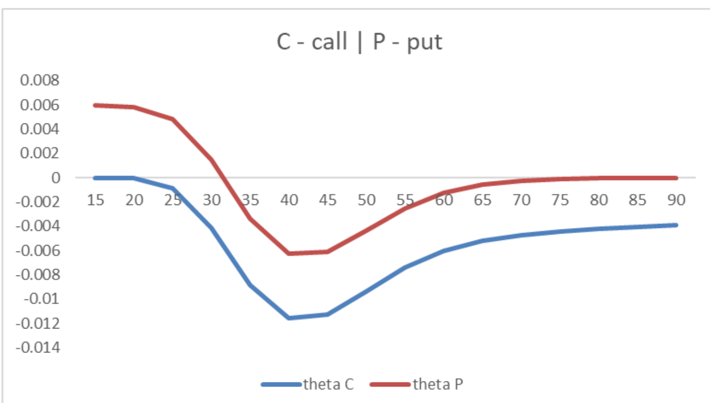
Note that $N'(x)$ is:

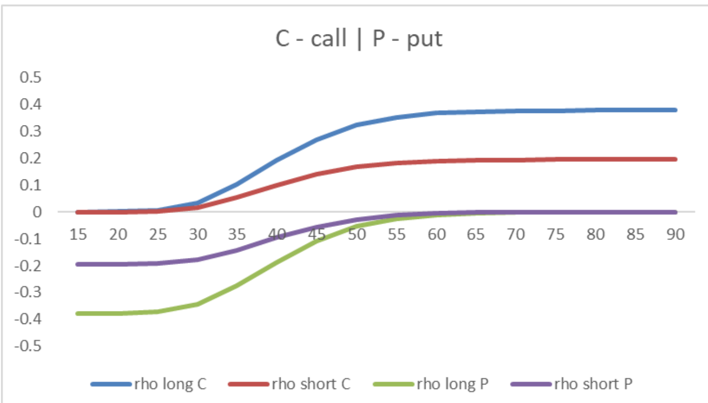
$$\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

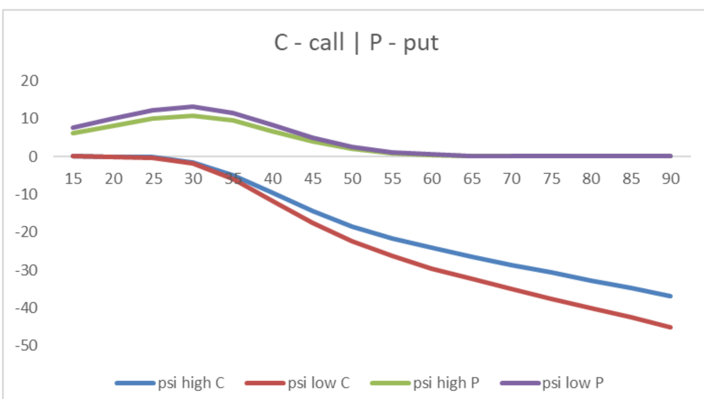
Greeks are calculated and graphed below for the same call and put options as in the table above.


C - call | P - put
1.5
1
0.5
0
-0.5
-1
-1.5
15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90
delta C
delta P


Call and Put
0.06
0.05
0.04
0.03
0.02
0.01
0
15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90
gamma


C - call | P - put
0.008
0.006
0.004
0.002
0
-0.002
-0.004
-0.006
-0.008
-0.01
-0.012
-0.014
15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90
theta C
theta P





Financial Modeling, Second Edition, Simon Benninga; MIT Press 2000, pp. 247-257

Derivatives Markets, Second Edition, Robert L. McDonald; Pearson Education Inc. 2006, pp. 410-412

Interest rate models and valuation

68. Describe the term structure models with drift

The continuously compounded instantaneous rate, denoted r_t, will change (over time) according to the following relationship:

$$dr = \sigma dw$$

where:

dr = change in interest rates over small time interval, dt

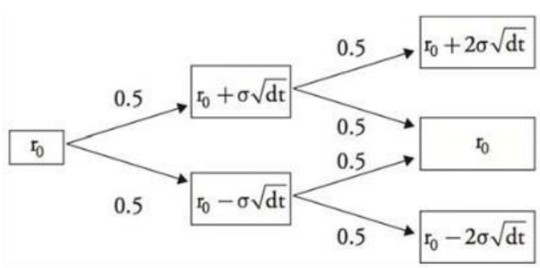
dt = small time interval (measured in years) (e.g., one month = 1/12)

σ = annual basis-point volatility of rate changes

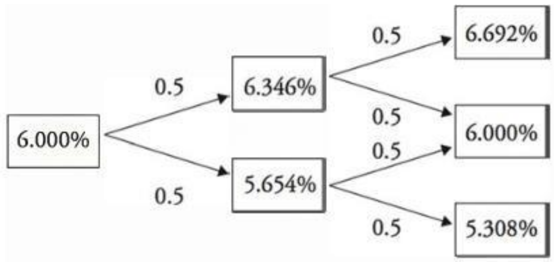
dw = normally distributed random variable with mean 0 and standard deviation $\sqrt{dt}$

For example, consider the evolution of interest rates on a monthly basis. Assume the current short-term interest rate is 6% and annual volatility is 120bps. Using the above notation, r_0 = 6%, σ = 1.2%, and dt = 1/12. Therefore, *dw* has a mean of 0 and standard deviation of $\sqrt{1/12}$ = 0.2887. After one month passes, assume the random variable *dw* takes on a value of 0.2 (drawn from a normal distribution with mean = 0 and standard deviation = 0.2887). Therefore, the change in interest rates over one month is calculated as: dr = 1.20% x 0.2 = 0.24% = 24 basis points. Since the initial rate was 6% and interest rates "changed" by 0.24%, the new spot rate in one month will be: 6% + 0.24% = 6.24%.

In Model 1, since the expected value of dw is zero [i.e., E(dw) = 0], the drift will be zero. Also, since the standard deviation of dw = $\sqrt{dt}$, **the volatility of the rate change = σ√dt**. This expression is also referred to **as the standard deviation of the rate**. In the preceding example, the standard deviation of the rate is calculated as: 1.2% x $\sqrt{1/12}$ =0.346% = 34.6 bps. We are now ready to construct an interest rate tree using Model 1.



$r_0 = r_0, hence\ no\ drift$



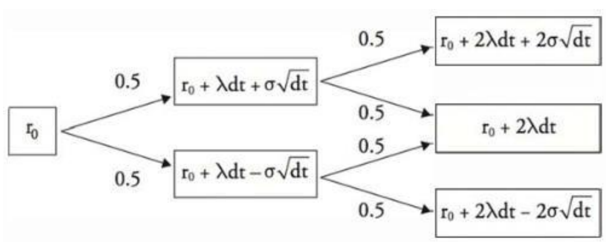
One obvious drawback to Model 1 is that there is always a positive probability that interest rates could become negative. There are two reasonable solutions for negative interest rates. First, the model could use distributions that are always non-negative, such as lognormal or chi-squared distributions. But this action may introduce other non-desirable characteristics such as skewness or inappropriate volatilities. Second, the interest rate tree can "force" negative interest rates to take a value of zero. This method may be preferred over the first method because it forces a change in the original distribution only in a very low interest rate environment whereas changing the entire distribution will impact a much wider range of rates. As a final note, it is ultimately up to the user to decide on the appropriateness of the model. For example, if the purpose of the term structure model is to price coupon-paying bonds, then the valuation is closely tied to the average interest rate over the life of the bond and the possible effect of negative interest rates (small probability of occurring or staying negative for long) is less important. On the other hand, option valuation models that have asymmetric payoffs will be more affected by the negative interest rate problem. Conclusions on Model 1:

- The no-drift assumption does not give enough flexibility to accurately model basic term structure shapes. The result is a downward-sloping predicted term structure due to a larger convexity effect. Recall that the convexity effect is the difference between the model par yield using its assumed volatility and the par yield in the structural model with assumed zero volatility (i.e. r(0,1) = r(0,2) = r(0,3) = e.g. 8%).
- Model 1 predicts a flat term structure of volatility, whereas the observed volatility term structure is hump-shaped, rising and then falling.
- Model 1 only has one factor, the short-term rate. Other models that incorporate additional factors (e.g., drift, time-dependent volatility) form a richer set of predictions.
- Model 1 implies that any change in the short-term rate would lead to a parallel shift in the yield curve, again, a finding incongruous with observed (non-parallel) yield curve shifts.

A natural extension to Model 1 is to add a positive drift term that can be economically interpreted as a positive risk premium associated with longer time horizons. We can augment Model 1 with a constant drift term, which yields Model 2:

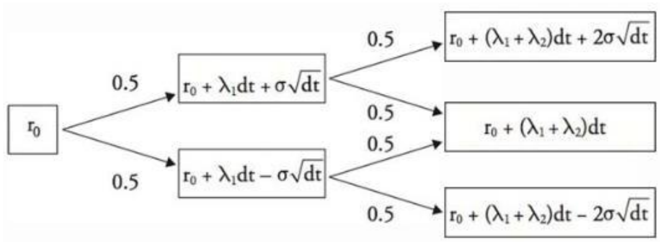
$$dr = \lambda dt + \sigma dw$$

The interest rate tree for Model 2 will look very similar to Model 1, but the drift term, λdt, will increase by λdt in the next period, 2λdt in the second period, and so on. This is visually represented on a tree below. Note that the tree recombines at time 2, but the value at time 2, $r_0 + 2\lambda dt$, is greater than the original rate, r_0, due to the positive drift.



Intuitively, the drift term can accommodate the typically observed upward-sloping nature of the term structure. In practice, a researcher is likely to choose r_0 and λ based on the calibration of observed rates. Hence, the term structure will fit better. The downside of this approach is that the estimated value of drift could be relatively high, especially if considered as a risk premium only. On the other hand, if the drift is viewed as a combination of the risk premium and the expected rate change, the model suggests that the expected rates in year 10 will be higher than year 9, for example. This view is more appropriate in the short run, since it is more difficult to justify increases in expected rates in the long run.

The model further generalizes the drift to incorporate time-dependency. That is, the drift in time 1 may be different than the drift in time 2; additionally, each drift does not have to increase and can even be negative. Thus, the model is more flexible than the constant drift model. Once again, the drift is a combination of the risk premium over the period and the expected rate change.



It is clear that if $\lambda 1 = \lambda 2$ then the Ho-Lee model reduces to Model 2. Also, it should not be surprising that $\lambda 1$ and $\lambda 2$ are estimated (*presumably, calibrated*) from observed market prices.

Broadly speaking, there are two types of models: arbitrage-free models and equilibrium models. The key factor in choosing between these two models is based on the need to match market prices. Arbitrage models are often used to quote the prices of securities that are illiquid or customized. For example, an arbitrage-free tree is constructed to properly price on-the-run Treasury securities (i.e., the model price must match the market price). Then, the arbitrage-free tree is used to predict off-the-run Treasury securities and is compared to market prices to determine if the bonds are properly valued. These arbitrage models are also commonly used for pricing derivatives based on observable prices of the underlying security (e.g., options on bonds). There are two potential detractors of arbitrage-free models. First, calibrating to market prices is still subject to the suitability of the original pricing model. For example, if the parallel shift assumption is not appropriate, then a better fitting model (by adding drift) will still be faulty. Second, arbitrage models assume the underlying prices are accurate. This will not be the case if there is an external, temporary, exogenous shock (e.g., oversupply of securities from forced liquidation, which temporarily depresses market prices). If the purpose of the model is relative analysis (i.e., comparing the value of one security to another), then using arbitrage-free models, which assume both securities are properly priced, is meaningless. Hence, for relative analysis, equilibrium models would be used rather than arbitrage-free models.

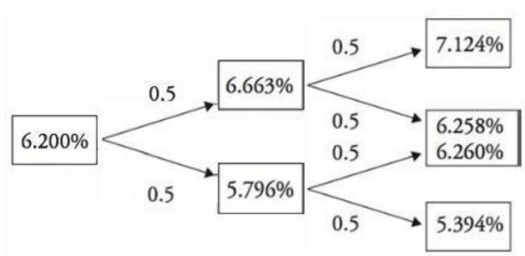
The Vasicek model assumes a mean-reverting process for short-term interest rates. The underlying assumption is that the economy has an equilibrium level based on economic fundamentals such as long-run monetary supply, technological innovations, and similar factors. Therefore, if the short-term rate is above the long-run equilibrium value, the drift adjustment will be negative to bring the current rate closer to its mean-reverting level. Mean reversion is a reasonable assumption but clearly breaks down in periods of extremely high inflation (i.e., hyperinflation) or similar structural breaks. The formal Vasicek model is as follows:

$$dr = k(\theta - r)dt + \sigma dw$$

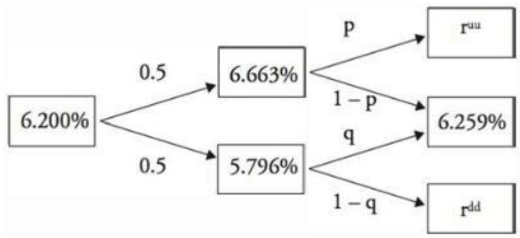
In this model, k measures the speed of the mean reversion adjustment; a high k will produce quicker (larger) adjustments than smaller values of k. A larger differential between the long-run and current rates will produce a larger adjustment in the current period. Similar to the previous discussion, the drift term, λ, is a combination of the expected rate change and a risk premium. The risk neutrality assumption of the long-run value of the short-term rate allows θ to be approximated with r_l, the long-run true interest rate, as:

$$\Theta \approx r_l + \lambda/k$$

Let's consider a numerical example with a reversion adjustment parameter of 0.03, annual standard deviation of 150 basis points, a true long-term interest rate of 6%, a current interest rate of 6.2%, and annual drift of 0.36%. The long-run value of the short-term rate assuming risk neutrality is approximately $\Theta \approx 0.06 + 0.0036/0.03 = 0.18$. It follows that the forecasted change in the short-term rate for the next period is $0.03(0.18 - 0.062)(1/12) = 0.000295$. And The volatility for the monthly interval is computed as $0.015x\sqrt{1/12} = 0.0043$ or 0.43% (43 basis points per month). The next step is to populate interest rate tree.



The most interesting observation is that the model is not recombining. The up-down path leads to a 6.258% rate while the down-up path leads to a 6.260% rate. In addition, the down-up path rate is larger than the up-down path rate because the mean reversion adjustment has to be larger for the down path, as the initial interest rate was lower (5.796% versus 6.663%). It is possible to modify the methodology so that a recombining tree is the end result. There are several ways to do this, but we will outline one straightforward method. The first step is to take an average of the two middle nodes (6.258% + 6.260%) / 2 = 6.259%. Next, we remove the assumption of 50% up and 50% down movements by generically replacing them with p(1-p) and q(1-q) as shown below:



The final step for recombining the tree is to solve for p and q and r^{uu} and r^{dd}. We can solve for the unknown values using a system of equations. First, we know that the average of p x r^{uu} and (1 - p) x 6.239% must equal to 6.663% + 0.03(18% - 6.663%)(1/12) = 6.691%. Second, we can use the definition of standard deviation to equate:

$$\sqrt{p(r^{uu} - 6.691\%)^2 + (1 - p)(6.259\% - 6.691\%)^2} = 1.50\% \times \sqrt{\frac{1}{12}}$$

We would then repeat the process for the bottom portion of the tree, solving for q and r^{dd}.

The previous discussion encompassed the rate change in the Vasicek model and the computation of the standard deviation when solving for the parameters in the recombining tree. To continue with the previous example, the current short-term rate is 6.2% with the mean-reversion parameter, k, of 0.03. The long-term mean-reverting level will eventually reach 18%, but it will take a long time since the value of k is quite small. Specifically, the current rate of 6.2% is 11.8% from its ultimate natural level and this difference will decay exponentially at the rate of mean reversion (11.8% is calculated as 18% - 6.2%). To forecast the rate in 10 years, we note that 11.8% x $e^{(-0.03x10)}$ = 8.74%. Therefore, the expected rate in 10 years is 18% – 8.74% = 9.26%.

In the Vasicek model, the expected rate in T years can be represented as the weighted average between the current short-term rate and its long-run horizon value. The weighting factor for the short-term rate decays exponentially by the speed of the mean-reverting parameter:

$$r_0 e^{-kT} + \theta(1 - e^{-kT})$$

A more intuitive measure for computing the forecasted rate in T years uses a factor's half-life, which measures the number of years to close half the distance between the starting rate and mean-reverting level. Numerically:

$$(0.18 - 0.062)e^{-0.03T} = \frac{1}{2}(0.18 - 0.062), \to T = \frac{ln(2)}{0.03} = 23.1\ years$$

In development of the mean-reverting model, the parameters r_0 and θ were calibrated to match observed market prices. Hence, the mean reversion parameter not only improves the specification of the term structure, but also produces a specific term structure of volatility. Specifically, the Vasicek model will produce a term structure of volatility that is declining. Therefore, short-term volatility is overstated and long-term volatility is understated. Furthermore, consider an upward shift in the short-term rate. In the mean-reverting model, the short-term rate will be impacted more than long-term rates. Therefore, the Vasicek model does not imply parallel shifts from exogenous liquidity shocks. Another interpretation concerns the nature of the shock. If the shock is based on short-term economic news, then the mean reversion model implies the shock dissipates as it approaches the long-run mean. *The larger the mean reversion parameter, the quicker the economic news is incorporated. Similarly, the smaller the mean reversion parameter, the longer it takes for the economic news to be assimilated into security prices.* In this case, the economic news is long-lived. In contrast, shocks to short-term rates in models without drift affect all rates equally regardless of maturity (i.e., produce a parallel shift).

Market Risk Measurement and Management, 2019 SchweserNotes, FRM exam prep Part II, Book 1; Reading 13, pp. 150-163

69. Describe term-structure models volatility and distribution

The Cox-Ingersoll-Ross (CIR) mean-reverting model suggests that the term structure of volatility increases with the level of interest rates and does not become negative. The lognormal model also has non-negative interest rates that proportionally increase with the level of the short-term rate. Following the notation convention of the previous topic, the generic continuously compounded instantaneous rate is denoted r_t and will change (over time) according to the following relationship:

$$dr = \lambda(t)dt + \sigma(t)dw$$

It is useful to note how this model augments Model 1 and the Ho-Lee model. The functional form of Model 1 (with no drift), $dr = \sigma dw$, now includes time-dependent drift and time-dependent volatility. The Ho-Lee model, $dr = \lambda(t)dt + \sigma dw$, now includes non-constant volatility. As in the earlier models, dw is normally distributed with mean 0 and standard deviation $\sqrt{dt}$. The relationships between volatility in each period could take on an almost limitless number of combinations. Consider the following model, which is known as Model 3:

$$dr = \lambda(t)dt + \sigma e^{-\alpha t}dw$$

where σ is volatility at time 0, which decreases exponentially to 0 for $\alpha > 0$. This is particularly useful for pricing multi-period derivatives like interest rate caps and floors. Each cap and floor is made up of single period caplets and floorlets (essentially interest rate calls and puts). The payoff to each caplet or floorlet is based on the strike rate and the current short-term rate over the next period. Hence, the pricing of the cap and floor will depend critically on the forecast of $\sigma(t)$ at several future dates. There are some parallels between Model 3 and the mean-reverting drift (Vasicek) model. Specifically, if the initial volatility for both models is equal and the decay rate is the same as the mean reversion rate, then the standard deviations of the terminal distributions are exactly the same. Similarly, if the time-dependent drift in Model 3 is equal to the average interest rate path in the Vasicek model, then the two terminal distributions are identical, an even stronger observation than having the same terminal standard deviation. There are still important differences between these models. First, Model 3 will experience a parallel shift in the yield curve from a change in the short-term rate. Second, the purpose of the model drives the choice of the model. If the model is needed to price options on fixed income instruments, then volatility dependent models are preferred to interpolate between observed market prices. On the other hand, if the model is needed to value or hedge fixed income securities or options, then there is a rationale for choosing mean reversion models. One criticism of time-dependent volatility models is that the market forecasts short-term volatility far out into the future, which is not likely. A compromise is to forecast volatility approaching a constant value (in Model 3, the volatility approaches 0). A point in favor of the mean reversion models is the downward-sloping volatility term structure.

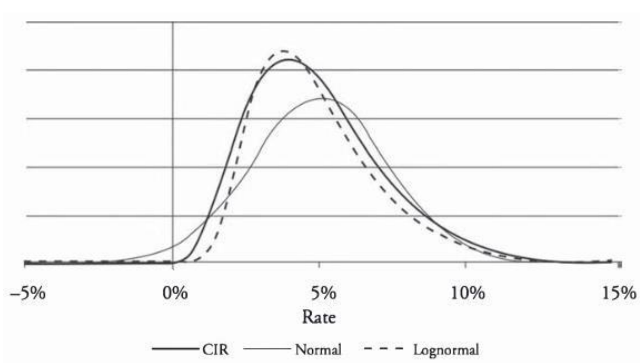
Another issue with the aforementioned models is that the basis-point volatility of the short-term rate is determined independently of the level of the short-term rate. This is questionable on two fronts. First, imagine a period of extremely high inflation (or even hyperinflation). The associated change in rates over the next period is likely to be larger than when rates are closer to their normal level. Second, if the economy is operating in an extremely low interest rate environment, then it seems natural that the volatility of rates will become smaller, as rates should be bounded below by zero or should be at most small, negative rates. In effect, interest rates of zero provide a downside barrier which dampens volatility. A common solution to this problem is to apply a model where the basis-point volatility increases with the short-term rate. Whether the basis-point volatility will increase linearly or non-linearly is based on the particular functional form chosen. A popular model where the basis-point volatility (i.e., annualized volatility of dr) increases proportional to the square root of the rate (i.e., $\sigma\sqrt{r}$) is the Cox-Ingersoll-Ross (CIR) model where dr increases at a decreasing rate and σ is constant. The CIR model is shown as follows:

$$dr = k(\theta - r)dt + \sigma\sqrt{r}dw.$$

A second common specification of a model where basis-point volatility increases with the short-term rate is the lognormal model (Model 4). An important property of the lognormal model is that the yield volatility, σ, is constant, but basis-point volatility increases with the level of the short-term rate. Specifically, basis-point volatility is equal to σr and the functional form of the model, where σ is constant and dr increases at σr is:

$$dr = ardt + \sigma rdw$$

For both the CIR and the lognormal models, as long as the short-term rate is not negative then a positive drift implies that the short-term rate cannot become negative. As discussed previously, this is certainly a positive feature of the models, but it actually may not be that important. For example, if a market maker feels that interest rates will be fairly flat and the possibility of negative rates would have only a marginal impact on the price, then the market maker may opt for the simpler constant volatility model rather than the more complex CIR. Figure below compares the distributions after ten years, assuming equal means and standard deviations for all three models.

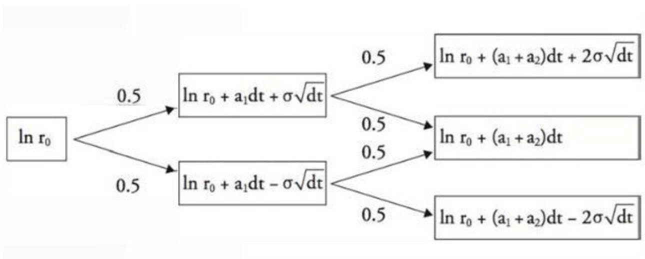


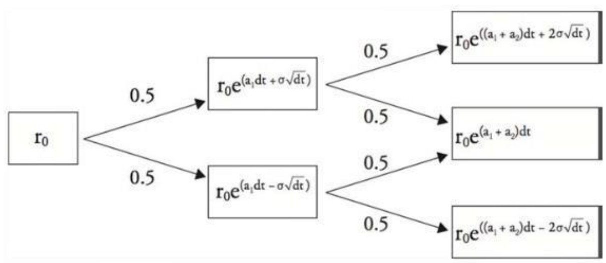
As mentioned before, the normal distribution will always imply a positive probability of negative interest rates. In addition, the longer the forecast horizon, the greater the likelihood of negative rates occurring. In contrast to the normal distribution, the lognormal and CIR terminal distributions are always non-negative and skewed right. This has important pricing implications particularly for out-of-the money options where the mass of the distributions can differ dramatically.

The lognormal model with drift is shown as follows:

$$d[ln(r)] = a(t)dt + \sigma dw$$

The natural log of the short-term rate follows a normal distribution and can be used to construct an interest rate tree based on the natural logarithm of the short-term rate. In the spirit of the Ho-Lee model, where drift can vary from period to period, the interest rate tree in Figure below is generated using the lognormal model with deterministic drift. If each node is exponentiated, the tree will display the interest rates at each node.



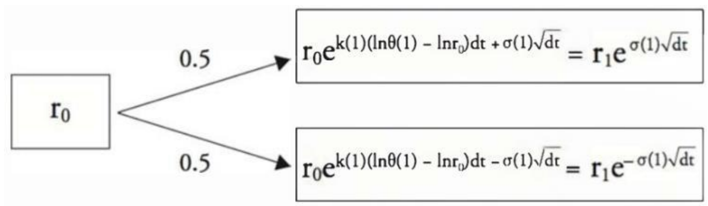


In contrast to the Ho-Lee model, where the drift terms are additive, the drift terms in the lognormal model are multiplicative. The implication is that all rates in the tree will always be positive since $e^x > 0$ for all x. Furthermore, since $e^x \approx 1 + x$, and if we assume $a_1 = 0$ and $dt = 1$, then: $r_0e^\sigma \approx r_0(1 + \sigma)$. Hence, volatility is a percentage of the rate. For example, if $\sigma = 20\%$, then the rate in the upper node will be 20% above the current short-term rate.

The lognormal distribution combined with a mean-reverting process is known as the Black-Karasinski model. This model is very flexible, allowing for time-varying volatility and mean reversion. In logarithmic terms, the model will appear as:

$$d[ln(r)] = k(t)[ln\theta(t) - ln(r)]dt + \sigma(t)dw$$

Thus, the natural log of the short-term rate follows a normal distribution and will revert to the long-run mean of ln[θ(t)] based on adjustment parameter k(t). In addition, volatility is time-dependent, transforming the Vasicek model into a time-varying one. The interest rate tree based on this model is a bit more complicated, but it exhibits the same basic structure as previous models.



Following the intuition of the mean-reverting model (*presumably, the averaging of non-recombining nodes part*), the tree will recombine in the second period only if:

$$k(2) = \frac{\sigma(1) - \sigma(2)}{\sigma(1)dt}.$$

The nodes can be "forced" to recombine by changing the probabilities in the second period to properly value the weighted average of paths in the next period. However, this adjustment varies the length of time between periods (i.e., by manipulating the dt variable). After choosing dt_1, dt_2 is determined with the following equation:

$$k(2) = \frac{1}{dt_2}\left[1 - \frac{\sigma(2)\sqrt{dt_2}}{\sigma(1)\sqrt{dt_1}}\right].$$

Market Risk Measurement and Management, 2019 SchweserNotes, FRM exam prep Part II, Book 1; Reading 14, pp. 164-173

70. Describe OIS discounting, credit issues, and funding costs.

A derivative (or a derivatives portfolio) should earn the risk-free rate if a no-arbitrage condition holds. Given that there are numerous proxies for the risk-free rate, its choice is important. In the United States, the rates on Treasury securities including Treasury bills, notes, and bonds are often considered the closest proxy for risk-free rates. Nevertheless, Treasury rates are seldom used as proxies for risk-free rates given that they are considered to be artificially low for three primary reasons:

1. Due to regulatory requirements, many financial institutions must purchase Treasury securities, driving up their demand and reducing their yield
2. Banks are required to hold significantly less capital in support of Treasury securities than in support of other securities
3. In the U.S., Treasury securities benefit from a favorable tax treatment.

Instead of Treasury rates, most market participants used the London Interbank Offered Rate (LIBOR) as a proxy for the risk-free rate prior to the 2007-2009 credit crisis. LIBOR is a short-term interest rate (one year or under) that creditworthy banks (typically rated AA or stronger) charge each other. LIBOR was historically considered near risk-free because the probability of one of these well-rated banks defaulting within one year was considered to be very small. The idea that LIBOR rates represent risk-free rates changed during the recent credit crisis. LIBOR rates increased materially during this period, and the spread between LIBOR and the U.S. Treasury bill rate increased to multiples of its pre-crisis level. At the same time, the use of collateralized derivative transactions increased, which reduced credit risk in transactions. This meant that LIBOR rates were no longer considered appropriate to discount lower risk derivatives. As a result, banks continued to use LIBOR rates as the risk-free discount rates for non-collateralized transactions because these trades required a higher discount rate to account for their risk. However, for collateralized transactions, banks changed the proxy for risk-free rates from LIBOR to overnight indexed swap (OIS) rates. The main driver of these differing choices is that derivatives should represent a bank's average funding costs. For collateralized transactions, which are funded by collateral, OIS rates are considered a good estimate of funding costs given that the federal funds rate (which is linked to the OIS rate) is the overnight interest typically paid on collateral. For non-collateralized transactions, the average funding costs should be estimated from LIBOR (or even higher) rates.

The overnight rate is the rate at which large financial institutions borrow from each other in the overnight market. In the United States, this rate is called the federal funds rate and is monitored and influenced by the central bank. If a financial institution borrows (lends) funds at the overnight rate, the rate it pays (earns) during the period is the geometric average of the overnight rates. An overnight indexed swap (OIS) is an interest rate swap where a fixed rate is exchanged for a floating rate, and where the floating rate is the geometric average of the overnight federal funds rates during the period. The fixed rate in the OIS is known as the OIS rate. The difference between the LIBOR rate and the OIS rate of the same maturity (e.g., three months) is known as the LIBOR-OIS spread. This spread is typically used as a gauge of market stress as well as the health of the financial system. Under normal market conditions, the spread has historically averaged 10 basis points. During the 2007-2009 financial crisis, the spread rose to over 350 basis points, indicating very high stress levels in the market as liquidity dried up and financial institutions became unwilling to lend to each other for short periods. The OIS rate is not considered perfectly risk-free for two reasons. First, a default between two participants in an overnight loan is still possible. However, the risk of such a default is very small, and institutions with credit problems would typically be shut out of the overnight market. Second, there might be a default on the OIS transaction itself. Any credit risk could be priced into the OIS rate; however, the risk of default is considered small, especially when the OIS is collateralized.

Constructing the OIS zero curve from OIS rates is based on the bootstrapping method. This means that the one-month OIS rate determines the one-month zero rate, the three-month OIS rate determines the three-month zero rate, etc. For longer maturity swaps using periodic settlements, the OIS rate defines a par yield bond. As an example, consider a three-year OIS rate of 3.1% with quarterly settlements. This implies that the three-year bond with a quarterly coupon payment is expected to sell for par. A limitation of OISs is that their maturities tend to be shorter than LIBOR swap maturities, which makes it more challenging to construct the OIS zero curve for longer maturities. However, the OIS zero curve can still be constructed for longer maturities by taking the spread between the longest maturity OIS for which there is reliable information and the corresponding LIBOR swap rate, and also assuming that this spread will remain constant for all longer maturities. For example, assume that there is no reliable data on OISs with maturities greater than four years, and that the four-year LIBOR swap rate is 3.85% and the OIS rate is 3.50%. The spread between the OIS and LIBOR swap is therefore 35 basis points, and the OIS rates are assumed to be 35 basis points lower than the LIBOR swap rates for all maturities exceeding four years. Determining the OIS zero curve is important since these OIS rates are used as the risk-free discount rates for derivatives cash flows. They are also used in deriving forward LIBOR rates to value FRAs and swaps. However, as will be shown in example below, there is a difference in the forward LIBOR rates calculated using LIBOR and OIS discounting.

For example, assume that the fixed rate in a two-year, annual pay LIBOR-for-fixed swap is 5%, and assume that a one-year LIBOR rate is 4%. Given that the swap should be valued at par using zero rates, we can set up the following equation per 100 value of the swap:

$$\frac{5}{1.04}+\frac{105}{(1+R)^2}=100.$$

Solving for R gives us R = 5.025%, which is the two-year LIBOR swap zero rate. Given the one-year and two-year zero rates, we can then calculate the one-year period forward LIBOR rate, or *F*, beginning in one year as follows:

$$F=\frac{1.05025^2}{1.04}-1=6.06\%.$$

We can check our work by setting up an equation to calculate *F* by making the value of the swap equal to zero. The values to be discounted are the payments made to the fixed by the floating side per 100 value of principal. Assuming that forward rates are realized, the swap value is:

$$\frac{5-4}{1.04}+\frac{5-100F}{1.05025^2}=\frac{5-4}{1.04}+\frac{5-6.06}{1.05025^2}=0.$$

Let’s assume that the bank calculated the OIS zero curve at 3.5% and 4.5% for the one-and two-year OIS zero rates assuming annual compounding. Notice that both of these rates are approximately 50 basis points lower than the LIBOR zero rates. Assuming that forward rates are realized, the swap value can be expressed as:

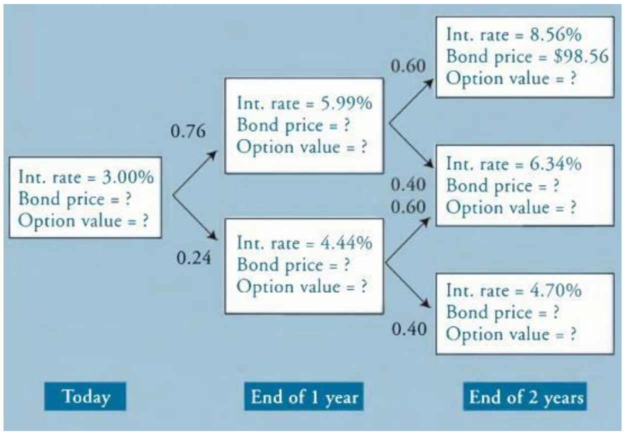
$$\frac{5-4}{1.035}+\frac{5-100F}{1.0405^2}=0,\rightarrow F=6.055\%.$$

The difference is half a basis point, which is arguably small but should not be ignored. The magnitude of the difference depends on the steepness of the zero curve and the maturity of the forward rate.

Market Risk Measurement and Management, 2019 SchweserNotes, FRM exam prep Part II, Book 1; Reading 14, pp. 164-173

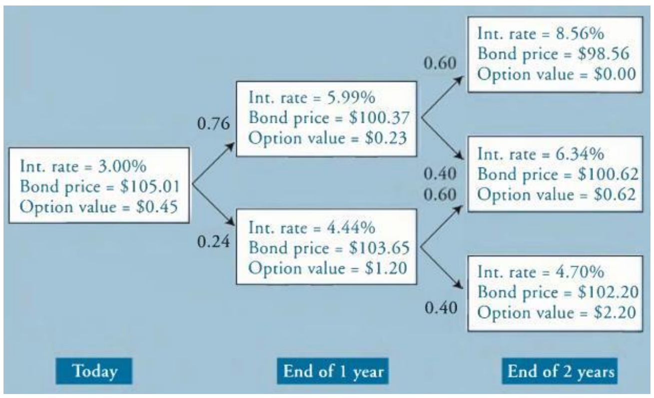
71. Give an example of valuing bond with embedded European call option

Assume that you want to value a European call option with two years to expiration and a strike price of $100.00. The underlying is a 7%, annual-coupon bond with three years to maturity. Figure below represents the first two years of the binomial tree for valuing the underlying bond. Assume that the risk-neutral probability of an up move is 0.76 in year 1 and 0.60 in year 2. Fill in the missing data in the binomial tree, and calculate the value of the European call option.



Step 1: Calculate the bond prices at each node using the backward induction methodology. At the middle node in year 2, the price is $100.62. You can calculate this by noting that at the end of year 2 the bond has one year left to maturity: N = 1; I / Y = 6.34; PMT = 7; FV = 100; CPT => PV = 100.62. At the bottom node in year 2, the price is $102.20: N = 1; I / Y = 4.70; PMT = 7; FV = 100; CPT => PV = 102.20. At the top node in year 1, the price is: $\frac{(\$105.56\times 0.6)+(\$107.62\times 0.4)}{1.0599} = \$100.37$. At the bottom node in year 1, the price is 103.65: $\frac{(\$107.62\times 0.6)+(\$109.20\times 0.4)}{1.0444} = \$103.65$; today, the price is: $\frac{(\$107.37\times 0.76)+(\$110.65\times 0.24)}{1.0300} = \$105.01$. As shown here, the price at a given node is the expected discounted value of the cash flows associated with the two nodes that "feed" into that node. The discount rate that is applied is the prevailing interest rate at the given node. Note that since this is a European option, you really only need the bond prices at the maturity date of the option (end of year 2) if you are given the arbitrage-free interest rate tree. However, it's good practice to compute all the bond prices. Step 2: Determine the intrinsic value of the option at maturity in each node. For example, the intrinsic value of the option at

the bottom node at the end of year 2 is \$2.20 = \$102.20 - \$100.00. At the top node in year 2, the intrinsic value of the option is zero since the bond price is less than the call price. Step 3: Using the backward induction methodology, calculate the option value at each node prior to expiration. For example, at the top node for year 1, the option price is \$0.23: $\frac{(\$0.00\times0.6)+(\$0.62\times0.4)}{1.0599} = \$0.23$. The option value today is computed as: $\frac{(\$0.23\times0.76)+(\$1.20\times0.24)}{1.0300} = \$0.45$.



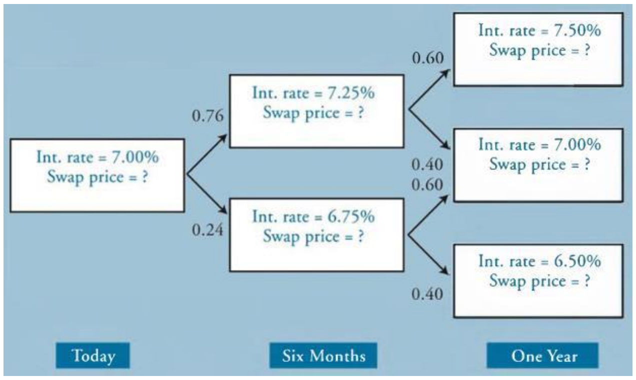
Market Risk Measurement and Management, 2019 SchweserNotes, FRM exam prep Part II, Book 1; Reading 11, pp. 127-128

72. Give an example of valuing constant maturity Swap (CMT)

Assume that you want to value a constant maturity Treasury (CMT) swap. The swap pays the following every six months until maturity:

$$\left(\frac{\$1{,}000{,}000}{2}\right) \times (y_{CMT} - 7\%)$$

y_{CMT} is a semiannually compounded yield, of a predetermined maturity, at the time of payment (y_{CMT} is equivalent to 6-month spot rates). Assume there is a 76% risk-neutral probability of an increase in the 6-month spot rate and a 60% risk-neutral probability of an increase in the 1-year spot rate. Fill in the missing data in the binomial tree, and calculate the value of the swap.



In six months, the top node and bottom node payoffs are, respectively:

$$payoff_{1,U} = \frac{\$1{,}000{,}000}{2} \times (7.25\% - 7.00\%) = \$1{,}250$$
$$payoff_{1,L} = \frac{\$1{,}000{,}000}{2} \times (6.75\% - 7.00\%) = -\$1{,}250$$

Similarly, in one year, the top, middle, and bottom payoffs are, respectively:

$$payoff_{2,U} = \frac{\$1{,}000{,}000}{2} \times (7.50\% - 7.00\%) = \$2{,}500$$
$$payoff_{2,M} = \frac{\$1{,}000{,}000}{2} \times (7.00\% - 7.00\%) = \$0$$
$$payoff_{2,L} = \frac{\$1{,}000{,}000}{2} \times (6.50\% - 7.00\%) = -\$2{,}500$$

The possible prices in six months are given by the expected discounted value of the 1-year payoffs under the risk-neutral probabilities, plus the 6-month payoffs ($1,250 and -$1,250). Hence, the 6-month values for the top and bottom node are, respectively:

$$V_{1,U} = \frac{(\$2{,}500 \times 0.6) + (\$0 \times 0.4)}{1 + 0.0725/2} + \$1{,}250 = \$2{,}697.53$$
$$V_{1,L} = \frac{(\$0 \times 0.6) + (-\$2{,}500 \times 0.4)}{1 + 0.0675/2} - \$1{,}250 = -\$2{,}217.35$$

Today the price is $1,466.63, calculated as follows

$$V_0 = \frac{(\$2{,}697.53 \times 0.76) + (-\$2{,}217.35 \times 0.24)}{1 + \frac{0.0700}{2}} = \$1{,}466.63.$$

Now assume that the market price of the CMT swap was instead \$1,464.40, which is \$2.23 less than the model price. In this case, the OAS to be added to each discounted risk-neutral rate in the CMT swap binomial tree turns out to be 20 basis points. In six months, the rates to be adjusted are 7.25% in the up node and 6.75% in the down node. Incorporating the OAS into the six-month rates generates the following new swap values:

$$V_{1,U} = \frac{(\$2{,}500 \times 0.6) + (\$0 \times 0.4)}{1 + 0.0745/2} + \$1{,}250 = \$2{,}696.13$$
$$V_{1,L} = \frac{(\$0 \times 0.6) + (-\$2{,}500 \times 0.4)}{1 + 0.0695/2} - \$1{,}250 = -\$2{,}216.42$$

Notice that the only rates adjusted by the OAS spread are the rates used for discounting values. The OAS does not impact the rates used for estimating cash flows. In this example, the discounted rate of 7% is adjusted by 20 basis points to 7.2%. The updated initial CMT swap value is:

$$V_0 = \frac{(\$2{,}696.13 \times 0.76) + (-\$2{,}216.42 \times 0.24)}{1 + 0.0720/2} = \$1{,}464.40.$$

Market Risk Measurement and Management, 2019 SchweserNotes, FRM exam prep Part II, Book 1; Reading 11, pp. 129-130

73. Formulate and discuss three continuous-time short-rate models: Rendleman-Bartter, Vasicek, and Cox-Ingersoll-Ross (CIS)

The Rendelman-Bartter Model

The simplest models of the short-term interest rate are those in which the interest rate follows arithmetic or geometric Brownian motion. For example, we could write

$$dr = \alpha dt + \sigma dZ$$

In this specification, the short-rate is normally distributed with mean $r_0 + \alpha t$ and variance $\sigma^2 t$. There are several objections to this model:

- The short-rate can be negative. It is not reasonable to think the nominal short-rate can be negative, since if it were, investors would prefer holding cash under a mattress to holding bonds.
- The drift in the short-rate is constant. If a > 0, for example, the short-rate will drift up over time forever. In practice if the short-rate rises, we expect it to fall; i.e., it ismean-reverting.
- The volatility of the short-rate is the same whether the rate is high or low. In practice, we expect the short-rate to be more volatile if rates are high.

The Rendleman and Bartter (1980) model, by contrast, assumes that the short-rate follows geometric Brownian motion:

$$dr = \alpha r dt + \sigma r dZ$$

While interest rates can never be negative in this model, they can be arbitrarily high. In practice we would expect rates to exhibit mean reversion; if rates are high, we expect them on average to decrease. The Rendleman-Bartter model, on the other hand, says that the probability of rates going up or down is the same whether rates are 100% or 1%.

The Vasicek Model

The Vasicek model incorporates mean reversion:

$$dr = \alpha(b - r)dt + \sigma dZ$$

This is an Ornstein-Uhlenbeck process. The drift term induces mean reversion. Suppose we set a = 20% and b = 10%. The parameter b is the level to which short-term interest rates revert. If r < b, the short-rate is expected to rise. For example, if r(t) = 5%, the instantaneous expected change in the interest rate is 0.01. If r > b, the short-rate is expected to decrease: If r(t) = 20%, the instantaneous expected change in the interest rate is −0.02.

Note also that the standard deviation of interest rates, σ, is independent of the level of the interest rate. This formulation implies that it is possible for interest rates to become negative and that the variability of interest rates is independent of the level of rates. For example, if σ = 1%, a one-standard-deviation move for the short-rate is 100 basis points, whatever the level of the rate.

In the Rendleman-Bartter model, the interest rate is lognormal, so it cannot be negative. In the Vasicek model, by contrast, rates can become negative because the variance is constant.

Vasicek used equation simply to illustrate the pricing methodology without claiming that it was a plausible empirical description of interest rates. The Vasicek model can have unreasonable pricing implications, including negative yields for long-term bonds.

Let the Sharpe ratio for interest rate risk be a constant, φ. With the Vasicek interest rate dynamics becomes

$$\frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} + [a(b-r)+\sigma\phi]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = 0$$

The bond price formula that solves this equation subject to the boundary condition *P (T, T, r)* = 1, and assuming α ≠ 0, is

$$P\,[t,T,r(t)] \;=\; A(t\,,T)e^{-B(t\,,T\,)r(t)}$$

where

$$A(t,T) = e^{\bar{r}(B(t,T)+t-T)-B(t,T)^2\sigma^2/4a}$$

$$B(t,T) = (1-e^{-a(T-t)})/a$$

$$\bar{r} = b + \sigma\phi/a - 0.5\sigma^2/a^2$$

with $\bar{r}$ being the yield to maturity on an infinitely lived bond.

The Cox-Ingersoll-Ross Model

The Cox-Ingersoll-Ross (CIR) model (Cox et al., 1985b) assumes a short-term interest rate model of the form:

$$dr = \alpha(b-r)dt + \sigma\sqrt{r}dZ$$

The standard deviation of the interest rate is proportional to the square root of the interest rate, instead of being constant as in the Vasicek model. Because of this subtle difference, the CIR model satisfies the objections to the earlier models:

- It is impossible for interest rates to be negative. If r = 0 (and assuming that b > 0 and a > 0), the drift in the rate is positive and the variance is zero, so the rate will become positive. If 2ab > σ^2, the interest rate will never reach zero.
- The volatility of the short-rate increases with the level of the short-rate.
- The short-rate exhibits mean reversion.

The assumption that the variance is proportional to √r also turns out to be convenient analytically-Cox, Ingersoll, and Ross (CIR) derive bond and option pricing formulas using this model. The Sharpe ratio in the CIR model takes the form

$$\varphi(r,t) = \bar{\varphi}\sqrt{r}/\sigma$$

With this specification for the risk premium and the CIR interest rate dynamics the partial differential equation for the bond price is

$$\frac{1}{2}\sigma^2 r\frac{\partial^2 P}{\partial r^2} + [a(b-r) + r\bar{\phi}]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = 0$$

The CIR bond price looks similar to that for the Vasicek dynamics but with A(t, T) and B(t, T) defined differently:

$$P\,[t,T,r(t)] \;=\; A(t\,,T)e^{-B(t\,,T\,)r(t)}$$

where

$$A(t,T) = \left[\frac{2\gamma e^{(a-\bar{\phi}+\gamma)(T-t)/2}}{(a-\bar{\phi}+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{2ab/\sigma^2}$$

$$B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(a-\bar{\phi}+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}$$

$$\gamma = \sqrt{(a-\bar{\phi})^2 + 2\sigma^2}$$

With the CIR process, the yield on a long-term bond approaches the value $\bar{r} = 2ab/(a-\varphi+\gamma)$ as time to maturity goes to infinity.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 25, pp. 760-763

74. Compare Vasicek and Cox-Ingersoll (CIR) models

How different are the prices generated by the CIR and Vasicek models? What is the role of the different variance specifications in the two models?

Figure 25.1 below illustrates the yield curves generated by the Vasicek and by the CIR models, assuming that the current short-term rate, r, is 5%, a = 0.2, and b = 10%. Volatility in the Vasicek model is 2% in the top panel and 10% in the bottom panel. The volatility, σ, has a different interpretation in each model. In the Vasicek model, volatility is absolute, whereas in the CIR model, volatility is scaled by the square root of the current interest rate. To make the CIR volatility comparable at the initial interest rate, it is set so that $\sigma_{CIR}\sqrt{r} = \sigma_{Vasicek}$, or 0.0894 in the top panel and 0.447 in the bottom panel. The interest rate risk premium is assumed to be zero.

The two models can exhibit very different behavior. The bottom panel has a relatively high volatility. For short-term bonds—with a maturity extending to about 2.5 years—the yield curves look similar. This is a result of setting the CIR volatility to match the Vasicek volatility. Beyond that point the two diverge, with Vasicek yields below CIR yields. The long-run interest rate in the Vasicek model is −0.025, whereas that in the CIR model is 0.0463. This difference is evident in Figure 25.1 below as the Vasicek yields approach zero (in the long run approaching −0.025).

What accounts for the difference in medium to long-term bonds? As discussed earlier, the pricing formulas are based on averages of interest rate paths. Some of the interest rate paths in the Vasicek model will be negative. Although the typical path will be positive because of mean reversion—rates will be pulled toward 10%—there will be paths on which rates are negative. Because of Jensen's inequality, these paths will be disproportionately important. Over sufficiently long horizons, large negative interest rates become more likely and this leads to negative yields. In the CIR model, this effect results in the long-run yield decreasing with volatility. Negative yields are impossible in the CIR model, however, since the short-term interest rate can never become negative.

In the top panel, with relatively low volatility, both yield curves are upward sloping. The effect of mean reversion outweighs that of volatility. In the long run, the Vasicek yield exceeds the CIR yield because volatility increases with the level of the interest rate in the CIR model. Consequently, the Jensen's inequality effect is more pronounced in the CIR model than in the Vasicek model.

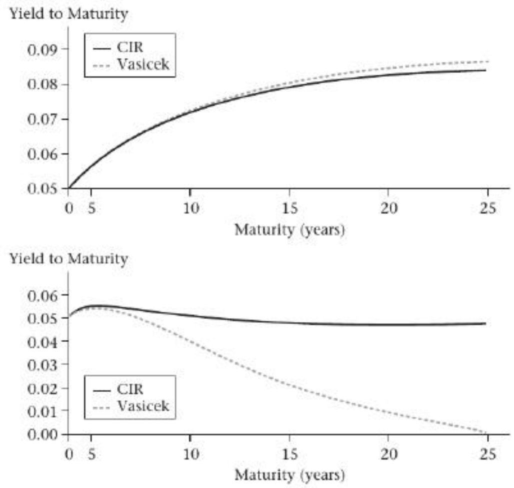


FIGURE 25.1

Yield curves implied by the Vasicek and CIR models, assuming that $r = 0.05$, $a = 0.2$, $b = 0.1$. In the top panel, $\sigma = 0.02$ in the Vasicek model and $\sigma = 0.02/\sqrt{0.05} = 0.0894$ in the CIR model. In the bottom panel, $\sigma = 0.10$ in the Vasicek model and $\sigma = 0.10/\sqrt{0.05} = 0.447$ in the CIR model. In all cases, $\phi = 0$.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 25, pp. 763-764

75. Calculate default-adjusted expected bond returns

Here we discuss the effects of default risk on the returns from holding bonds to maturity. The expected return on a bond that may possibly default is different from the bond's promised return. The latter is defined as the bond's yield to maturity, the internal rate of return calculated from the bond's current market price and its promised coupon payments and promised eventual return of principal in the future. The bond's expected return is less easily calculated: We need to take into account both the bond's probability of future default and the percentage of its principal that holders can expect to recover in the case of default. To complicate matters still further, default can happen in stages, through the gradual degradation of the issuing company's creditworthiness.

We'll use a Markov model to solve for the expected return on a risky bond. Our adjustment procedure takes into account all three of the factors mentioned: the probability of default, the transition of the issuer from one state of creditworthiness to another, and the percentage recovery of face value when the bond defaults. After illustrating the problem and using Excel to solve a small-scale problem, we use some publicly available statistics to program a fuller spreadsheet model. Finally, we show that this model can be used to derive bond betas, the CAPM's risk measure for securities. Before proceeding, we define a number of terms:

- A bond is issued with a given amount of principal or face value. When the bond matures, the bondholder is promised the return of this principal. If the bond is issued at par, then it is sold for the principal amount
- A bond bears an interest rate called the coupon rate. The periodic payment promised to the bondholders is the product of the coupon rate times the bond's face value
- At any given moment, a bond will be sold in the market for a market price. This price may differ from the bond's coupon rate
- The bond's yield to maturity (YTM) is the internal rate of return of the bond, assuming that it is held to maturity and that it does not default

American corporate bonds are rated by various agencies on the basis of the bond issuer's ability to make repayment on the bonds. The classification scheme for two of the major rating agencies, Standard & Poor's (S&P) and Moody's is given in the following table:

Long-Term Senior Debt Ratings

Investment – Grade Ratings			***Speculative – Grade Ratings***		
S&P	***Moody's***	***Interpretation***	***S&P***	***Moody's***	***Interpretation***
AAA	Aaa	Highest Quality	BB+	Ba1	Likely to fulfill obligations; ongoing uncertainty
			BB	Ba2	
			BB-	Ba3	
AA+	Aa1	High Quality	B+	B1	High – risk obligations
AA	Aa2		B	B2	
AA-	Aa3		B-	B3	
A+	A1	Strong Payment Capacity	CCC+	Caa	Current vulnerability to default
A	A2		CCC		
A-	A3		CCC-		
BBB+	Baa1	Adequate payment capacity	C	Ca	In bankruptcy or default, or other marked shortcomings
BBB	Baa2		D	D	
BBB-	Baa3				

When a bond defaults, its holders will typically receive some payoff, though less than the promised bond coupon rate and return of principal. We refer to the percent of face value paid off in default as the *recovery percentage*.

- Besides default risk, bonds are also subject to term-structure risk: The prices of bonds may show significant variations over time as a result of changing term structure. This statement will be especially true for long-term bonds. We will abstract from term-structure risk, confining ourselves only to a discussion of the effects of default risk on bond expected returns.
- Just to complicate matters, in the United States the convention is to add to a bond's listed price the *prorated coupon* between the time of the last

coupon payment and the purchase date. The sum of these two is termed the *invoice price* of the bond; the invoice price is the actual cost at any moment to a purchaser of buying the bond. In our discussion we use the term *market price* to denote the invoice price.

The bond's yield to maturity is not its expected return: It is clear that both a bond's rating and the anticipated payoff to bondholders in the case of bond default should affect its expected return. All other things being equal, we would expect that if two newly issued bonds have the same term to maturity, then the lower-rated bond (having the higher default probability) should have a higher coupon rate. Similarly, we would expect that an issued and traded bond whose rating has been lowered would experience a decrease in price. We might also expect that the lower is the anticipated payoff in the case of default; the lower will be the bond's expected return.

As a simple illustration, we calculate the expected return of a one-year bond that can default at maturity. We use the following symbols:

F = face value of the bond
P = price of bond
c = annual coupon rate of the bond
π = probability that the bond will not default at end of year
λ = fraction of face value that bondholders collect upon default
ER = expected return

This bond's expected end-of-year cash flow is $\pi \times (1 + c) \times F + (1 - \pi) \times \lambda \times F$ and its expected return is given by:

$$ER = \frac{\pi(1+c) + (1-\pi)\lambda F}{P} - 1$$

This calculation is illustrated in the following spreadsheet:

	A	B	C	D	E	F	G	H
1			Expected Return on A One-Year Bond					
2		With An Adjustment For Default Probability						
3								
4	*F*	100						
5	*P*	98						
6	*c*	16%						
7	*π*	90%						
8	*λ*	80%						
9								
10	***Expected Cash Flow***	112.40	<=	B7*(1+B6)*B4+(1-B7)*B8*B4				
11	***ER***	14.69%	<=	B10/B5-1				

We now introduce multiple periods into the problem. Here we define a basic model using a very simple set of ratings. We suppose that at any date there are four possible bond "ratings":

- A – The highest rating.
- B – The next highest rating.
- D – The bond is in default for the first time (and hence pays off Π of the face value).
- E – The bond was in default in the previous period; it therefore pays off 0 in the current period and in any future periods.

The *transition probability* matrix Π is given by:

$$\Pi = \begin{bmatrix} \pi_{AA} & \pi_{AB} & \pi_{AD} & 0 \\ \pi_{BA} & \pi_{BB} & \pi_{BD} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The probabilities π_{ij} indicate the probability that in one period the bond will go from a rating of *i* to a rating of *j*. We will use the following transition probability matrix:

$$\Pi = \begin{bmatrix} .99 & .01 & 0 & 0 \\ .03 & .96 & .01 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What does this matrix Π mean?

- If a bond is rated A in the current period, there is a probability of 0.99 that it will still be rated A in the next period. There is a probability 0.01 that it will be rated B in the next period, but it is impossible for the bond to be rated A today and D or E in the subsequent period. While it is possible to go from ratings A and B to any of ratings A, B, and D; it is *not* possible to go from A or B to E. This statement is true because E denotes that default took place in the previous period.
- In the example of Π, a bond that starts off with a rating of B can – in a subsequent period – be rated A (with a probability of 0.03); be rated B (with a probability of 0.96); or rated D (and hence in default) with a probability of .01

- A bond that is currently in state D (i.e., first-time default); will necessarily be in E in the next period. Thus, the third row of our matrix Π will always be [0 0 0 1]
- Once the rating is in E, it remains there permanently. Therefore, the fourth row of the matrix Π also will always be [0 0 0 1]

The matrix P defines the transition probabilities over one period. The two-period transition probabilities are given by the matrix product $\Pi \times \Pi$:

$$\Pi \times \Pi = \begin{bmatrix} .99 & .01 & 0 & 0 \\ .03 & .96 & .01 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} .99 & .01 & 0 & 0 \\ .03 & .96 & .01 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} .9804 & .0195 & .0001 & 0 \\ .0585 & .9219 & .0096 & .0100 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus if a bond is rated B today, there is a probability of 5.85 percent that in two periods it will be rated A, a probability of 92.19 percent that in two periods it will be rated B, a probability of 0.96 percent that in two periods it will default (and hence be rated D), and a probability of 1 percent that in two periods it will be rated E. The last rating means, of course, that the bond went into default in the first period.

We can use the array function **MMult** function of Excel to calculate multiyear transition probability matrices:

	G	H	I	J	K	L	M	N	O	P	Q	R
4	One - Period Transition Matrix				Two - Period Transition Matrix				Three - Period Transition Matrix			
5	0.99	0.01	0	0	0.9804	0.0195	0.0001	0	0.9712	0.0285	0.0002	0.0001
6	0.03	0.96	0.01	0	0.0585	0.9219	0.0096	0.01	0.0856	0.8856	0.0092	0.0196
7	0	0	0	1	0	0	0	1	0.0000	0.0000	0.0000	1.0000
8	0	0	0	1	0	0	0	1	0.0000	0.0000	0.0000	1.0000

In general, the year – t transition matrix is given by the matrix power Π^t. Calculating these matrix powers by the procedure that we have illustrated is cumbersome, so we first define a VBA function that can compute powers of matrices:

```
Function matrixpower (matrix, n)
      If n = 1 Then
            matrixpower = matrix
      Else
      matrixpower =
```

```
                    Application.MMult(matrixpower(mat
                rix, n - 1) , matrix)
        End If
End Function
```

The use of this function is illustrated in the following spreadsheet. The function **Matrixpower** allows a one-step computation of the power of any transition matrix:

	A	B	C	D	E	F	G
1		Using Function Matrixpower					
2							
3		One-period transition matrix					
4		0.99	0.01	0	0		
5		0.03	0.96	0.01	0		
6		0	0	0	1		
7		0	0	0	1		
8							
9	*t*	10	{=MATRIXPOWER(B4:E7,B9)}				
10							
11		0.915901	0.080152	0.000739	0.003209		
12		0.240456	0.675445	0.007013	0.077087		
13		0	0	0	1		
14		0	0	0	1		
15							

From this example it follows that if a bond started out with an A rating, there is a probability of 0.3 percent that the bond will be in default at the end of ten periods, and there is a probability of 0.07 percent that it will default before the tenth period.

Recall that c denotes the bond's coupon rate and λ denotes the percentage payoff of face value if the bond defaults. The payoff vector of the bond depends on whether the bond is currently in its last period N or whether $t < N$:

$$Payoff(t) = \begin{cases} \begin{bmatrix} c \\ c \\ \lambda \\ 0 \end{bmatrix} & if\ t < N \\ \begin{bmatrix} 1+c \\ 1+c \\ \lambda \\ 0 \end{bmatrix} & if\ t = N \end{cases}$$

The first two elements of each vector denote the payoff in non-defaulted states, the third element λ is the payoff if the rating is D, and the fourth element 0 is the payoff if the bond rating is E. The distinction between the two vectors depends, of course, on the repayment of principal in the terminal period.

Before we can define the expected payoffs, we need to define one further vector, which will denote the *initial state of the bond*. This current-state vector is a vector with a 1 for the current rating of the bond and zeros elsewhere. For example, if the bond has rating A at date 0, then Initial = [1 0 0 0]; if it has date – 0 rating of B, then Initial = [0 1 0 0]. We can now define the expected bond payoff in period *t*:

$$E[Payoff(t)] = Initial \times \Pi^t \times Payoff(t)$$

We continue using the numerical λ from the previous part above, and we further suppose that λ = 0.8, meaning that a defaulted bond will pay off 80 percent of face value in the first period of default. We consider a bond having the following characteristics:

- The bond is currently rated B
- Its coupon rate *c* = 7 percent
- The bond has five more years to maturity
- The bond's current market price is 98 percent of its face value

	A	B	C	D	E	F	G	H
1			Calculating The Expected Bond Return					
2								
3	*P*	98%		*Payoff (t < N)*			*Payoff (t = N)*	
4	*c*	7%		7%			107%	
5	*λ*	80%		7%			107%	
6	*N*	5		80%			80%	
7	*Rating*	B		0			0	
8								
9			*A*	*B*	*C*	*D*		
10	*Transition Matrix*	*A*	0.99	0.01	0	0		
11		*B*	0.03	0.96	0.01	0		
12		*C*	0	0	0	1		
13		*D*	0	0	0	1		
14								
15	*Initial Vector*		0	1	0	0		
16								
17	*Year*	0	1	2	3	4	5	
18	*Expected Payoffs*	-0.98	0.0773	0.076308	0.07535787	0.074447616	1.027389845	
19	*Expected Yield*	**7.24%**						
20								

The spreadsheet above shows the facts in the preceding list as well as the payoff vectors of the bond at dates before maturity (in cells D4:D7) and on the maturity date (cells G4:G7). The transition matrix is given in cells C10:F13, and the initial vector is given in C15:F15.

The expected bond payoffs are given in cells B19:G19. Before we explain how they were calculated, we note the important economic fact that – if the expected payoffs are as given – then the bond's expected return is calculated by IRR(B18:G18). As cell B19 shows, this expected return is 7.245 percent.

As indicated in the previous part above, the period – t expected bond payoff is given by the following formula

$E[Payoff(t)] = Initial \times \Pi^t \times Payoff(t)$. The formula in row 18 uses **IF** statement to implement this formula as:

$$if\begin{pmatrix} t_i = N, mmult\left(initial\ vector, mmult(matrixpower(transition\ matrix, t_i), payoff(t_i = N))\right), \\ mmult\left(initial\ vector, mmult(matrixpower(transition\ matrix, t_i), payoff(t_i < N))\right) \end{pmatrix}$$

Here's what these statements mean:

- If the current year, t_i is equal to the bond term N, then the expected payoff on the bond is $mmult(C15{:}F15, mmult(matrixpower(C10{:}F13, C17), G4{:}G7))$
- If the current year, t_i is less than the bond term, then the expected payoff on the bond is $mmult(C15{:}F15, mmult(matrixpower(C10{:}F13, C17), D4{:}D7))$

Copying this formula gives the whole vector of expected bond payoffs. The actual formula in Cell B19 is IRR (B18:G18), i.e. default-adjusted expected bond return.

A vexatious problem in corporate finance is the computation of bond betas. The model presented here can be easily used to compute the beta of a bond. Recall that the capital asset pricing model's *security market line* (SML) is given by:

$$E(r_d) = r_f + \beta_d\left(E(R_m) - r_f\right)$$

where $E\ (r_d)$ is expected return on debt, r_f is return on riskless debt, and $E\ (r_m)$ is return on equity market portfolio.

If we know expected return on debt, we can calculate β of the debt. Provided we know the risk – free rate and the expected rate of return on the market. Suppose, for example, that the market risk premium is 8.4 percent, and that risk – free rate is 5 percent. Then a bond having an expected return of 8 percent will have a β of .357 as illustrated below:

	A	B	C	D	E	F
1			Calculating A Bond's Beta			
2						
3	*Market Risk Premium*	$E\ (r_m)$	8.40%			
4	*risk-free rate*	r_f	5%			
5	*expected bond return*	$E\ (r_d)$	8%			
6	*implied bond beta*	β	**0.357**	<= (C5-C4)/C3		
7						

Financial Modeling, Second Edition, Simon Benninga; MIT Press 2000, pp. 334-348

Hedging strategies

76. Describe and formulate market maker's strategy for delta-hedging a call and making the hedge Gamma neutral

Delta hedging is a fundamental technique in option pricing. The idea is to replicate an option by a portfolio of stocks and bonds, with the portfolio proportions determined by the Black-Scholes formula.

Suppose we decide to replicate an at-the-money European call option which has 12 weeks to run until expiration. The stock on which the option is written has S_0 = \$40 and exercise price X = \$35, the interest rate is r = 4%, and the stock' s volatility is σ = 25%. The Black-Scholes price of this option is 5.44:

	A	B	C	D	E	F	G	H
1	DELTA HEDGING A CALL							
2	S, current stock price	40.00		Initial pricing of call using Black-Scholes formula				
3	X, exercise	35.00						
4	r, interest rate	2.00%						
5	k, dividend yield	0.00%						
6	T, expiration	0.2308	<-- =12/52					
7	Sigma	25%						
8								
9	BS value	5.44	<-- =bsmertoncall(B2,B3,B6,B4,B5,B7)					
10								
11				Hedging portfolio				
12	Weeks until expiration	Time until expiration	Stock price	Stock = =C13*deltacall(C13,$B$3, B13,$B$4,0,$B$7)	Investment in stock	Bond	Porfolio value	Portfolio cash flow
13	12	0.2308	40.000	35.48		-30.04	5.44	5.44
14	11	0.2115	38.042	30.19	-3.5498	-26.50	3.69	0.00
15	10	0.1923	38.884	33.17	2.3135	-28.82	4.35	0.00
16	9	0.1731	38.568	32.62	-0.2814	-28.55	4.07	0.00
17	8	0.1538	38.501	32.87	0.3044	-28.87	4.00	0.00
18	7	0.1346	37.768	30.87	-1.3759	-27.50	3.36	0.00
19	6	0.1154	39.383	36.54	4.3488	-31.86	4.67	0.00
20	5	0.0962	40.406	39.29	1.8031	-33.68	5.61	0.00
21	4	0.0769	39.626	38.34	-0.1859	-33.51	4.84	0.00
22	3	0.0577	39.216	38.20	0.2496	-33.77	4.43	0.00
23	2	0.0385	39.745	39.58	0.8646	-34.65	4.93	0.00
24	1	0.0192	41.522	41.52	0.1756	-34.83	6.69	0.00
25	0	0.0000	43.199				8.35	
26								
27	Hedged position payoff	8.35	<-- =G25					
28	Actual call payoff	8.20	<-- =MAX(C25-B3,0)					
29								
30								
31	**Formulas**							
32	Cell D14: =C14*deltacall(C14,$B$3,B14,$B$4,0,$B$7)							
33	Cell E14: =D14-D13*C14/C13							
34	Cell F14: =F13*EXP($B$4/52)-E14							
35	Cell G14: =D14+F14							
36	Cell H14: =(D13*C14/C13-D14)+F13*EXP($B$4*(B13-B14))-F14							

At initial date, the stock and bond positions are set using the Black-Scholes formula: Stock = $SN(d_1)$, Bond = $-X^*\exp(-rT)N(d_2)$.

At each subsequent date t, the stock position is adjusted to $S_t{}^*\Delta_{call}$. The bond position is adjusted so that the net cash flow of the portfolio is zero.

At the final date, the stock and bond portfolios are liquidated.

Note that we use the formula BSMertoncall but with the dividend yield k = 0%, so that this is, in effect, a regular BS call option. In the spreadsheet above, we create this option by replicating, on a week-to-week basis, the BS option pricing formula using delta hedging.

- At the beginning, 12 weeks before the option’ s expiration, we determine our stock/bond portfolio according to the formula $Call = SN(d_1) - Xe^{-rT}N(d\_2)$, so that we have a dollar amount $SN(d_1)$ of shares in the portfolio and have borrowing of $Xe^{-rT}N(d_2)$. Having determined the portfolio holdings at the beginning of the 12-week period, we now determine our portfolio holdings for each of the successive weeks as follows:
- In each successive week we set the stock holdings in the portfolio according to the formula $SN(d_1)$, but we set the portfolio borrowing so that the net cash flow of the portfolio is zero. Note that $SN(d_1) = S\Delta_{Call}$, hence the name “delta hedging.”
- At the end of the 12-week period, we liquidate the portfolio.

The delta hedge would be perfect if we rebalanced our portfolio continuously. However, here we have rebalanced only weekly. Had we a perfect hedge, the portfolio would have paid off $max[S_{Terminal} - X, 0]$ (cell B27); the actual hedge payoff (cell B28) is slightly different. Using the technique of Data Table on a blank cell, we replicate this simulation to check the deviation between the desired payoff and the hedge position payoff:

	K	L	M	N	O
12	Simulation	Delta hedge payoff	Max(S_T-X,0)	Hedge—actual payoff	
13		5.4024	5.3669	0.0355	<-- =L13-M13, data table header
14	1	3.4743	3.5591	-0.0848	
15	2	0.3252	0.0000	0.3252	
16	3	0.0971	0.0000	0.0971	
17	4	-0.0193	0.0000	-0.0193	
18	5	11.8112	12.2037	-0.3925	
19	6	3.5610	3.4877	0.0733	
20	7	10.6744	10.7604	-0.0860	
21	8	8.6602	8.8020	-0.1417	
22	9	10.3202	10.2422	0.0780	
23	10	4.3520	4.2742	0.0777	
24	11	2.3202	2.1388	0.1814	
25	12	0.0123	0.0000	0.0123	
26	13	6.4462	6.2541	0.1920	
27	14	2.7854	3.5831	-0.7978	
28	15	10.6339	10.7932	-0.1592	
29	16	3.1851	3.0200	0.1651	
30	17	1.8986	2.0582	-0.1596	
31	18	13.6581	13.8167	-0.1586	
32	19	7.9048	7.8179	0.0869	
33	20	7.5720	7.6190	-0.0471	

Making the Hedge Gamma Neutral

Another strategy is to add another asset to the hedge position, in an effort to neutralize the gamma. In the example below we have added an out-of-the-money put to the position to neutralize the large call gamma:

	A	B	C	D	E
1	**COLLAR HEDGE, DELTA & GAMMA** **in this example we costlessly neutralize a large call gamma**				
2		**Call**	**Put**	**Another put**	
3	S	48.00	48.00	48.00	
4	X	70.00	49.04	35.00	
5	r	5.00%	5.00%	5.00%	
6	k, dividend yield	0.00%	0.00%	0.00%	
7	T	0.0200	0.0200	0.0200	
8	Sigma	40.00%	40.00%	40.00%	
9					
10	Option prices	0.00	1.66	0.00	
11					
12	Delta	0.0000	-0.6304	0.0000	<-- =deltaput(D3,D4,D7,D5,D6,D8)
13	Gamma	494,472,087	0	1,118,872	<-- =gamma(D3,D4,D7,D5,D6,D8)
14					
15					
16	**Bank position: short call with X = 70.00 + long put with X = 49.04 + put with X = 35.00**				
17	Call, X = 70.00	-1			
18	Put, X = 49.04	1			
19	Put, X = 35.00	441.938			
20					
21	Position delta	-0.6304	<-- {=SUMPRODUCT(TRANSPOSE(B17:B19),B12:D12)}		
22	Position gamma	0.1553	<-- {=SUMPRODUCT(TRANSPOSE(B17:B19),B13:D13)}		
23					
24	Position cost				
25	Without second put	1.6604	<-- =B17*B10+B18*C10		
26	With second put	1.6604	<-- =B17*B10+B18*C10+B19*D10		
27					
28	Traditional collar delta	-0.6304	<-- =-B12+C12		

This can be done at very little cost, since the put in question is almost costless (cell D10). Of course, it may not always be possible to costlessly neutralize the gamma. In this case we will have to make some compromises.

Financial Modeling, Fourth Edition, Simon Benninga; MIT Press 2014, pp. 474-476

77. Describe Naïve Hedging strategy

To see the usefulness of forward contracts in hedging the interest rate risk of an FI, consider a simple example of a **naive hedge** (the hedge of a cash asset on a direct dollar-for-dollar basis with a forward or futures contract). Suppose an FI portfolio manager holds a 20-year, $1 million face value bond on the balance sheet. At time 0, these bonds are valued by the market at $97 per $100 face value, or $970,000 in total. Assume the manager receives a forecast that interest rates are expected to rise by 2 percent from their current level of 8 to 10 percent over the next three months. Knowing that rising interest rates mean that bond prices will fall, the manager stands to make a capital loss on the bond portfolio. The manager is an expert in duration and has calculated the 20-year maturity bonds' duration to be exactly 9 years. Thus, the manager can predict a capital loss, or change in bond values (ΔP), from the duration equation:

$$\frac{\Delta P}{P} = -D * \frac{\Delta R}{1+R}$$

where

ΔP - capital loss on bonds

P – Initial value of bond position = 970 000

D – Duration of the bonds = 9 Years

ΔR – change in forecast yield = .02

$$\frac{\Delta P}{970\,000} = -9 * \frac{0.02}{1.08}$$

$$\Delta P = -\$161\,666.67$$

As a result, the FI portfolio manager expects to incur a capital loss on the bond portfolio of $161,666.67 (as a percentage loss (ΔP/P)=16.67%) or as a drop in price from $97 per $100 face value to $80.833 per $100 face value. To offset this loss—in fact, to reduce the risk of capital loss to zero—the manager may hedge this position by taking an off-balance-sheet hedge, such as selling $1 million face value of 20-year bonds for forward delivery in three months' time. 8 Suppose at time 0 the portfolio manager can find a buyer willing to pay $97 for every $100 of 20-year bonds delivered in three months' time. Now consider what happens to the FI portfolio manager if the gloomy forecast of a 2 percent rise in interest rates proves to be true. The portfolio manager's bond position has fallen in value by 16.67 percent, equal to a capital loss of $161,667. After the rise in interest rates, the manager can buy $1 million face value of

20-year bonds in the spot market at $80.833 per $100 of face value, a total cost of $808,333, and deliver these bonds to the forward contract buyer. Remember that the forward contract buyer agreed to pay $97 per $100 of face value for the $1 million of face value bonds delivered, or $970,000. As a result, the portfolio manager makes a profit on the forward transaction of

970 000 – 808 333 = 161 667

Price paid from forward buyer to forward seller	cost of purchasing Bonds in the spot market At *t* = month 3 for delivery

As you can see, the on-balance-sheet loss of $161,667 is exactly offset by the off balance- sheet gain of $161,667 from selling the forward contract. In fact, for any change in interest rates, a loss (gain) on the balance sheet is offset by a gain (loss) on the forward contract. Indeed, the success of a hedge does not hinge on the manager's ability to accurately forecast interest rates. Rather, the reason for the hedge is the lack of ability to perfectly predict interest rate changes. The hedge allows the FI manager to protect against interest rate changes even if they are unpredictable. Thus, the FI's net interest rate exposure is zero; in the parlance of finance, it has immunized its assets against interest rate risk.

Financial Institutions Management, 6th ed., Anthony Saunders, Marcia Millon Cornett; McGraw-Hill, 2008; Chapter 23, pp. 696-697

78. Describe MicroHedging strategy

An FI is **microhedging** when it employs a futures or a forward contract to hedge a particular asset or liability risk. For example, earlier we considered a simple example of microhedging asset-side portfolio risk, where an FI manager wanted to insulate the value of the institution's bond portfolio fully against a rise in interest rates. An example of microhedging on the liability side of the balance sheet occurs when an FI, attempting to lock in a cost of funds to protect itself against a possible rise in short-term interest rates, takes a short (sell) position in futures contracts on CDs or T-bills. In microhedging, the FI manager often tries to pick a futures or forward contract whose underlying deliverable asset is closely matched to the asset (or liability) position being hedged. The earlier example, where we had an exact matching of the asset in the portfolio with the deliverable security underlying the forward contract (20-year bonds) was unrealistic. Such exact matching cannot be achieved often, and this produces a residual unhedgable risk termed basis risk. Basis risk arises mainly because the prices of the assets or liabilities that an FI wishes to hedge are imperfectly correlated over time with the prices on the futures or forward contract used to hedge risk.

Financial Institutions Management, 6th ed., Anthony Saunders, Marcia Millon Cornett; McGraw-Hill, 2008; Chapter 23, pp. 697

79. Describe MacroHedging strategy with futures

Macrohedging occurs when an FI manager wishes to use futures or other derivative securities to hedge the entire balance sheet duration gap. This contrasts to microhedging, where an FI manager identifies specific assets and liabilities and seeks individual futures and other derivative contracts to hedge those individual risks. Note that macrohedging and microhedging can lead to quite different hedging strategies and results. In particular, a macrohedge takes a whole portfolio view and allows for individual asset and liability interest sensitivities or durations to net each other out. This can result in a very different aggregate futures position than when an FI manager disregards this netting or portfolio effect and hedges individual asset and liability positions on a one-to-one basis.

The number of futures contracts that an FI should buy or sell in a macrohedge depends on the size and direction of its interest rate risk exposure and the return risk trade-off from fully or selectively hedging that risk. FI's net worth exposure to interest rate shocks was directly related to its leverage adjusted duration gap as well as its asset size. Again, this is:

$$\Delta E = -[D_A - kD_L] * A * \frac{\Delta R}{1+R}$$

Where

ΔE - Change in an FI's net worth
D_A- Duration of its asset portfolio
D_L- Duration of its liability portfolio
k - Ratio of an FI's liabilities to assets (L/A)
A - Size of an FI's asset portfolio
$\frac{\Delta R}{1+R}$–Shock to interest rates

Financial Institutions Management, 6th ed., Anthony Saunders, Marcia Millon Cornett; McGraw-Hill, 2008; Chapter 23, pp. 698-699

80. What problem arises with Basis Risk

Because spot bonds and futures on bonds are traded in different markets, the shift in yields, $\Delta R/(1+R)$, affecting the values of the on-balance-sheet cash portfolio may differ from the shift in yields, $\Delta R_F/(1+R_F)$, affecting the value of the underlying bond in the futures contract; that is, changes in spot and futures prices or values are not perfectly correlated. This lack of perfect correlation is called basis risk. Previously, we assumed a simple world of no basis risk in which $\Delta R/(1+R)=R_F/(1+R_F)$.

Basis risk occurs for two reasons. First, the balance sheet asset or liability being hedged is not the same as the underlying security on the futures contract. For instance, in Example 23–2 we hedged interest rate changes on the FI's entire balance sheet with T-bond futures contracts written on 20-year maturity bonds with duration of 9.5 years. The interest rates on the various assets and liabilities on the FI's balance sheet and the interest rates on 20-year T-bonds do not move in a perfectly correlated (or one-to-one) manner. The second source of basis risk comes from the difference in movements in spot rates versus futures rates. Because spot securities (e.g., government bonds) and futures contracts (e.g., on the same bonds)are traded in different markets, the shift in spot rates may differ from the shift in futures rates (i.e., they are not perfectly correlated).To solve for the risk-minimizing number of futures contracts to buy or sell, N_F, while accounting for greater or less rate volatility and hence price volatility in the futures market relative to the spot or cash market, we look again at the FI's on balance-sheet interest rate exposure:

$$\Delta E = -[D_A - kD_L] * A * \frac{\Delta R}{1+R}$$

and its off-balance-sheet futures position:

$$\Delta F = -D_F[N_F - kP_F] * \frac{\Delta R_F}{1+R_F}$$

Setting

$$\Delta E = \Delta F$$

and solving for N_F, we have:

$$N_F = \frac{[D_A - kD_L] * A * \frac{\Delta R}{1+R}}{D_F * P_F * \frac{\Delta R_F}{1+R_F}}$$

Let *br* reflect the relative sensitivity of rates underlying the bond in the futures market relative to interest rates on assets and liabilities in the spot market, that is, *br* = $[\Delta R_F /(1 + R_F)]/[\Delta R /(1+R)]$. Then the number of futures contracts to buy or sell is:

$$N_F = \frac{[D_A - kD_L] * A}{D_F * P_F * br}$$

The only difference between this and the previous formula is an adjustment for basis risk (*br*), which measures the degree to which the futures price (yield) moves more or less than spot bond price (yield).

Financial Institutions Management, 6th ed., Anthony Saunders, Marcia Millon Cornett; McGraw-Hill, 2008; Chapter 23, p. 707

81. Describe strategy of Credit Risk hedging with forwards

A credit forward is a forward agreement that hedges against an increase in default risk on a loan (a decline in the credit quality of a borrower) after the loan rate is determined and the loan is issued. Common buyers of credit forwards are insurance companies and common sellers are banks. The credit forward agreement specifies a credit spread (a risk premium above the risk-free rate to compensate for default risk) on a benchmark bond issued by an FI borrower. For example, suppose the benchmark bond of a bank borrower was rated BBB at the time a loan was originated. Further, at the time the loan was issued, the benchmark bonds had a 2 percent interest rate or credit spread (representing default risk on the BBB bonds) over a U.S. Treasury bond of the same maturity. To hedge against an increase in the credit risk of the borrower, the bank enters into (sells)a credit forward contract when the loan is issued. We define CS F as the credit spread over the U.S. Treasury rate on which the credit forward contract is written (equals 2 percent in this example). Table below illustrates the payment pattern resulting from this credit forward. In Table below , CS T is the actual credit spread on the bond when the credit forward matures, for example, one year after the loan was originated and the credit forward contract was entered into, MD is the modified duration on the benchmark BBB bond, and A is the principal amount of the forward agreement.

Credit Spread at End of Forward Agreement	Credit Spread Seller (Bank)	Credit Spread Buyer (Counterparty)
$CS_T>CS_F$	Receives $(CS_T-CS_F) \times MD \times A$	Pays $(CS_T-CS_F) \times MD \times A$
$CS_F>CS_T$	Pays $(CS_T-CS_F) \times MD \times A$	Receives $(CS_T-CS_F) \times MD \times A$

From the payment pattern established in the credit forward agreement, table shows that the credit forward buyer (an insurance company) bears the risk of an increase in default risk on the benchmark bond of the borrowing firm, while the credit forward seller (the bank lender) hedges itself against an increase in the borrower's default risk. That is, if the borrower's default risk increases so that when the forward agreement matures the market requires a higher credit spread on the borrower's benchmark bond, CS_T, than that originally agreed to in the forward contract, CS F (i.e., $CS_T > CS_F$), the credit forward buyer pays the credit forward seller, which is the bank, (CS T - CS F) * MD * A. For example, suppose the credit spread between BBB bonds and U.S. Treasury bonds widened to 3 percent from 2 percent over the year, the modified duration (MD) of the benchmark BBB bond was five years, and the size of the forward contract A was $10 million. Then the gain on the credit forward contract to the seller (the bank) would be $500,000 [(3% - 2%) * 5 * $10,000,000]. This amount could be used to offset the loss in market value of the loan due to the rise in the borrower's default risk. However, if the borrower's default risk and credit spread decrease over the year, the credit forward seller pays the credit forward buyer (CS_F - CS_T) * MD * A. [However, the maximum loss on the forward contract (to the bank seller) is limited, as will be explained below.]

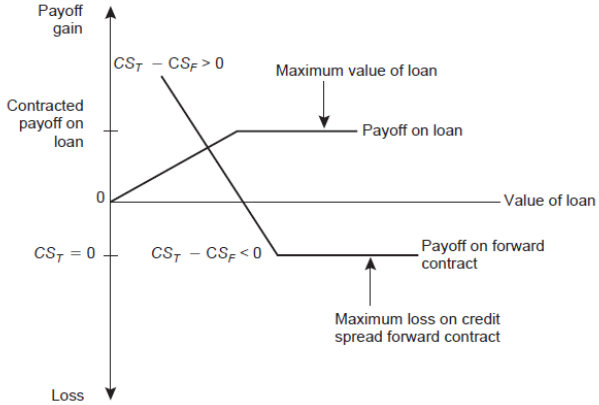


Figure above illustrates the impact on the bank from hedging the loan. If the default risk on the loan increases, the market or present value of the loan falls below its value at the beginning of the hedge period. However, the bank hedged the change in default risk by selling a credit forward contract. Assuming the credit spread on the borrower's benchmark bond also increases (so that $CS_T > CS_F$), the bank receives (CS_T - CS_F) * MD * A on the forward contract. If the characteristics of the benchmark bond (i.e., change in credit spread, modified duration, and principal value) are the same as those of the bank's loan to the borrower, the loss on the balance sheet is offset completely by the gain (off the balance sheet) from the credit forward (i.e., in our example a $500,000 market value loss in the loan would be offset by a $500,000 gain from selling the credit forward contract).

Financial Institutions Management, 6th ed., Anthony Saunders, Marcia Millon Cornett; McGraw-Hill, 2008; Chapter 23, p. 716-718

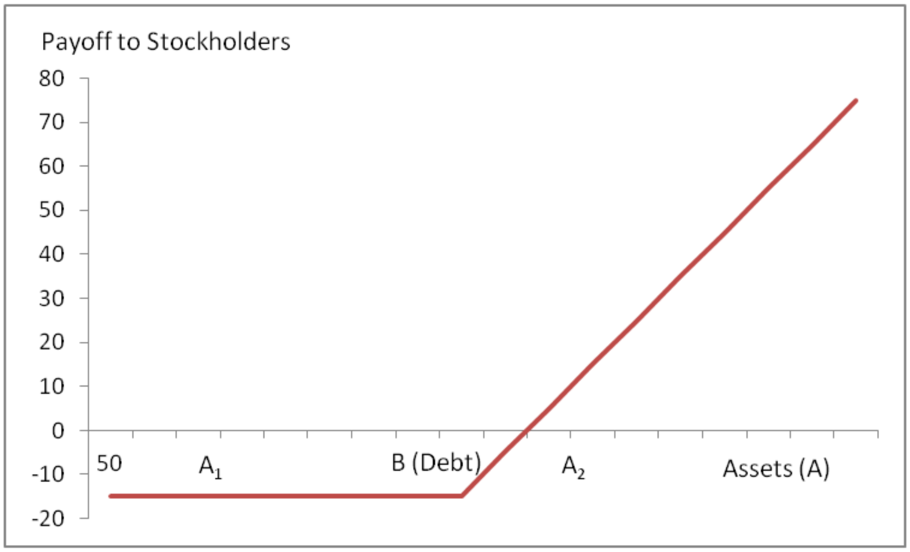
82. Show that payoffs to Bondholders and Stockholders may be described as options

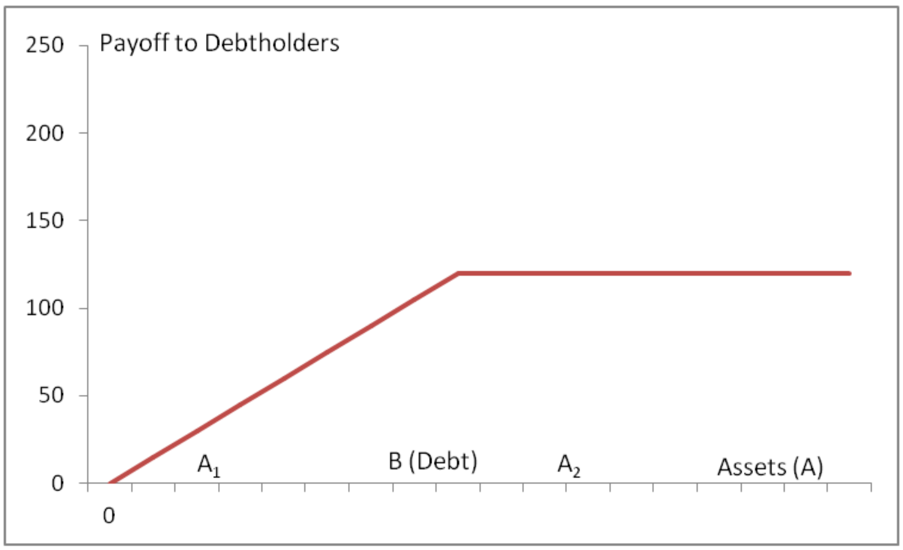
Look at the payoff function for the borrower, where S is the size of the initial equity investment in the firm, B is the value of outstanding bonds or loans (assumed for simplicity to be issued on a discount basis), and A is the market value of the assets of the firm.

If the investments in turn out badly such that the firm's assets are valued at point A_1, the limited-liability stockholder–owners of the firm will default on the firm's debt, turn its assets (such as A_1) over to the debt holders, and lose only their initial stake in the firm (S). By contrast, if the firm does well and the assets of the firm are valued highly (A_2), the firm's stockholders will pay off the firm's debt and keep the difference ($A_2 - B$). Clearly, the higher A_2 is relative to B, the better off are the firm's stockholders. Given that borrowers face only a limited downside risk of loss of their equity investment but a very large potential upside return if things turn out well, equity is analogous to buying a call option on the assets of the firm.

Consider the same loan or bond issue from the perspective of the FI or bondholder. The maximum amount the FI or bondholder can get back is B, the promised payment. However, the borrower who possesses the default or repayment option would rationally repay the loan only if $A > B$, that is, if the market value of assets exceeds the value of promised debt repayments. A borrower whose asset value falls below B would default and turn over any remaining assets to the debtholders. The payoff function to the debt holder is shown in figure below.

After investment of the borrowed funds has taken place, if the value of the firm's assets lies to the right of B , the face value of the debt – such as A_2 – the debt holder or FI will be paid off in full and receive B. On the other hand, if asset values fall in the region to the left of B – such as A_1 – the debt holder will receive back only those assets remaining as collateral, thereby losing $B - A_1$. Thus, the value of the loan from the perspective of the lender is always the minimum of B or A, or min [B, A]. That is, the payoff function to the debt holder is similar to writing a put option on the value of the borrower's assets with B, the face value of debt, as the exercise price. If $A > B$, the loan is repaid and the debt holder earns a small, fixed return (similar to the premium on a put option), which is the interest rate implicit in the discount bond. If $A < B$, the borrower defaults and the debt holder stands to lose both interest and principal. In the limit, default for a firm with no assets left results in debtholders' losing all their principal and interest. In actuality, if there are also costs of bankruptcy, the debt holder can potentially lose even more than this.





Financial Institutions Management, 6th ed., Anthony Saunders, Marcia Millon Cornett; McGraw-Hill, 2008; Chapter 11, pp. 332-333

83. Describe mechanics of hedging bond portfolio with options

Suppose that an FI manager has purchased a $100 zero-coupon bond with exactly two years to maturity. A zero-coupon bond, if held to maturity, pays its face value of $100 on maturity in two years. Assume that the FI manager pays $80.45 per $100 of face value for this zero-coupon bond. This means that if held to maturity, the FI's annual yield to maturity (R2) from this investment would be:

$$BP_2 = \frac{100}{(1+R_2)^2} \rightarrow R_2 = 11.5\%$$

Suppose also that, at the end of the first year, interest rates rise unexpectedly. As a result, depositors, seeking higher returns on their funds, withdraw deposits. To meet these unexpected deposit withdrawals, the FI manager is forced to liquidate (sell) the two-year bond before maturity, at the end of year 1. Treasury securities are important liquidity sources for an FI. Because of the unexpected rise in interest rates at the end of year 1, the FI manager must sell the bond at a low price.

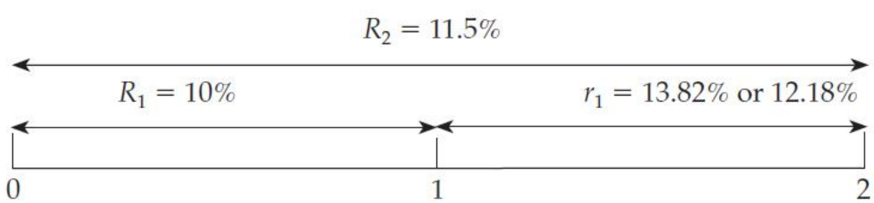
Assume when the bond is purchased, the current yield on one-year discount bonds (R_1) is R_1= 10 percent. Also, assume that at the end of year one, the one year interest rate (r_1) is forecasted to rise to either 13.82 percent or 12.18 percent. If one-year interest rates rise from R_1=10 percent when the bond is purchased tor$_1$=13.82 percent at the end of year 1, the FI manager will be able to sell the zero-coupon bond with one year remaining to maturity for a bond price, BP, of:

$$BP_1 = \frac{100}{(1+r_1)} = \frac{100}{(1.1382)} = \$87.86$$

If, on the other hand, one-year interest rates rise to 12.18 percent, the manager can sell the bond with one year remaining to maturity for:

$$BP_1 = \frac{100}{(1+r_1)} = \frac{100}{(1.1218)} = \$89.14$$

In these equations, r_1 stands for the two possible one-year rates that might arise one year into the future. That is:



Assume the manager believes that one-year rates (r_1) one year from today will be 13.82 percent or 12.18 percent with an equal probability. This means that the expected one-year rate one year from today would be:

$$[E(r_1)] = 0.5 * (0.1382) + 0.5 * (0.1218) = 0.13 = 13\%$$

Thus, the expected price if the bond has to be sold at the end of the first year is:

$$E(P_1) = \frac{100}{(1.13)} = \$88.50$$

Assume that the FI manager wants to ensure that the bond sale produces at least$88.50 per $100; otherwise, the FI has to find alternative and very costly sources of liquidity (for example, the FI might have to borrow from the central bank’s discount window and incur the direct and indirect penalty costs involved). One way for the FI to ensure that it receives at least $88.50 on selling the bond at the end of the year is to buy a put option on the bond at time 0 with an exercise price of $88.50 at time (year) 1. If the bond is trading below $88.50 at the end of the year—say, at $87.86—the FI can exercise its option and put the bond back to the writer of the option, who will have to pay the FI $88.50. If, however, the bond is trading above $88.50—say, at $89.14—the FI does not have to exercise its option and instead can sell the bond in the open market for $89.14.The FI manager will want to recalculate the fair premium to pay for buying this put option or bond insurance at time 0.

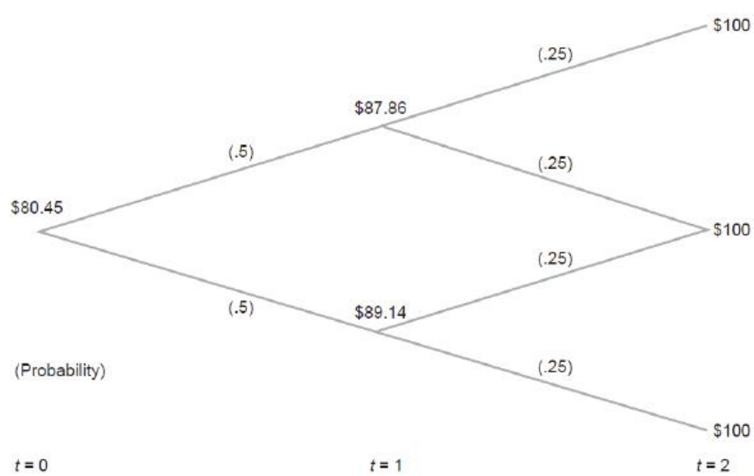


Figure shows the possible paths (i.e., the binomial tree or lattice) of the zero-coupon bond’s price from purchase to maturity over the two-year period.

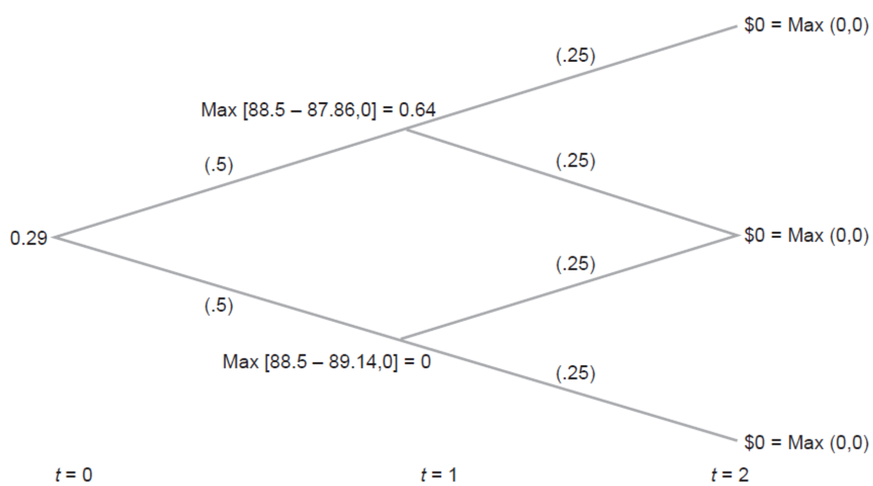
The FI manager purchased the bond at $80.45with two years to maturity. Given expectations of rising rates, there is a 50 percent probability that the bond with one year left to maturity will trade at $87.86 and a 50 percent probability that it will trade at $89.14. Note that between t=1, or one year left to maturity, and maturity (t=2), there must be a pull to par on the bond; that is, all paths must lead to a price of $100 on maturity. The value of the option is shown in figure below. The option can be exercised only at the end of year 1 (t=1). If the zero-coupon bond with one year left to maturity trades at $87.86, the option is worth $88.50 $87.86 in time1 dollars, or $0.64. If the bond trades at $89.14, the option has no value since the bond could be sold at a higher value than the exercise price of $88.50 on the open market. This suggests that in time 1 dollars, the option is worth:

$$0.5 * 0.64 + 0.5 * 0 = \$0.32$$

However, the FI is evaluating the option and paying the put premium at time t = 0, that is, one year before the date when the option might be exercised. Thus, the fair value of the put premium (P) the FI manager should be willing to pay is the discounted present value of the expected payoff from buying the option. Since one-year interest rates (R_1) are currently 10 percent, this implies:

$$P = \frac{\$0.32}{(1 + R_1)} = \frac{\$0.32}{(1.1)} = \$0.29$$

or a premium, P, of approximately 29 cents per $100 bond option purchased.



Financial Institutions Management, 6th ed., Anthony Saunders, Marcia Millon Cornett; McGraw-Hill, 2008; Chapter 24, pp. 737-740

84. Describe mechanics of hedging interest rate risk using options

Our previous simple example showed how a bond option could hedge the interest rate risk on an underlying bond position in the asset portfolio. Next, we determine the put option position that can hedge the interest rate risk of the overall balance sheet; that is, we analyze macrohedging rather than microhedging.

FI's net worth exposure to an interest rate shock could be represented as:

$$\Delta E = -(D_A - kD_L) * A * \frac{\Delta \mathrm{R}}{1 + R}$$

Suppose the FI manager wishes to determine the optimal number of put options to buy to insulate the FI against rising rates. An FI with a positive duration gap would lose on-balance-sheet net worth when interest rates rise. In this case, the FI manager would buy put options. That is, the FI manager wants to adopt a put option position to generate profits that just offset the loss in net worth due to an interest rate shock (where E 0 is the FI's initial equity (net worth)position).

Let ΔP be the total change in the value of the put option position in T-bonds.

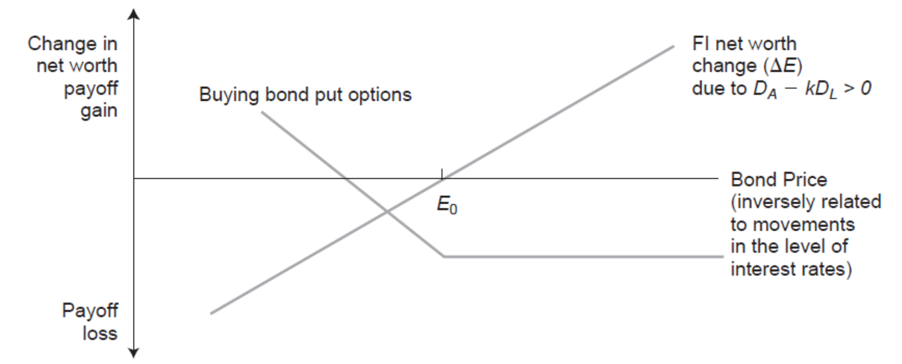
This can be decomposed into:

$$\Delta P = N_P * \Delta p$$

where N p is the number of $100,000 put options on T-bond contracts to be purchased(the number for which we are solving) and Δ p is the change in the dollar value for each $100,000 face value T-bond put option contract.

The change in the dollar value of each contract (Δp) can be further decomposed into:

$$\Delta p = \frac{dp}{dB} * \frac{dB}{dR} * \Delta R$$



This decomposition needs some explanation. The first term (*dp / dB*) shows the change in the value of a put option for each $1 change in the underlying bond. This is called the delta of an option (δ), and its absolute value lies between 0 and1. For put options, the delta has a negative sign since the value of the put option falls when bond prices rise. 8 The second term (*dB / dR*) shows how the market value of a bond changes if interest rates rise by one basis point. This value of one basis point term can be linked to duration. Specifically, we know from that:

$$\frac{dB}{B} = -MD * dR$$

That is, the percentage change in the bond's price for a small change in interest rates is proportional to the bond's modified duration (MD). Equation (3) can be rearranged by cross multiplying as:

$$\frac{dB}{dR} = -MD * B$$

Thus, the term *dB / dR* is equal to minus the modified duration on the bond (MD)times the current market value of the T-bond (B) underlying the put option contract.

As a result, we can rewrite equation (2) as:

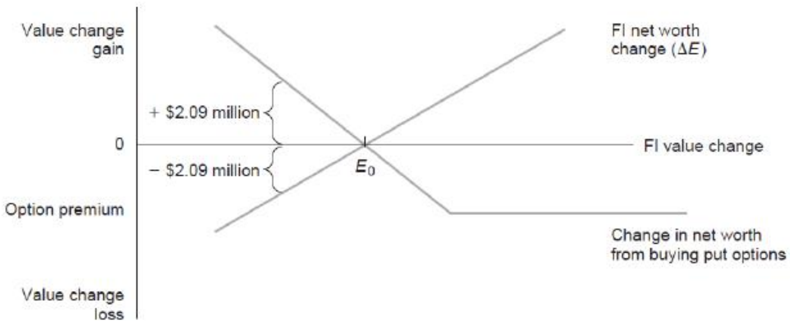
$$\Delta p = [(-\delta) * (-MD) * B * \Delta R]$$

Where Δ R is the shock to interest rates (i.e., the number of basis points by which rates change). Since we know that MD = D /(1 + R), we can rewrite last equation as:

$$\Delta p = \left[(-\delta) * (-D) * B * \frac{\Delta R}{1 + \mathrm{R}}\right]$$

Thus, the change in the total value of a put position (ΔP) is:

$$\Delta P = N_p * \left[\delta * D * B * \frac{\Delta R}{1 + \mathrm{R}}\right]$$



The term in brackets is the change in the value of one $100,000 face-value T-bond put option as rates change, and N p is the number of put option contracts. To hedge net worth exposure, we require the profit on the off-balance-sheet put options (ΔP) to just offset the loss of on-balance-sheet net worth (ΔE) when interest rates rise (and thus, bond prices fall). That is:

$$\Delta P = -\Delta E$$

$$N_p * \left[\delta * D * B * \frac{\Delta R}{1 + \mathrm{R}}\right] = (D_A - kD_L) * A * \frac{\Delta \mathrm{R}}{1 + R}$$

Canceling ΔR/(1 + R) on both sides, we get:

$$N_p * [\delta * D * B *] = (D_A - kD_L) * A$$

Solving for N_p – the number of put options to buy – we have:

$$N_p = \frac{(D_A - kD_L) * A}{[\delta * D * B *]}$$

Financial Institutions Management, 6th ed., Anthony Saunders, Marcia Millon Cornett; McGraw-Hill, 2008; Chapter 24, pp. 743-746

85. Describe mechanics of hedging credit risk using options

Options also have a potential use in hedging the credit risk of an FI. Relative to their use in hedging interest rate risk, option use to hedge credit risk is a relatively new phenomenon. Although FIs are always likely to be willing to bear some credit risk as part of the intermediation process (i.e., exploit their comparative advantage to bear such risk), options may allow them to modify that level of exposures electively. FI can seek an appropriate credit risk hedge by selling credit forward contracts. Rather than using credit forwards to hedge, an FI has at least two alternative credit option derivatives with which it can hedge its on-balance-sheet credit risk.

A credit spread call option is a call option whose payoff increases as the(default) risk premium or yield spread on a specified benchmark bond of the borrower increases above some exercise spread, S. An FI concerned that the risk on a loan to that borrower will increase can purchase a credit spread call option to hedge the increased credit risk.

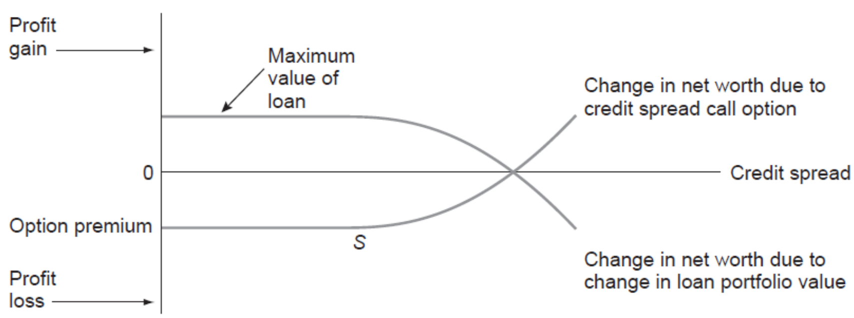
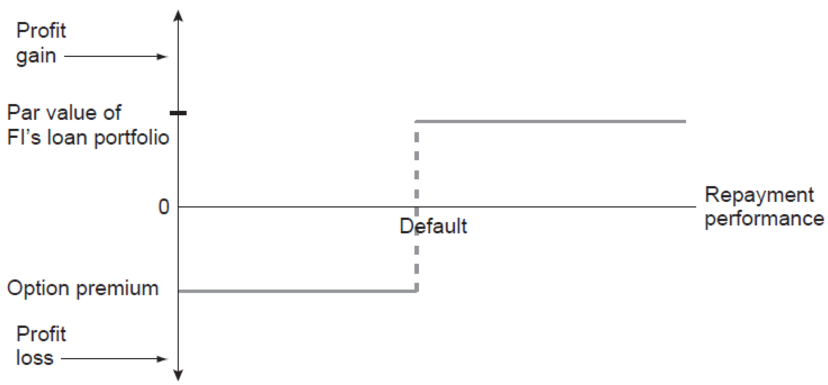


Figure illustrates the change in the FI's capital value and its payoffs from the credit spread call option as a function of the credit spread. As the credit spread increases on an FI's loan to a borrower, the value of the loan, and consequently the FI's net worth, decreases. However, if the credit risk characteristics of the benchmark bond (i.e., change in credit spread) are the same as those on the FI's loan, the loss of net worth on the balance sheet is offset with a gain from the credit spread call option. If the required credit spread on the FI's loan decreases (perhaps because the credit quality of the borrower improves over the loan period), the value of the FI's loan and net worth increases (up to some maximum value), but the credit spread call option will expire out of the money. As a result, the FI will suffer a maximum loss equal to the required (call) premium on the credit option, which will be offset by the market value gain of the loan in the portfolio (which is reflected in a positive increase in the FI's net worth).

A digital default option is an option that pays a stated amount in the event of a loan default (the extreme case of increased credit risk). As shown in below, the FI can purchase a default option covering the par value of a loan (or loans) in its portfolio. In the event of a loan default, the option writer pays the FI the par value of the defaulted loans. If the loans are paid off in accordance with the loan agreement, however, the default option expires unexercised. As a result, the FI will suffer a maximum loss on the option equal to the premium (cost) of buying the default option from the writer (seller).



Financial Institutions Management, 6th ed., Anthony Saunders, Marcia Millon Cornett; McGraw-Hill, 2008; Chapter 24, pp. 749-751

Real Options Analysis

86. Define Real Options. Illustrate how an investment project is a call option

In finance, an option is a contract which gives the buyer (the owner) the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price on or before a specified date. Real Options are options written on Real Assets instead of Financial Assets.

A call option gives the holder the right to buy an asset at a certain price within a specific period of time. If price rises, option is executed and option holder receives underlying asset for pre specified price. Notice similarity between Call Option and Investment Project

Investment Project		Call Option
Initial Investment	=	Strike Price
Present Value of future Cash Flows	=	Underlying Asset

Investment Project is a Call Option with Initial Investment as Strike Price and Present Value of Future Cash Flows as Underlying Asset.

Derivatives Markets, 2nd ed., Robert L. McDonald; Pearson, 2006; Chapter 17, p. 547

87. Give examples of real options uses in practice

Below are some examples of how real options have been or should be used in different industries.

Automobile and Manufacturing Industry In automobile manufacturing, General Motors (GM) applies real options to create switching options in producing its new series of autos. This is essentially the option to use a cheaper resource over a given period of time. GM holds excess raw materials and has multiple global vendors for similar materials with excess contractual obligations above what it projects as necessary. The excess contractual cost is outweighed by the significant savings of switching vendors when a certain raw material becomes too expensive in a particular region of the world. By spending the additional money in contracting with vendors as well as meeting their minimum purchase requirements, GM has essentially paid the premium on purchasing a switching option. This is important especially when the price of raw materials fluctuates significantly in different regions around the world. Having an option here provides the holder a hedging vehicle against pricing risks.

Oil and Gas Industry In the oil and gas industry, companies spend millions of dollars to refurbish their refineries and add new technology to create an option to switch their mix of outputs among heating oil, diesel, and other petrochemicals as a final product, using real options as a means of making capital and investment decisions. This option allows the refinery to switch its final output to one that is more profitable based on prevailing market prices, to capture the demand and price cyclicality in the market.

Real Estate Industry In the real estate arena, leaving land undeveloped creates an option to develop at a later date at a more lucrative profit level. However, what is the optimal wait time? In theory, one can wait for an infinite amount of time, and real options provide the solution for the optimal timing option.

High-Tech and e-Business Industry In e-business strategies, real options can be used to prioritize different e-commerce initiatives and to justify those large initial investments that have an uncertain future. Real options can be used in e-commerce to create incremental investment stages, options to abandon, and other future growth options, compared to a large one-time investment (invest a little now, wait and see before investing more).

All these cases where the high cost of implementation with no apparent payback in the near future seems foolish and incomprehensible in the traditional discounted cash flow sense are fully justified in the real options sense when taking into account the strategic options the practice creates for the future, the uncertainty of the future operating environment, and management's flexibility in making the right choices at the appropriate time.

Real Option Analysis, Johnathan Munn; Wiley Finance, 2002; Chapter 1, pp. 26-27

88. Why Are Real Options Important?

An important point is that the traditional discounted cash flow approach assumes a single decision pathway with fixed outcomes, and all decisions are made in the beginning without the ability to change and develop over time. The real options approach considers multiple decision pathways as a consequence of high uncertainty coupled with management's flexibility in choosing the optimal strategies or options along the way when new information becomes available. That is, management has the flexibility to make midcourse strategy corrections when there is uncertainty involved in the future. As information becomes available and uncertainty becomes resolved, management can choose the best strategies to implement. Traditional discounted cash flow assumes a single static decision, while real options assume a multidimensional dynamic series of decisions, where management has the flexibility to adapt given a change in the business environment.

Another way to view the problem is that there are two points to consider, one, the initial investment starting point where strategic investment decisions have to be made; and two, the ultimate goal, the optimal decision that can ever be made to maximize the firm's return on investment and shareholder's wealth. In the traditional discounted cash flow approach, joining these two points is a straight line, whereas the real options approach looks like a map with multiple routes to get to the ultimate goal, where each route is conjoint with others. The former implies a one-time decision-making process, while the latter implies a dynamic decision-making process wherein the investor learns over time and makes different updated decisions as time passes and events unfold.

Real Option Analysis, Johnathan Munn; Wiley Finance, 2002; Chapter 3, pp. 82-83
Project Valuation using Real Options, Dr. Prasad Kodukula, PMP Chandra Papudesu; J. Ross Publishing, 2006, Chapter 8, pp. 168-174

89. Describe real options analysis assumptions

One important assumption behind the options pricing models is that no "arbitrage" opportunity exists. This means that in efficient financial markets, you cannot buy an asset at one price and simultaneously sell it at a higher price. Professional investors supposedly will buy and sell assets quickly, closing any price gaps, thereby making arbitrage opportunities rare. Critics argue that a "no arbitrage" condition is impossible with real assets because they are not as liquid as financial assets, and therefore option pricing models are inappropriate for real options valuation.

Another big assumption regarding use of Real Options is that competition will not have a significant impact on cash flows of the company. Problem occurs when there is more than one company, developing same product and first one to complete will obtain strategic advantage. Thus, Real Option Analysis is fair for Monopolistic market, whereas in real life case, there are plenty of competitors.

Project Valuation using Real Options, Dr. Prasad Kodukula, PMP Chandra Papudesu; J. Ross Publishing, 2006, Chapter 8, pp. 83-85

90. How game theory can be used in real options analysis?

Game theory is a study of strategic decision making. More formally, it is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers". To be fully defined, a game must specify the following elements: the players of the game (there must be at least two players), the information and actions available to each player at each decision point (list of strategies that each player is able to choose), and the payoffs for each outcome. These elements are used, along with a solution concept of their choosing, to deduce a set of equilibrium strategies for each player such that, when these strategies are employed, no player can profit by unilaterally deviating from their strategy. These equilibrium strategies determine equilibrium to the game—a stable state in which either one outcome occurs or a set of outcomes occurs with known probability.

An investment decision in competitive markets can be seen, in its essence, as a “game” among firms, since in their investment decisions firms implicitly take into account what they think will be the other firms’ reactions to their own actions, and they know that their competitors think the same way. Consequently, as game theory aims to provide an abstract framework for modeling situations involving interdependent choices, so a merger between these two theories appears to be a logic step.

Strategic Investment, Real Options and Games, Han T. J. Smit, Lenos Trigeorgis, Prinston University Press, 2006, Chapter 1, pp. 13-31

91. Define and give an example of option to abandon

The option to abandon is embedded in virtually every project. This option is especially valuable where the net present value (NPV) is marginal but there is a great potential for losses. As the uncertainty surrounding the payoff clears and if the payoff is not attractive, you can abandon the project early on without incurring significant losses. The losses can be minimized by selling off the project assets either on the spot or preferably by prearranged contracts. The contingent decision in this option is to abandon the project if the expected payoff (the underlying asset value) falls below the project salvage value, the strike price. This option therefore has the characteristics of a put option.

Example

Bio Pharma is a drug-related-products company with a number of initiatives in its R&D pipeline. One of the new patented product ideas from its R&D lab has become a lead candidate for a development effort because of its potential market demand. The total estimated cost to launch the product, including its development, is estimated to be $95 million. Code-named Bluneon, it faces stiff competition, however, from other major projects in the pipeline. The vice president of the division concerned decides to create a strategic abandonment option.

The discounted cash flow (DCF) analysis on Bluneon's market potential shows that the present value of the payoff discounted at an appropriate market risk-adjusted discount rate would be $100 million. At any time during the next five years of development, based on the results, Bio Pharma can either continue with the development effort or sell off its intellectual property for $65 million (considered the salvage value) to a strategic partner. This technology is of importance to the partner, because it can upsell it to its existing customer base. The annual volatility of the logarithmic returns of the future cash flows is calculated to be 35%, and the continuous annual riskless interest rate over the next five years is 5%. What is the value of the abandonment option?

Identify the input parameters

S_0 = \$100 million

X = \$65 million

T = 5 years

$\sigma = 35\%$

$r = 5\%$

δt = 1 year

Calculate the option parameters

$u = exp(\sigma\sqrt{\delta t}) = exp(0.35 * \sqrt{1}) = 1.419$

$d = 1/u = 1/1.419 = 0.705$

$p = (exp(r\delta t) - d)/(u - d)$

$p = [exp(0.05 * 1) - 0.705]/(1.419 - 0.705) = 0.485$

Build a binomial tree, as shown in Figure 7-1, using one-year time intervals for five years and calculate the asset values over the life of the option. Start with S_0 at the very first node on the left and multiply it by the up factor and down factor to obtain S_0u (\$100 million * 1.419 = \$142 million) and S_0d (\$100 million * 0.705 = \$70 million), respectively, for the first time step. Moving to the right, continue in a similar fashion for every node of the binomial tree until the last time step. In Figure 7-1, the top value at each node represents the asset value at that node.

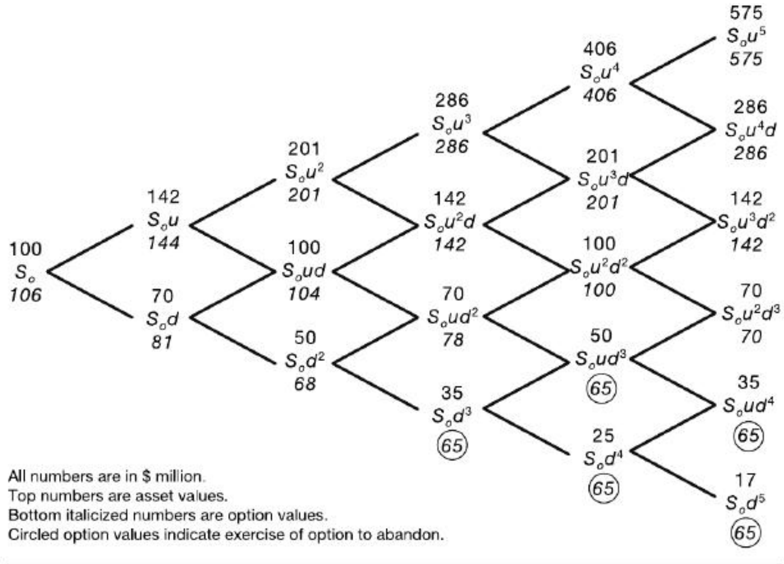


Figure 7-1. Binomial Tree for Option to Abandon for Bio Pharma

Figure 7-1 shows the option values (bottom italicized numbers) at each node of the binomial tree by backward induction. Each node represents the value maximization of abandonment versus continuation. At every node, you have an option to either abandon the project for a salvage value of $65 million or continue keeping the option open until it expires.

a. Start with the terminal nodes that represent the last time step. At node S_0u^5, the expected asset value is $575 million, compared to the salvage value of $65 million. Since you want to maximize your return, you would continue rather than abandon the project. Thus, the option value at this node is $575 million.
b. At node S_0ud^4, the expected asset value is $35 million, compared to the salvage value of $65 million; therefore, it makes sense to sell off the asset and abandon the project, which makes the option value at this node $65 million.
c. Next, move on to the intermediate nodes, one step away from the last time step. Starting at the top, at node Sou4, calculate the expected asset value for keeping the option open. This is simply the discounted (at the risk-free rate) weighted average of potential future option values using the risk-neutral probability as weights:

$$[p(S_0u^5) + (1 - p)(S_0u^4d)]e^{(-r\delta t)} =$$
$$= [0.485(\$575m) + (1 - 0.485)(\$286m)]e^{(-0.05)(1)}$$
$$= \$406\,m$$

Since this value is larger than the salvage value of $65 million, you would keep the option open and continue; therefore, the option value at S_0u^4 is $406 million.

d. Similarly, at node S_0ud^3, the expected asset value for keeping the option open, taking into account the downstream optimal decisions, is:

$$[0.485(\$70m) + (1 - 0.485)(\$65m)]e^{(-0.05)(1)} = \$64m$$

Since this value is smaller than the salvage value of $65 million, you would sell the assets for the salvage value and abandon the project.

e. Complete the option valuation binomial tree all the way to time = 0 using the approach outlined above.

Analyze the results

The payoff of the project based on the DCF method without flexibility is $100 million, but the cost to develop and launch the product is $95 million, leaving a relatively small project NPV of $5 million. ROA, however, shows a total project value of $106 million, yielding an additional $6 million ($106 million – $100 million) due to the abandonment option. This is exactly the same value obtained by using the Black-Scholes equation for this put option. Thus, the project NPV more than doubled ($5 million + $6 million = $11 million) because of the abandonment option. This can make an important difference to the survival of the Bluneon product in Bio Pharma's R&D portfolio.

Project Valuation using Real Options, Dr. Prasad Kodukula, PMP Chandra Papudesu; J. Ross Publishing, 2006, Chapter 7, pp. 102-108

92. Define and give an example of option to expand

The option to expand is common in high-growth companies, especially during economic booms. For some projects, the initial NPV can be marginal or even negative, but when growth opportunities with high uncertainty exist, the option to expand can provide significant value. You may accept a negative or low NPV in the short term because of the high potential for growth in the future.

Without considering an expansion option, great opportunities may be ignored due to a short-term outlook. Investment for expansion is the strike price that will be incurred as a result of exercising the option. The option would be exercised if the expected payoff is greater than the strike price, thereby making it a call option.

Example

Voodoo Video Vision has just introduced video-on-demand services in two major metropolitan areas in the United States. The initial market response has been lukewarm but not totally disappointing. The DCF valuation of the project free cash flows using a risk-adjusted discount rate currently indicates a present value of $80 million over the project life. Using the project proxy approach, the annual volatility of the logarithmic returns on these cash flows is calculated to be 30%. The company believes there is great potential for its services in seven other metropolitan areas and decides to explore the option to expand. This move is expected to result in a threefold expansion of current operations, at a cost of expansion of $200 million. What is the value of the expansion option over the next four years, if the continuous annual riskless interest rate for that time period is 5%?

Identify the input parameters

S_0 = *$80 million*
T = 4 years
σ = 30%
r = 5%
Expansion factor = 3.0
Cost of expansion = $200 million
δt = 1 year

Calculate the option parameters

$u = exp(\sigma\sqrt{\delta t}) = exp(0.30 * \sqrt{1}) = 1.35$
$d = 1/u = 1/1.35 = 0.741$
$p = (exp(r\delta t) – d)/(u – d)$
$p = [exp(0.05 * 1) – 0.741]/(1.35 – 0.741) = 0.51$

Build a binomial tree, as shown in Figure 7-5, using one-year time intervals for four years and calculate the asset values over the life of the option. Start with S_0 at the very first node on the left and multiply it by the up factor and down factor to obtain S_0u ($80 million * 1.350 = $108 million) and S_0d ($80 million * 0.741 = $59 million), respectively, for the first-time step. Moving to the right, continue in a similar fashion for every node of the binomial tree until the last time step. In Figure 7-5, the top value at each node represents the asset value at that node.

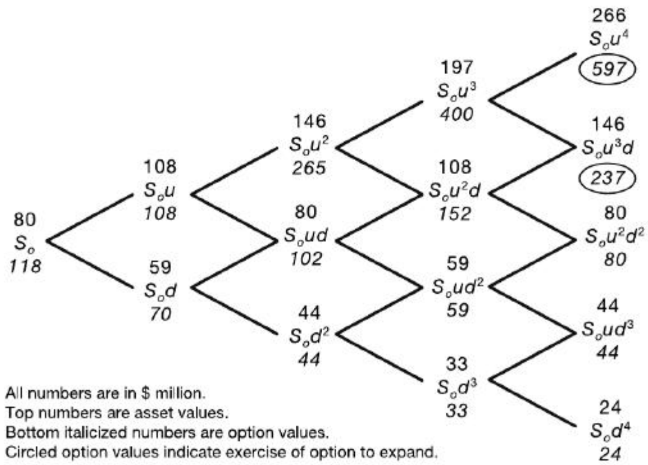


Figure 7-5. Binomial Tree for Option to Expand for Voodoo Video Vision

Figure 7-5 shows the option values (bottom italicized numbers) at each node of the binomial tree calculated by backward induction. Each node represents the value maximization of continuation versus expansion by threefold at a cost of $200 million. At every node, you have an option to either continue the operation and keep the option open for the future or expand it by three times by committing the investment for expansion.

A. Start with the terminal nodes that represent the last time step. At node S_0u^4, the expected asset value is $266 million. However, if you invest $200 million and expand the operation by threefold, the asset value would be: (3 * $266 million) – $200 million = $597 million. Since you want to maximize your return, you would expand rather than continue, because expansion results in an asset value of $597 million, whereas continuation of the status quo would yield a value of only $266 million. Thus, the option value at this node would become $597 million.
B. At node $S_0u^2d^2$, the expected asset value with no expansion is $80 million. However, if you invest $200 million and grow the operation by threefold, the asset value would be: (3 * $80 million) – $200 million = $40 million. To maximize your return, you would continue your operations without expansion, because that gives you an asset value of $80 million, whereas expansion would result in only $40 million.
C. Next, move on to the intermediate nodes, one step away from the last time step. Starting at the top, at node S_0u^3, calculate the expected asset

value for keeping the option open and accounting for the downstream optimal decisions. This is simply the discounted (at the risk-free rate) weighted average of potential future option values using the risk-neutral probability. That value, for example, at node S_0u^3, is:

$$[p(S_0u^4) + (1 - p)(S_0u^3d)]e^{(-r\delta t)} =$$
$$= [0.51(\$597m) + (1 - 0.51)(\$237m)]e^{(-0.05)(1)}$$
$$= \$400\,m$$

Since this value is less than the $400 million that corresponds to the alternative to continue, you would not exercise the expand option, and the option value at this node would be $400 million.

D. Similarly, at node S_0ud^2, the expected asset value for keeping the option open, taking into account the downstream optimal decision, is:

$$[0.510(\$80m) + (1 - 0.510)(\$44m)]e^{(-0.05)(1)} = \$59m$$

If, on the other hand, you exercise the option to expand the operation by three times at a cost of $200 million, the expected asset value would be:

$$(3.0 \times \$59) - \$200m = -\$23m$$

Maximizing $59 million versus –$23 million, you would not exercise the expand option. Therefore, the option value at node S_0ud^3 would be $59 million.

E. Complete the option valuation binomial tree all the way to time = 0.

Analyze the results

Let us first compare the value of the expansion option based on DCF versus ROA. The present value of the cash flows for the current operations based on the risk adjusted DCF method is $80 million. If the operation were to be expanded today, the additional value created would be:

$$3.0(\$80\,million) - \$80\,million = \$160\,million$$

Since the investment is $200 million, the NPV of the expansion project would be:

$$\$160\,million - \$200\,million = -\$40\,million$$

This means that your decision would be not to expand. However, ROA suggests that the expanded project's worth (taking into account the investment cost of $200 million) is $118 million. This means that the NPV of the expansion project is \$38 million after subtracting the present value of the cash flows associated with the current operations (\$118 million – \$80 million). The expansion option value calculated using the Black-Scholes equation for a call option is \$36 million, which is fairly close to the value (\$38 million) obtained through binomial lattice calculations. Comparing this with the baseline NPV of –\$40 million for the project, the additional value provided by the expansion option, therefore, is:

$$\$38\,million - (-\$40\,million) = \$78\,million$$

The difference is substantial and is the value added to the project because of the real options approach which management can take into consideration in decision making. Management may decide to keep the option of expansion open at this time and exercise it when the uncertainty clears and conditions become favorable.

Some of the practical considerations with this option relate to the expansion factor, volatility of the underlying asset value, cost of expansion (strike price), and time until the option expires.

The expansion factor is based on the underlying asset value, because it represents the cash flows from not only the current operation but also the expansion. It is estimated through correlations between the current operation and future expansion. The volatility of the asset value is kept constant due to the implicit assumption that it is the same for the current as well as the expanded operation. This is an important consideration because the framework of the option used in this example will not be applicable if the cash flow structure (asset value and its volatility) is significantly different for the expansion operation. For example, if Voodoo Video Vision is considering expanding its operations into China, the underlying asset value and its volatility would be expected to follow a different profile due to the difference in demographics compared to the current operation in the United States.

Voodoo Video Vision’s estimated cost of expansion (\$200 million) and the expansion factor (3.0) may change over time for the same demographics even within the United States. You can easily incorporate these changes into the binomial lattice and recalculate the results because of the flexibility it provides. In this example, however, considering that the company has already launched the project in two markets, it presumably has relatively good estimates for these factors. It is also implicitly assumed in this analysis that Voodoo Video Vision has access to capital and the other resources needed for expansion; otherwise the option valuation calculation may be meaningless after all. The framing of a real option problem should be rooted in reality.

Although the option to expand is implicit in most operations, the ROA calculation helps to quantify the value of the option. If the expansion option is indeed valuable, then management can take the necessary steps to keep the option alive. To clear the uncertainty, management can have a process in place to periodically gather information on market demand or arrange for a market study to proactively gather information on the asset value for better decision making.

In framing an expansion option, as is the case for many other options, a practitioner most likely is challenged with what value should be used for the option's time to expire, one of the inputs to the options model. You may not want to choose a long-time frame because competitors can enter the market and change the cash flow profile. Competition can be a big factor if your technology is not protected by patents or the barriers to entry are low. Another factor that works against long option time frames is the ever-decreasing product life in today's marketplace — especially for technology products. By using longer option lifetimes, you may end up overvaluing the option. On the other hand, a shorter time to expiration may not be desirable either, as there may not be enough time for the uncertainty surrounding the cash flows to clear, which would make the option valuation exercise meaningless. It is therefore important to think through the time implications and use the appropriate value for the option's time to expire.

In the Voodoo Video Vision example, a four-year time frame is used because it is assumed that competition will not have a significant impact on cash flows if the company expands within four years. A sensitivity analysis as a function of the option's time to expire would reveal that the option value will change as follows:

2 years	\$62 million
3 years	\$67 million
4 years	**\$78 million**
5 years	\$83 million

In the Voodoo Video Vision example, the option is framed as an expansion option. However, the expansion can be valued as a separate project by itself instead of as an expansion of the current operation. In that case, it can be considered an option to wait, where the underlying asset would be valued at \$160 million (3.0 * \$80 million – \$80 million) and the other input parameters kept the same. The solution of this option yields exactly the same results as the option to expand. You may want to treat the expansion of current operations as an option to wait if the underlying asset value and its volatility have a different structure compared to the current operation, a scenario mentioned earlier in this section.

Growth companies and novel products are good candidates to be considered for expansion options. Such assets have very high market uncertainty and hence start out on a small scale, but as uncertainty clears, they can be expanded if conditions are favorable.

Project Valuation using Real Options, Dr. Prasad Kodukula, PMP Chandra Papudesu; J. Ross Publishing, 2006, Chapter 7, pp. 110-116

93. Define and give an example of option to contract

The option to contract is significant in today's competitive marketplace, where companies need to downsize or outsource swiftly as external conditions change. Organizations can hedge themselves through strategically created options to contract. The option to contract has the same characteristics as a put option, because the option value increases as the value of the underlying asset decreases.

Example

Contracting Cars recently introduced a new product line and built two assembly plants in the United States. Now it appears that a Chinese automaker is going to introduce a similar line of cars in the U.S. market at a 30% cheaper cost. Contracting Cars is contemplating scaling down its operations by either selling or outsourcing one of the two plants to gain efficiencies through consolidation within the next five years. It frames an option to contract the size of its current operation by a factor of two and gain $250 million in savings because of lower general overhead expenses. Using the traditional DCF analysis and appropriate risk-adjusted discount rate, the present value of the future free cash flows of both plants is $600 million. The annual volatility of the logarithmic returns on the future cash flows is estimated to be 35%, and the continuous annual risk-free interest rate is 5% over the option life. What would be the value of the option to contract?

Identify the input parameters

S_0 = *$600 million*

T = *5 years*

σ = *35%*

r = *5%*

Contraction factor = 0.5
Savings of contraction = \$250 million
δt = 1 year

Calculate the option parameters
$u = exp(\sigma\sqrt{\delta t}) = exp(0.35 * \sqrt{1}) = 1.419$
$d = 1/u = 1/1.419 = 0.705$
$p = (exp(r\delta t)–d)/(u – d)$
$p = [exp(0.05 * 1) – 0.705]/(1.419 – 0.705) = 0.485$

Build a binomial tree, as shown in Figure 7-6, using one-year time intervals for five years and calculate the asset values over the life of the option. Start with S_0 at the very first node on the left and multiply it by the up factor and down factor to obtain S_0u (\$600 million * 1.419 = \$851 million) and S_0d (\$600 million * 0.705 = \$423 million), respectively, for the first time step. Moving to the right, continue in a similar fashion for every node of the binomial tree until the last time step. In Figure 7-6, the top value at each node represents the asset value at that node.

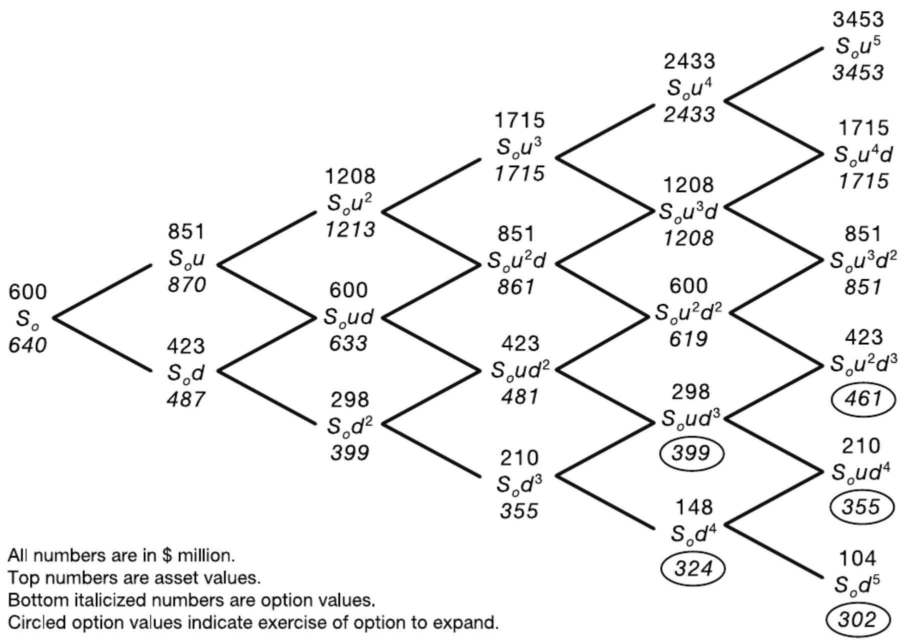


Figure 7-6. Binomial Tree for Option to Contract for Contracting Cars

Figure 7-6 shows the option values (bottom italicized numbers) at each node of the binomial tree calculated by backward induction. Each node represents the value maximization of continuation versus contraction by 50% with savings of $250 million. At every node, you have an option to either continue the operation and keep the option open or contract it by half.

a. Start with the terminal nodes that represent the last time step. At node S_0u^5, the expected asset value is $3,453 million. However, if you shrink the operation by half and save $250 million, the asset value would be: (0.5 * $3,453 million) + $250 million = $1,977 million. Since your objective is to maximize your return ($3,453 million versus $1,977 million), you would continue rather than contract. Thus, the option value at this node becomes $3,453 million.
b. At node $S_0u^2d^3$, the expected asset value without contraction is $423 million. However, if you shrink the operation by half and save $250 million, the asset value would be: (0.5 * $423 million) + $250 million = $461 million. Since you want to maximize your return ($423 million versus $461 million), you would contract rather than continue. Your option value at $S_0u^2d^3$ would, therefore, be $461 million.
c. Next, move on to the intermediate nodes, one step away from the last time step. Starting at the top, at node Sou4, calculate the expected asset value for keeping the option open and accounting for the downstream optimal decisions. This is simply the discounted (at the risk-free rate) weighted average of potential future option values using the risk-neutral probability. That value at node S_0u^4 is:

$$[p(S_0u^5) + (1 - p)(S_0u^4d)]\mathrm{e}^{(-r\delta t)} =$$
$$= [0.485(\$3{,}453m) + (1 - 0.485)(\$1{,}715m)]e^{(-0.05)(1)}$$
$$= \$2{,}433m$$

If the option is exercised to shrink the operation by half to gain a savings of $250 million, the expected asset value would be:

$$(0.5 \times \$2{,}433\, million) + \$250\, million = \$1{,}467\, million$$

Since this value is less than the $2,433 million corresponding to the alternative to continue, you would not exercise the contract option, and the option value at this node would be $2,433 million.
d. Similarly, at node S_0ud^3, the expected asset value for keeping the option open, taking into account the downstream optimal decisions, is:

$$[0.485(\$461m) + (1 - 0.485)(\$355m)]e^{(-0.05)(1)} = \$387m$$

If, on the other hand, you exercise the contract option to shrink the operation by half to gain a savings of $250 million, the expected asset value would be:

$$(0.5 \times \$298m) + \$250m = \$399m$$

Maximizing $387 million versus $399 million, you would exercise the contract option. Therefore, the option value at node S_0ud^3 would be $399 million.

e. Complete the option valuation binomial tree all the way to time = 0.

Analyze the results

Let us first compare the NPV of the asset based on DCF versus ROA. The present value of the project based on the risk-adjusted DCF method is $600 million, compared to the real options value (ROV) of $640 million. The difference of $40 million is the value added to the project from the ROA which management can take into consideration in making its decision today on contraction. The Black-Scholes equation provides a value of $33 million for this put option, which is relatively close to the binomial lattice option value. If Contracting Cars wanted to contract its operations today, the present value of the contracted project would be:

$$(0.5 \times \$600\ million) + \$250\ million = \$550\ million$$

Since this is less than the expected present value of $600 million for the current operation, Contracting Cars would not contract. However, the looming competition from China may have a significant impact on this expected payoff. Because of this uncertainty, it is possible that one of the plants would have to be shut down. How do you calculate the impact of that uncertainty on the value of the operation? This is where ROA gives us the answers. You can capture the value and impact of the uncertainty through ROA. In the foregoing example, we used a volatility factor of 35%, but you can easily calculate the option values for various volatility factors using the same approach as illustrated above. The option values from such calculations are shown in Table 7-2. The option value increases by almost 50% as the volatility increases 10% (from 35% to 45%) and decreases by the same factor as the volatility decreases 10% (from 35% to 25%).

Table 7-2. Option to Contract: Option Values Versus Volatility

Volatility	Option Value
0%	$0
10%	$0.7
15%	$5
25%	$21
35%	$40
45%	$59
55%	$78
100%	$148

Volatility of the asset value has a significant impact on the value of any option. The higher the volatility of the underlying asset value, the higher the range of asset values at any given time frame on the binomial tree. More important, as the volatility increases, the option value also increases. On the contrary, as the volatility factor approaches zero, the u and d factors will approach 1.0 and the binomial tree collapses into a straight line (Figure 7-7). At zero volatility, for the example above, the ROV will be $600 million, which is the same as the baseline payoff. This means that when the uncertainty is zero, real options do not add any value. If Contracting Cars is absolutely certain about its future payoff of $600 million, there is no real option value. Thus, DCF can be considered a special case of ROA where the uncertainty is zero. This also shows that ROA is not a substitute for but a logical extension of the DCF that takes project valuation to the next level of sophistication.

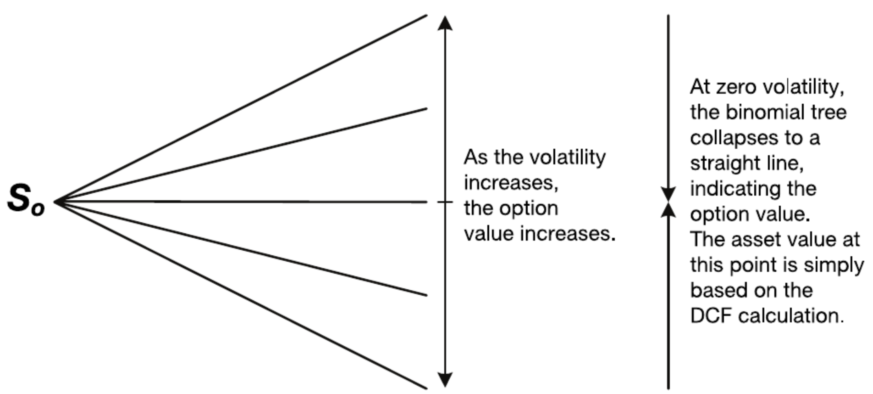


Figure 7-7. Binomial Tree: Same as Discounted Cash Flow at Zero Volatility

One of the other inputs in the Contracting Cars example is the contraction factor. As in the case of an expansion option, it is important for the practitioner to choose the contraction factor that accurately reflects the cash flow stream of the contracted operation. Contracting Cars has two assembly plants, and management wants to consider selling one of them as part of the contraction. The contraction factor of 0.5 is appropriate only if the two plants are contributing equally to the cash flow streams. If the cash flows of the two plants are different, the contraction factor needs to be adjusted accordingly.

All real options have real value only if management will make the value-maximizing decision when faced with choices analyzed by ROA. In the real world, however, decisions are made based not just on quantitative data that can be obtained through an ROA calculation but on political and emotional considerations. The option to contract is especially sensitive to these issues, as it may involve loss of jobs and selling of assets. Therefore, caution should be exercised in framing and valuing an option to contract.

Project Valuation using Real Options, Dr. Prasad Kodukula, PMP Chandra Papudesu; J. Ross Publishing, 2006, Chapter 7, pp. 116-121

94. Define and give an example of option to choose

The option to choose consists of multiple options combined as a single option. The multiple options are abandonment, expansion, and contraction. The reason it is called a chooser option is that you can choose to keep the option open and continue with the project or choose to exercise any one of the options to expand, contract, or abandon. The main advantage with this option is the choice. This is a unique option in the sense that, depending upon the choice to be made, it can be considered a put (abandonment or contraction) or call (expansion) option.

Example

Multiple Choice Drugs is faced with the dilemma of choosing among four strategies (continuation, expansion, contraction, or total abandonment) for one of its manufacturing operations. The present value of the projected future free cash flows for this operation using DCF analysis with the appropriate risk-adjusted discount rate is $200 million. The volatility of the logarithmic returns on the projected future cash flows is 25%, and the risk-free rate is 5% over the next five years. At any time during this time period, the company can expand by 30% by investing $50 million, contract one-quarter of its current operations to save $40 million, or abandon the operation altogether by selling the property for a salvage value of $100 million. What is the value of the chooser option for Multiple Choice Drugs?

Identify the input parameters

S_0 = \$200 million
T = 5 years
σ = 25%
r = 5%
Expansion factor = 1.3
Cost of expansion = \$50 million
Contraction factor = 0.75
Savings of contraction = \$40 million
Salvage value = \$100 million
δt = 1 year

Calculate the option parameters

$u = \exp(\sigma\sqrt{\delta t}) = \exp(0.25 * \sqrt{1}) = 1.284$
$d = 1/u = 1/1.284 = 0.779$
$p = (\exp(r\delta t) - d)/(u - d)$
$p = [\exp(0.05 * 1) - 0.779]/(1.284 - 0.779) = 0.539$

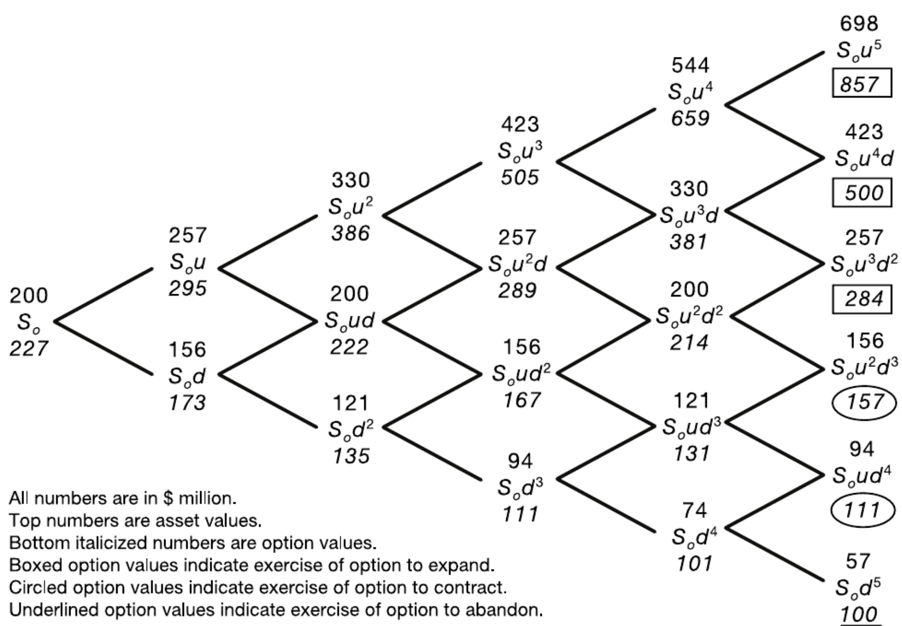


Figure 7-8. Binomial Tree for Option to Choose for Multiple Choice Drugs

Build a binomial tree, as shown in Figure 7-8, using one-year time intervals for five years and calculate the asset values over the life of the option. Start with S_0 at the very first node on the left and multiply it by the up factor and down factor to obtain S_0u ($200 million * 1.284 = $257 million) and S_0d ($200 million * 0.779 = $156 million), respectively, for the first time step. Moving to the right, continue in a similar fashion for every node of the binomial tree until the last time step. In Figure 7-8, the top value at each node represents the asset value at that node.

Figure 7-8 shows the option values (bottom italicized numbers) at each node of the binomial tree calculated by backward induction. Each node represents the value maximization of different mutually exclusive options available. At every node, you have the option to either continue the operation and keep the option open for the future or:

- Abandon for a salvage value of $100 million
- Expand 30% with an investment cost of $50 million
- Contract 25% to save $40 million

This means you need to calculate the asset values for each of the above options at each node and compare them against the continuation alternative. If continuation turns out to provide the maximum return, you would keep the option open for the future. Otherwise, you would exercise the option that provides you the maximum return.

A. Start with the terminal nodes that represent the last time step. At node S_0u^5, the expected asset value is $698 million. Now calculate the asset values for exercising each of the available options:
    a. Abandon: $100 million
    b. Expand: (1.3 * $698 million) – $50 million = $857 million
    c. Contract: (0.75 * $698 million) + $40 million = $564 million

    Maximization shows that the option to expand would be exercised at this node, so the option value here becomes $857 million. It turns out that you also would exercise the option to expand at nodes S_0u^4d and $S_0u^3d^2$.
B. At nodes $S_0u^2d^3$ and S_0ud^4, you would exercise the contract option, and at node S_0d^5 you would exercise the abandon option, because those actions provide the maximum value, which now become the option values at those nodes.
C. Next, move on to the intermediate nodes, one step away from the last time step. Starting at the top, at node Sou4, calculate the expected asset value for keeping the option open and accounting for the downstream optimal decisions. This is simply the discounted (at the risk-free rate) weighted average of potential future option values using the risk-neutral probability. That value, for example, at node S_0u^4, is:

$$[p(S_0u^5) + (1 - p)(S_0u^4d)]e^{(-r\delta t)} =$$
$$= [0.539(\$857m) + (1 - 0.539)(\$500m)]e^{(-0.05)(1)}$$
$$= \$659m$$

Now calculate the asset value for exercising each of the available options:

a. Abandon: $100 million
b. Expand: (1.3 * $544 million) – $50 million = $657 million
c. Contract: (0.75 * $644 million) + $40 million = $448 million

Maximization shows that you would keep the option open at this node. Therefore, the option value at this point becomes $659 million.

D. Similarly, at node Soud 3, calculate the expected asset value for keeping the option open and accounting for the downstream optimal decisions:

$$[0.539(\$157m) + (1 - 0.539)(\$111m)]e^{(-0.05)(1)} = \$131m$$

Now calculate the asset value for exercising each of the available options:

a. Abandon: $100 million
b. Expand: (1.3 * $121 million) – $50 million = $107 million
c. Contract: (0.75 * $121 million) + $40 million = $131 million

Maximization shows that you would contract the project or keep the option open (both have equal values). Therefore, the option value at this node becomes $131 million.

E. Complete the option valuation binomial tree all the way to time = 0.

The NPV of the project based on the risk-adjusted DCF method is $200 million, compared to the ROV of $227 million. The difference of $27 million is substantial and is the value added to the project by real options which management can take into consideration in making the project decisions. Figure 7-8 shows the strategic choices you would make at different points during the option life. It appears that you would either continue the project keeping the options open, abandon, contract, or expand depending on the expected asset values.

If Multiple Choice Drugs considers the individual options separately, rather than a combined chooser option, the ROV can be calculated as follows:

a. Abandon: $1 million
b. Expand: $24 million
c. Contract: $3 million

As you might expect, the combined option ($27 million) has more value than any one of the individual options. Summation of the individual options ($28 million) may not necessarily be the same as the combined chooser option. This is because the individual options are mutually exclusive and independent of each other. For example, you cannot abandon and expand the project at the same time. The value of a chooser option will always be less than or equal to the summation of the individual options that make up the chooser option. At each node of the binomial lattice, among the choices to abandon, expand, or contract versus continue the project, you choose whichever provides the maximum value; you do not add up the individual option values.

As with other options discussed earlier, you can change the salvage value, analyze the impact of volatility, calculate the probability of exercising a given option at a given time, and so on with the chooser option also. The binomial method gives you the flexibility and makes the calculations visible, so the results can be easily understood and communicated to management. A chooser option need not include all three choices (abandonment, expansion, and contraction). You can have an option that only has two of these choices and still value it in the same manner.

Project Valuation using Real Options, Dr. Prasad Kodukula, PMP Chandra Papudesu; J. Ross Publishing, 2006, Chapter 7, pp. 121-126

95. Describe compound options (options to stage) and differential between sequential and parallel types

Many project initiatives (research and development, capacity expansion, launching of new services, etc.) are multistage project investments where management can decide to expand, scale back, maintain the status quo, or abandon the project after gaining new information to resolve uncertainty. For example, a capital investment project divided into multiple phases, including permitting, design, engineering, and construction, can either be terminated or continued into the next phase depending upon the market conditions at the end of each phase. These are compound options where exercising one option generates another, thereby making the value of one option contingent upon the value of another option. A compound option derives its value from another option — not from the underlying asset. The first investment creates the right but not the obligation to make a second investment, which in turn gives you the option to make a third investment, and so on. You have the option to abandon, contract, or scale up the project at any time during its life.

A compound option can either be sequential or parallel, also known as simultaneous. If you must exercise an option in order to create another one, it is considered a sequential option. For example, you must complete the design phase of a factory before you can start building it. In a parallel option, however, both options are available at the same time. The life of the independent option is longer than or equal to the dependent option. A television broadcaster may be building the infrastructure for digital transmission and acquiring the required broadcast spectrum at the same time, but cannot complete testing of the infrastructure without the spectrum license. Acquiring the spectrum — an option itself — gives the broadcaster the option to complete the infrastructure and launch the digital broadcast service. For both sequential and compound options, valuation calculations are essentially the same except for minor differences.

Project Valuation using Real Options, Dr. Prasad Kodukula, PMP Chandra Papudesu; J. Ross Publishing, 2006, Chapter 8, pp. 146

96. Define and give an example of sequential compound option

The same six-step process introduced in previous questions and used in solving simple options is also applied to value sequential compound options with a few modifications.

Example

Sweet'n Sour Cola is considering investment in a bottling plant for HyperCola, its new beverage that recently came off its R&D pipeline. Despite sales in test markets for a short period of time, there still is some market uncertainty regarding future sales; therefore, the company wants to use the options approach to value the project for a go/no-go investment decision. The project is divided into three sequential phases: land acquisition and permitting (simply referred to as permitting hereafter), design and engineering (referred to as design hereafter), and construction. Each phase has to be completed before the next phase can begin. Sweet 'n Sour Cola wants to bring HyperCola to market in no more than seven years. The construction will take two years to complete, and hence the company has a maximum of five years to decide whether to invest in the construction. The design phase will take two years to complete, and since design is a prerequisite to construction, the company has a maximum of three years to decide whether to invest in the design and engineering phase. The permitting process will take two years to complete, and since it must be completed before the design phase can begin, the company has a maximum of one year from today to decide on permitting. Permitting is expected to cost $30 million, design $90 million, and construction another $210 million. Discounted cash flow analysis using an appropriate risk-adjusted discount rate values the plant, if it existed today, at $250 million. The annual volatility of the logarithmic returns for the future cash flows for the plant is estimated to be 30%, and the continuous annual risk-free interest rate over the next five years is 6%.

Identify the input parameters
S_0 = $250 million
X1, X2, and X3 = $30, $90, and $210 million, respectively
T1, T2, and T3 = 1, 3, and 5 years, respectively (cumulative option life for each stage)
$\sigma = 30\%$
$r = 6\%$
$\delta t = 1$ year

Calculate the option parameters
$u = exp(\sigma\sqrt{\delta t}) = exp(0.30 * \sqrt{1}) = 1.35$
$d = 1/u = 1/1.35 = 0.741$
$p = (exp(r\delta t) - d)/(u - d)$
$p = [exp(0.06 * 1) - 0.741]/(1.35 - 0.741) = 0.527$

Build a binomial tree, as shown in Figure 8-1, using one-year time intervals for five years and calculate the asset values over the life of the option. Start with S_0 and multiply it by the up factor and the down factor to obtain S_0u and S_0d, respectively. For the first time step: S_0u = $250 million * 1.350 = $337 million; S_0d = $250 million * 0.741 = $185 million. Moving to the right, continue in a similar fashion for every node of the binomial tree until the last time step. In Figure 8-1, the top value at each node represents the asset value at that node.

There are three sequential options available on this project. Construction is dependent on design, which in turn is dependent on permitting. The option value calculations are done in sequence, starting with the longest option. First, you calculate the option values for the construction option using the binomial tree, as discussed in the previous chapter. The option values of the longest option (construction) then become the underlying asset values for the preceding option (design), for which you calculate the option values using backward induction. These option values become the underlying asset values for the next preceding option (permitting), for which you employ backward induction to calculate its option values.

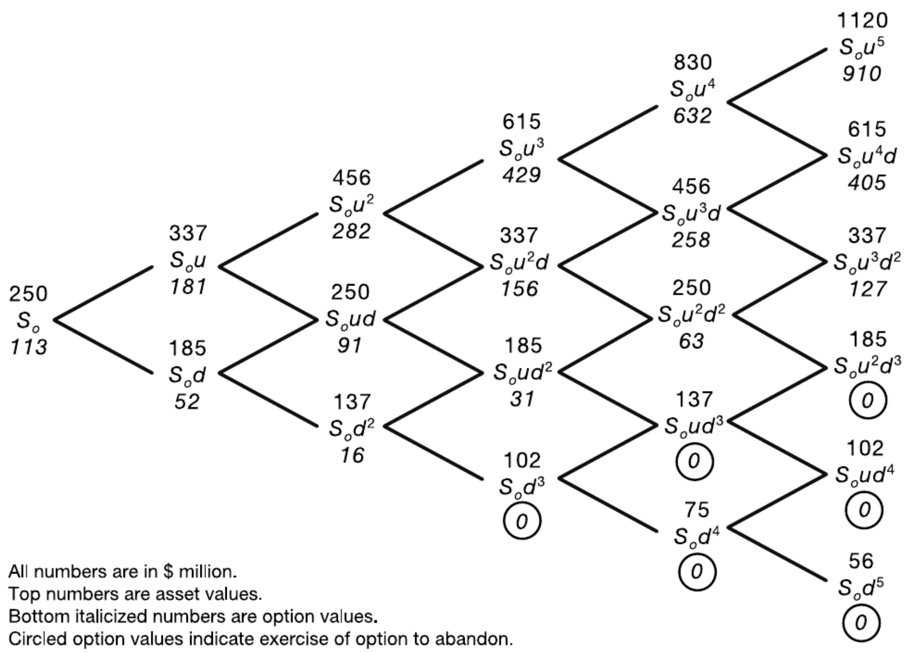


Figure 8-1. Binomial Tree for the Longest Option of the Sequential Compound Option

Figure 8-1 shows the option values (bottom italicized numbers) for the longest dependent option (that is, plant construction) at each node of the binomial tree calculated by backward induction. Each terminal node represents the value maximization of exercising the option by investing $210 million versus letting the option expire. Each intermediate node represents the value maximization of continuation versus exercising the option.

A. Start with the terminal nodes that represent the last time step. At node S_0u^5, the expected asset value is $1,120 million. If you invest $210 million to build the plant, the net payoff will be $1,120 million – $210 million = $910 million. Since your objective is to maximize your return, you would exercise your option by investing. Thus the option value at this node becomes $910 million.
B. At node Sod 5, the expected asset value is $56 million. Since this is less than the investment cost of $210 million, you would not invest and would let the option expire. Your option value would, therefore, be $0. Next, move on to the intermediate nodes, one step away from the last time step. Starting at the top, at node Sou4, calculate the expected asset value for keeping the option open and accounting for the downstream

optimal decisions. This is simply the discounted (at the risk-free rate) weighted average of potential future option values using the risk-neutral probabilities. That value at node S_0u^4 is:

$$[p(S_0u^5) + (1 - p)(S_0u^4d)]\mathrm{e}^{(-r\delta t)} =$$
$$= [0.527(\$910m) + (1 - 0.527)(\$405m)]e^{(-0.06)(1)}$$
$$= \$632m$$

If, on the other hand, the option is exercised by investing $210 million, the expected asset value would be:

$$\$830\ million - \$210\ million = \$620\ million$$

Since this value is less than the $632 million corresponding to the alternative to continue, you would not exercise the option, and the option value at this node would be $632 million.

C. Similarly, at node S_0u^3d, the expected asset value for keeping the option open, taking into account the downstream optimal decisions, is:

$$[0.527(\$405m) + (1 - 0.527)(\$127m)]\mathrm{e}^{(-0.06)(1)} = \$258m$$

If, on the other hand, you exercise the option to invest, the expected asset value would be:

$$\$456\ million - \$210\ million = \$246\ million$$

Maximizing $258 million versus $246 million, you would keep the option open. Therefore, the option valuation at node Sou3d would be $258 million.

D. Complete the option valuation binomial tree all the way to time = 0 using the approach outlined above.

Calculate the option values for the predecessor option (design) for its three-year life using the option values of the successor option (construction) as the underlying asset values. Exercising the option to design creates the option to construct the plant; hence the construction option values are treated as the underlying asset values for this calculation. Figure 8-2 shows the underlying asset values (top numbers) for the first three years, which are the same as the option values (bottom italicized numbers in Figure 8-1) for the construction option. Calculate the option values (bottom italicized numbers in Figure 8-2) at each node of the binomial tree by backward induction. Each node represents the value maximization of investing versus continuation, where you have the option to either invest $90 million in the project and benefit from the payoff or continue to keep the option open until it expires. At the terminal nodes, you would invest if the payoff is greater than the investment of $90 million; otherwise, you would let the option expire worthless.

A. Start with the terminal nodes that represent the last time step. At node S_0u^3, the expected asset value is $429 million. If you invest $90 million to design the plant, the net payoff will be $429 million – $90 million = $339 million. Since your objective is to maximize your return, you would exercise your option by investing. Thus the option value at this node becomes $339 million.
B. At node S_0ud^2, the expected asset value is $31 million. Since this is less than the investment cost of $90 million, you would not invest and would let the option expire. Your option value would, therefore, be $0 million.

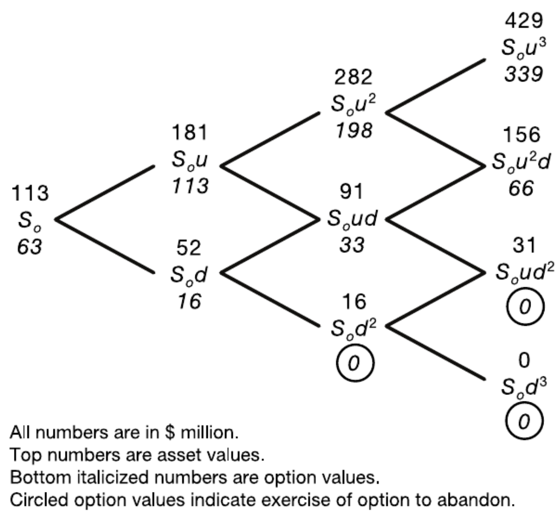


Figure 8-2. Binomial Tree for the Predecessor Option of the Sequential Compound Option

C. Next, move on to the intermediate nodes, one step away from the last time step. Starting at the top, at node S_0u^2, calculate the expected asset value for keeping the option open and accounting for the downstream optimal decisions. This is simply the discounted (at the risk-free rate) weighted average of potential future option values using the risk-neutral probabilities:

$$[p(S_0u^3) + (1 - p)(S_0u^2d)]\mathrm{e}^{(-r\delta t)} =$$
$$= [0.527(\$339m) + (1 - 0.527)(\$66m)]e^{(-0.06)(1)}$$
$$= \$198m$$

If, on the other hand, the option is exercised by investing $90 million, the expected asset value would be:

$$\$282\,million - \$90\,million = \$192\,million$$

Since this value is less than the $198 million corresponding to the alternative to continue, you would not exercise the option, and the option value at this node would be $198 million.

D. Similarly, at node Soud, the expected asset value for keeping the option open, taking into account the downstream optimal decisions, is:

$$[0.527(\$6m) + (1-0.527)(\$0m)]e^{(-0.06)(1)} = \$33m$$

If, on the other hand, you exercise the option to invest, the expected asset value would be:

$$\$91\,million - \$90\,million = \$1\,million$$

Maximizing $33 million versus $1 million, you would keep the option open. Therefore, the option value at node S_0ud would be $33 million.

E. Complete the option valuation binomial tree all the way to time = 0 using the approach outlined above.

Calculate the option values for the next predecessor option (permitting), which happens to be the shortest option in this example. The life of this option is one year, and the underlying asset values of this option are the same as the option values for the successor option (design). Exercising the option to apply for a permit creates the option to design the plant, and so the design option values are treated as the underlying asset values for this calculation. Figure 8-3 shows the underlying asset values (top numbers) for the first year, which are the same as the option values (bottom italicized numbers in Figure 8-2) from the design option. Calculate the option values (bottom italicized numbers in Figure 8-3) at each node of the binomial tree by backward induction. Each node represents the value maximization of investing versus continuation, where you have the option to either invest $30 million to obtain the permit and benefit from the payoff or continue to keep the option open until it expires. At the terminal nodes, you would invest if the payoff is greater than the investment of $30 million; otherwise, you would let the option expire worthless.

A. Start with the terminal nodes that represent the last time step. At node S_0u, the expected asset value is $113 million. If you invest $30 million to apply for permits, the net payoff will be $113 million – $30 million = $83 million. Since your objective is to maximize your return, you would exercise your option by investing. Thus, the option value at this node becomes $83 million.

B. At node Sod, the expected asset value is $16 million. If you invest $30 million to apply for permits, the net payoff will be $16 million – $30 million = –$14 million. Since your objective is to maximize your return, you would not exercise your option. Thus the option value at this node becomes $0 million.

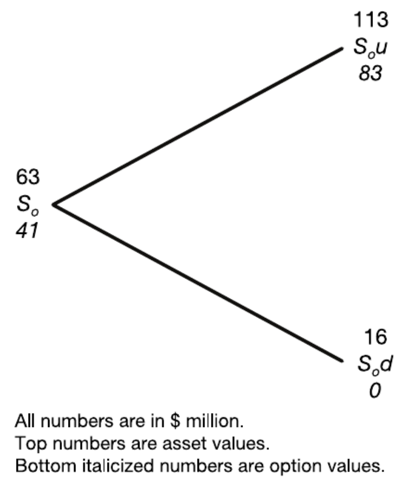


Figure 8-3. Binomial Tree for the Shortest Option of the Sequential Compound Option

C. Next, move on to the root node, one step away from the last time step. At node So, calculate the expected asset value for keeping the option open and accounting for the downstream optimal decisions. This is simply the discounted (at the risk-free rate) weighted average of potential future option values using the risk-neutral probabilities:

$$[p(S_0u) + (1 - p)(S_0d)]e^{(-r\delta t)} =$$
$$= [0.527(\$113m) + (1 - 0.527)(\$0m)]e^{(-0.06)(1)}$$

$$= \$41m$$

> If, on the other hand, the option is exercised by investing $30 million, the expected asset value would be:
>
> $$\$63\ million - \$30\ million\ =\ \$33\ million$$
>
> Since this value is less than the $41 million corresponding to the alternative to continue, you would not exercise the option, and the option value at this node would be $41 million.

A combined lattice integrating the option values from the three options over the entire option life is presented in Figure 8-4.

Let us first calculate the net present value (NPV) of the project by taking the difference between the expected payoff ($250 million) and the investment costs. Assuming Phase I investment ($30 million) is incurred at time = 0, the present values of the costs of the two subsequent phases can be calculated using a slightly higher rate (9%) than the risk-free rate. Thus, the present values of Phase II and Phase III costs are $75 million and $147 million, respectively. The project NPV, therefore, is:

$$\$250m - (\$30m + \$75m + \$147m) = -\$2m$$

By strictly considering the NPV alone, this project would be rejected for investment. The real options value (ROV), however, shows a total project value of $41 million, yielding an additional value of $43 million [$41 million – (–$2 million)] due to the compound option. Thus, consideration of the flexibility embedded in the project makes it attractive and more likely to be accepted. If the market uncertainty clears by the end of the design phase and the project payoff is expected to be significantly higher than the $210 million required for construction, Sweet ’n Sour Cola can move forward with the project; otherwise, it may abandon it or shelve it for later consideration. (Note that the prior investment of $120 million becomes a sunk cost and would not be considered in valuation of the project at that point.) Multiphase projects have a particular advantage in an options framework, when competitors face significant barriers to entry and there is a great deal of uncertainty about the market demand. The disadvantages include higher costs due to loss of economies of scale and loss of market to a competitor that may have entered the market full scale.

A closer examination of the HyperCola options results indicates that the ROVs for the options to permit, design, and construct are $41 million, $63 million, and $113 million, respectively. The value of the options increases simply because of the increase in uncertainty as a function of time. The option value can also increase due to resolution of technical uncertainty, as you exercise options in sequence and move toward project completion. For example, as mentioned before, the option value of launching a new technology increases after a pilot test. Another classic example of sequential options where technical uncertainty is resolved is drug development, where FDA approval follows successful clinical trials. However, in the HyperCola example, no efforts are made during the option life to resolve any technical uncertainty. In fact, presumably no technical uncertainty exists, because the company has already developed the formula for the cola and has the know-how to build the plant.

In this example, option lives of one, three, and five years are used for permitting, design, and construction, respectively. The individual option lives represent the amount of time the company has to make a go/no-go decision on the next phase and invest in it. For instance, Sweet 'n Sour Cola has a maximum of three years to decide to go forward with the design and invest in it. Then the company has an additional two years to decide on and start the construction. The total and individual option lives basically depend on the market competition and the amount of time it takes to complete each of the project phases.

For some projects, the option to stage may not be explicit or intuitive. However, when uncertainty is high, a project can be redesigned into appropriate phases based on real options analysis. You can also calculate the option values for multiple scenarios and choose the one that offers the highest value for project execution. As the number of phases increases, the options calculations become more complex. Any compound option obviously can be treated as a simple option, the value of which will be most conservative and can be treated as a floor for the option value. A key problem with that comparison is that for the simple options framework, you would have to assume that all the investment occurs at one time, which is not true for its parallel counterpart. Furthermore, in turning a compound option into a simple option for comparison purpose, you should be cautious about defining the option life. In the HyperCola example, if you assume that the entire investment is made at one time, then Sweet 'n Sour Cola has only one year to decide on that investment, so that the product can come to market in seven years (since it takes six years to complete the entire project). Thus, the appropriate option life for a simple option for HyperCola would be one year.

The Black-Scholes model can also be used to solve sequential options by solving for the options sequentially, starting with the longest option first. As in the binomial model, the option value of the successor option becomes the asset value for the predecessor option. The Black-Scholes-derived ROVs for the HyperCola project are computed to be \$9, \$37, and \$107 million for permitting, design, and construction phases, respectively. Breaking a project into phases offers advantages when you can afford to delay the project possibly due to competitors facing high barriers to entry, significant investment costs especially toward the front end of the project, and potential future opportunities for expansion. However, you may lose economies of scale, resulting in higher costs, and allow the competition to capture the market. Whereas the availability of the sequential compound options is obvious in multiphase projects, formal valuation of the options provides quantitative data to help management make rational decisions.

Project Valuation using Real Options, Dr. Prasad Kodukula, PMP Chandra Papudesu; J. Ross Publishing, 2006, Chapter 8, pp. 146-156

97. Define and give an example of parallel compound option

A parallel compound option example, a binomial solution (essentially the same as used for the preceding sequential option), and analysis of the results is shown below with a few modifications required for solving a parallel compound option.

Example

KlearKom, a mid-size telecommunications company, is considering offering third-generation (3G) wireless services to its customers in a specific market. Upgrading the existing network to meet the 3G requirements will involve an investment of $500 million, and the license for the required radio spectrum is estimated to cost $100 million. Discounted cash flow analysis using an appropriate risk-adjusted discount rate places the NPV of the expected future cash flows at $600 million, and the annual volatility factor for this payoff is calculated to be 35%. Based on the competition, KlearKom estimates that it has three years to make a go/no-go decision on this project. The continuous annual risk-free interest rate over this period is 5%. KlearKom can start the infrastructure upgrade at any time, but the spectrum license must be obtained before the upgrade can be tested and the service launched. This creates a parallel option, which the company can take advantage of in valuation of the project to make a better investment decision that takes into account the payoff uncertainty. Since both options — the option to purchase the 3G license and the option to invest in the network upgrade — are alive during the same time frame, and the license purchase must be exercised before the network upgrade, this constitutes a parallel compound option. What is the value of this option?

Identify the input parameters

S_0 *= $600 million*
X1 and X2 = $100 and $500 million, respectively
T1 and T 2 = 3 years
$\sigma = 35\%$
$r = 5\%$
$\delta t = 1$ *year*

Calculate the option parameters

$u = exp(\sigma\sqrt{\delta t}) = exp(0.35 * \sqrt{1}) = 1.419$
$d = 1/u = 1/1.419 = 0.705$
$p = (exp(r\delta t)–d)/(u – d)$
$p = [exp(0.05 * 1) – 0.705]/(1.419 – 0.705) = 0.485$

Build a binomial tree, as shown in Figure 8-5, using one-year time intervals for three years and calculate the asset values over the life of the option. Start with S_0 and multiply it by the up factor and the down factor to obtain S_0u and S_0d, respectively. For the first time step: Sou = $600 million * 1.419 = $851 million; Sod = $600 million * 0.705 = $423 million. Moving to the right, continue in a similar fashion for every node of the binomial tree until the last time step. In Figure 8-5, the top value at each node represents the asset value at that node.

There are two parallel options for this project: the option to upgrade the network and the option to obtain a license. Both have the same lifetime, but the license must be obtained before the upgrade is completed for launch. The option calculations are basically the same as for the sequential option. The option values for the dependent option, the network upgrade, are calculated first, which then become the underlying asset values for the option to obtain the license.

Figure 8-5 shows the option values (bottom italicized numbers) for the dependent option (network upgrade) at each node of the binomial tree by backward induction. At the terminal nodes, you would invest if the payoff is greater than the investment of $500 million; otherwise, you would let the option expire worthless. At each intermediate node, you have the option to either invest $500 million in the project and benefit from the payoff or continue to keep the option open until it expires.

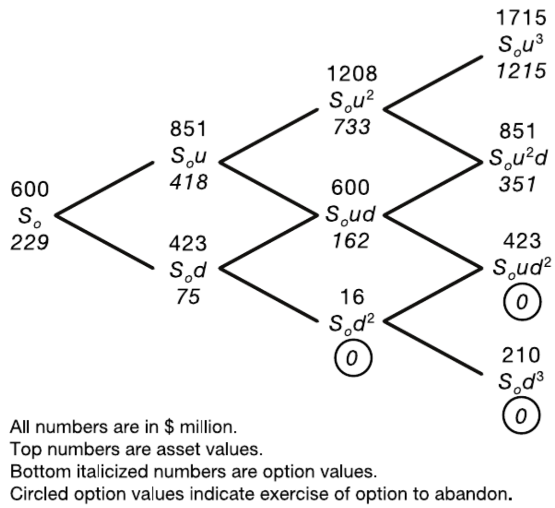


Figure 8-5. Binomial Tree for the Independent Option of the Parallel Compound Option

A. Start with the terminal nodes that represent the last time step. At node S_0u^3, the expected asset value is $1,715 million, which is greater than $500 million. Therefore, you would invest and realize a net payoff of $1,215 million ($1,715 million – $500 million), thereby making the option value at this node $1,215 million.

B. At node S_0d^3, the expected asset value is $210 million, which is smaller than the investment value of $500 million; therefore, it does not make sense to invest in the project, which puts the option value at this node at $0 million.

C. Next, move on to the intermediate nodes, one step away from the last time step. Starting at the top, at node S_0u^2, calculate the expected asset value for keeping the option open. This is simply the discounted (at the risk-free rate) weighted average of potential future option values using the risk-neutral probabilities:

$$[p(S_0u^3) + (1 - p)(S_0u^2d)]\mathrm{e}^{(-r\delta t)} =$$
$$= [0.485(\$1{,}215m) + (1 - 0.485)(\$351m)]e^{(-0.05)(1)}$$
$$= \$733m$$

The expected asset value at this node is $1,208 million. Exercising the option at this node will provide an option value of $708 million ($1,208 million – $500 million). Since the value of keeping the option open is larger, you would keep the option open and continue; therefore, the option value at Sou2 is $733 million.

D. Similarly, at node Sod 2, the expected asset value for keeping the option open, taking into account the downstream optimal decisions, is:

$$[0.485(\$351m)+(1-0.485)(\$0m)]e^{(-0.05)(1)}=\$162m$$

If, on the other hand, you exercise the option to invest, the expected asset value would be:

$$\$600\ million-\ \$500\ million\ =\ \$100\ million$$

Maximizing $162 million versus $100 million, you would keep the option open. Therefore, the option value at node S_0d^2 would be $162 million.

E. Complete the option valuation binomial tree all the way to time = 0 using the approach outlined above.

Calculate the option values for the independent option (purchase the 3G license) using the dependent option (network upgrade) values as the underlying asset values. The top numbers in Figure 8-6 show the underlying asset values, which are the same as the option values (bottom italicized numbers in Figure 8-5) from the network upgrade option. Calculate the option values (bottom italicized numbers in Figure 8-6) at each node of the binomial tree by backward induction. Each node represents the value maximization of investing versus continuation, where you have the option to either invest $100 million in the project or keep the option open. At the terminal nodes, you would invest if the payoff is greater than the investment of $100 million; otherwise, you would let the option expire worthless.

A. Start with the terminal nodes that represent the last time step. At node S ou3, the expected asset value is $1,215 million. Since this is greater than $100 million, you would invest and realize a net payoff of $1,115 million ($1,215 million – $100 million). The option value at this node now would be $1,115 million.

B. At node Soud 2, the expected asset value is $0; therefore, no investment would be made, resulting in an option value of $0 at that node.

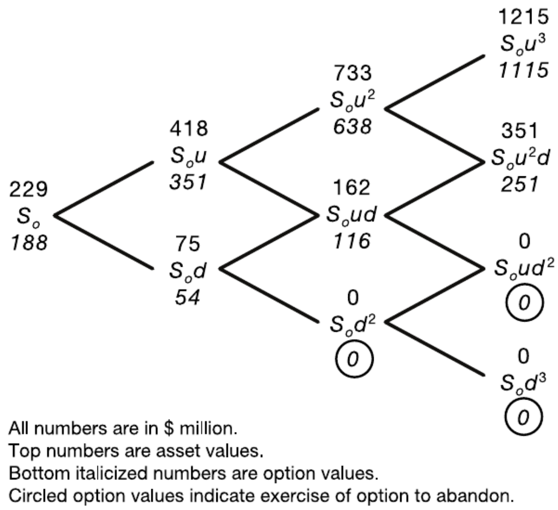


Figure 8-6. Binomial Tree for the Dependent Option of the Parallel Compound Option

C. Next, move on to the intermediate nodes, one step away from the last time step. Starting at the top, at node Sou2, calculate the expected asset value for keeping the option open. This is simply the discounted (at the risk-free rate) weighted average of potential future option values using the risk-neutral probability as weights:

$$[p(S_0u^3) + (1 - p)(S_0u^2d)]e^{(-r\delta t)} =$$
$$= [0.485(\$1{,}115m) + (1 - 0.485)(\$251m)]e^{(-0.05)(1)}$$
$$= \$638m$$

The expected asset value at this node is $733 million. Exercising the option at this node will provide an option value of $633 million ($733 million – $100 million). Since the value of keeping the option open is larger than this, you would keep the option open and continue; therefore, the option value at S_0u^2 is $638 million.

D. Similarly, at node S_0ud, the expected asset value for keeping the option open, taking into account the downstream optimal decisions, is:

$$[0.485(\$251m) + (1 - 0.485)(\$0m)]e^{(-0.05)(1)} = \$116m$$

The expected asset value at this node is $162 million. Exercising the option at this node will provide an option value of $62 million ($162 million – $100 million). Since the value of keeping the option open is larger than this, you would keep the option open and continue; therefore, the option value at S_0ud is $116 million.

E. Complete the option valuation binomial tree all the way to time = 0 using the approach outlined above.

The project NPV is calculated to be $0 million for a payoff of $600 million, assuming the total investment of $600 million ($500 million + $100 million) occurs at time = 0. This project would not be considered for investment because of this NPV. Real options analysis (ROA), however, shows an ROV of $188 million due to the compound option. Thus, consideration of the flexibility embedded in this project makes it more likely to be accepted for investment. Although the option available in this example seems similar to the sequential compound option discussed in the previous section, there is a minor difference. In both cases, the predecessor option must be exercised to take advantage of the successor option. But for KlearKom, both the options — the option to invest in the network upgrade and the option to purchase the 3G license — are alive during the same time frame, and one must be exercised (purchase of the license) before the exercise of the other is completed, thereby creating a parallel compound option. The infrastructure construction can be started at any time, but the spectrum license must be obtained before the network upgrade can be tested and the service launched. This creates a parallel option, which the company can take advantage of in valuation of the project to make a better investment decision that takes into account the payoff uncertainty. The options solution method is essentially the same for sequential and compound options, yielding exactly the same ROV. The difference lies in framing the option.

Project Valuation using Real Options, Dr. Prasad Kodukula, PMP Chandra Papudesu; J. Ross Publishing, 2006, Chapter 8, pp. 156-162

98. Define and give an example of Rainbow option

A key input parameter of any ROA problem is the volatility factor that represents the uncertainty associated with the underlying asset value. Typically, it is calculated as an aggregate factor built from many of the uncertainties that contribute to it. For example, the aggregate volatility used in the ROA of a product development project is representative of and a function of multiple uncertainties, including the unit price, number of units sold, unit variable cost, etc. If one of the sources of uncertainty has a significant impact on the options value compared to the others or if management decisions are to be tied to a particular source of uncertainty, you may want to keep the uncertainties separate in the options calculations. For instance, if you own a lease on an undeveloped oil reserve, you face two separate uncertainties: the price of oil and the quantity of oil in the reserve. You may want to treat them separately in evaluating the ROV.

When multiple sources of uncertainty are considered, the options are called rainbow options, and this warrants the use of different volatility factors — one for each source of uncertainty — in the options calculations. The options solution method is basically the same as for a single volatility factor except that it involves a quadrinomial tree instead of a binomial. This is because the asset can take one of four values as you move from one node to the nodes of the next time step in the lattice. The following illustration shows the solution to an options problem with two sources of uncertainty.

Schizo Petro Chemco is contemplating building a chemical plant to produce specialty polymers that can reduce construction costs. The raw materials for this plant are by-products of petroleum refining and are supplied by nearby refineries. The present value of the expected cash flows from future polymer sales is estimated to be $160 million. Plant construction is expected to cost $200 million. The project payoff is influenced by two types of uncertainty: market demand for the final product (specialty polymers), the annual volatility (σ_1) of which is estimated to be 30%, and the oil prices that dictate the cost of the raw materials, which are shown to exhibit an annual volatility (σ_2) of 20%. What is the value of the option to wait to invest given an option life of two years, over which the continuous annual risk-free rate is expected to be 5%?

Identify the input parameters

S_0 = \$160 million
X = \$200 million
T = 2 years
$\sigma_1 = 30\%$
$\sigma_2 = 20\%$
$r = 5\%$
δt = 1 year

Calculate the option parameters

$u_1 = exp(\sigma_1\sqrt{\delta t}) = exp(0.3 * \sqrt{1}) = 1.350$
$d_1 = 1/u_1 = 1/1.350 = 0.741$
$u_2 = exp(\sigma 2\sqrt{\delta t}) = exp(0.2 * \sqrt{1}) = 1.221$
$d_2 = 1/u_2 = 1/1.221 = 0.819$
$p_1 = (exp(r\delta t) - d_1)/(u_1 - d_1) = [exp(0.05*1) - 0.741]/(1.350 - 0.741) = 0.510$
$p_2 = (exp(r\delta t) - d_2)/(u_2 - d_2) = [exp(0.05*1) - 0.819]/(1.221 - 0.819) = 0.577$

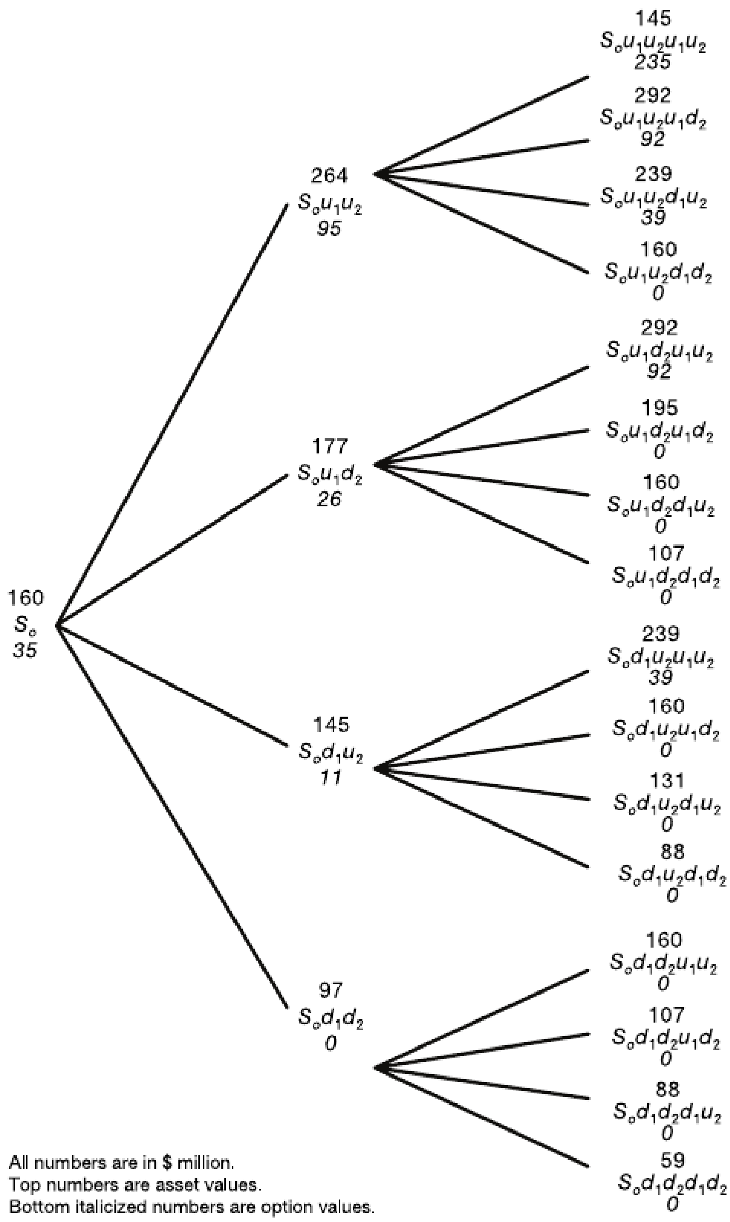


Figure 8-7. Quadrinomial Tree for the Compound Rainbow Option

Build a quadrinomial tree, as shown in Figure 8-7, using one-year time intervals for two years and calculate the asset values over the life of the option. The lattice in this example is a quadrinomial instead of a binomial, because the two volatility factors yield two up and down factors each. At any given node of the tree, therefore, there are four possible asset values in the next time period. The factors used to determine these values are: u_1u_2, u_1d_2, d_1u_2, d_1d_2. Start with S_0 and multiply it separately by the four up/down factors to obtain $S_0u_1u_2$, $S_0u_1d_2$, $S_0d_1u_2$, and $S_0d_1d_2$ for the first-time step. For example, $S_0u_1u_2$ = $160 million * 1.350 * 1.221 = $264 million; $S_0d_1d_2$ = $160 million * 0.741 * 0.819 = $97 million. Continue in a similar fashion for every node of the quadrinomial tree for the next step, which is also the last step in this example, as shown in Figure 8-7, where the top value at each node represents the asset value.

Figure 8-7 shows the option values (bottom italicized numbers) at each node of the quadrinomial tree calculated by backward induction. Each terminal node represents the value maximization of exercising the option by investing $200 million versus letting the option expire. Each intermediate node represents the value maximization of continuation versus exercising the option.

A. Start with the terminal nodes that represent the last time step first. At node $S_0u_1u_2u_1u_2$, the expected asset value is $435 million. If you invest $200 million to build the plant, the net payoff will be $435 million – $200 million = $235 million. Since your objective is to maximize your return, you would exercise the option by investing. Thus the option value at this node becomes $235 million.
B. At node $S_0d_1d_2d_1d_2$, the expected asset value is $59 million. Since this is less than the investment cost of $200 million, you would not invest and would let the option expire. The option value at this node would, therefore, be $0.
C. Next, move on to the intermediate nodes, one step away from the last time step. Starting at the top, at node Sou1u2, calculate the expected asset value for keeping the option open and accounting for the downstream optimal decisions. This is simply the discounted (at the risk-free rate) weighted average of potential future option values using the risk-neutral probabilities. That value at node $S_0u_1u_2$ is:

$$[p_1p_2(S_0u_1u_2u_1u_2) + p_1(1-p_2)(S_0u_1u_2u_1d_2) + (1-p_1)p_2(S_0u_1u_2d_1u_2) + (1-p_1)(1-p_2)(S_0u_1u_2d_1d_2)]e^{(-r\delta t)}$$

$$= [(0.510)(0.577)(\$235m) + 0.510(1-0.577)(\$92m) + (1-0.510)(0.577)(\$39m) + (1-0.510)(1-0.577)(\$0m)]e^{(-0.05)(1)} = \$95m$$

If, on the other hand, the option is exercised by investing $200 million, the expected asset value would be:

$$\$264\ million - \$200\ million = \$64\ million$$

Since this value is less than the $95 million corresponding to the alternative to continue, you would not exercise the option, and the option value at this node would be $95 million.

D. Similarly, at node $S_0d_1u_2$, the expected asset value for keeping the option open, taking into account the downstream optimal decisions, is:

$$\begin{aligned}[(0.510)(0.577)(\$39m) + (0.510)(1-0.577)(\$0m) \\ + (1-0.510)(0.577)(\$0m) \\ + (1-0.510)(1-0.577)(\$0m)]e^{(-0.05)(1)} \\ = \$11m\end{aligned}$$

If, on the other hand, you exercise the option to invest, the expected asset value would be:

$$\$145\ million - \$200\ million = -\$55\ million$$

Maximizing $11 million versus –$55 million, you would keep the option open. Therefore, the option value at node $S_0d_1u_2$ would be $11 million.

E. Complete the option valuation quadrinomial tree to the next step on the left at time = 0.

Let us first compare the NPV of the asset based on discounted cash flow (DCF) versus ROA. The project payoff value based on the DCF calculation is expected to be $160 million at an investment cost of $200 million, resulting in an NPV of –$40 million. The project value based on ROA, considering the two sources of uncertainty separately, is $35 million. The difference of $75 million [$35 million – (–$40 million)] is the value added to the project because of the ROV that management can take into consideration in making its investment decision. If the option value is calculated using only one aggregate volatility factor of 20% or 30%, the value of the option to wait for this example would be $10 or $19 million, respectively. This shows that unless the multiple sources of uncertainty are considered, the option value may not be completely realized. The solution to a rainbow option problem requires a quadrinomial method because of multiple sources of uncertainty, and the calculations are cumbersome compared to a recombining binomial tree. In the Schizo Petro Chemco example, an option life of two years was used to keep the illustration simple. With longer lives, however, the calculations become even more complex due to the growing quadrinomial tree. Although the calculations are complex with this approach, the underlying theme in computing the option value remains the same as with the binomial method. The standard Black-Scholes equation cannot accommodate the multiple sources of uncertainty and is not useful for rainbow options.

For the sake of simplicity, practitioners typically combine all the known uncertainties that drive the asset value and estimate one aggregate volatility factor for the asset. As discussed in Chapter 6, Monte Carlo simulation is a commonly used method for this purpose. You may, however, want to treat the uncertainties separately, if the controlling variables are independent of each other, evolve differently over time, and especially impact the asset value in different directions. Separate treatment of the different sources of uncertainty gives you better insight into what variables have the highest impact on the option value. It also helps you to easily re-evaluate the project value when one of the two uncertainties clears. Most important, it will help you capture the true value of the option embedded in the project.

Strictly speaking, the quadrinomial lattice is appropriate when the sources of uncertainty are not related to each other. In the Schizo Petro Chemco example, the two sources of volatility (cost of the raw material and market demand for the final product) may be correlated, but we assumed the correlation to be insignificant for the purpose of illustration. In real world situations, the different sources of uncertainty, especially market uncertainty, may not be completely independent, but as long as the correlation is not significant, the quadrinomial method is expected to provide good approximation of the true option value. Private uncertainty related to the technical effectiveness of a project, on the other hand, is independent of market uncertainty and can be accounted for in options valuation as demonstrated with an example presented in a later section of this chapter. As you will see, decision tree analysis is used to address the private uncertainty, and either a binomial or quadrinomial approach is used to account for the market uncertainty, depending on whether it is a simple or rainbow option, respectively. The quadrinomial approach uses the nonrecombining lattice instead of the recombining lattice, which is a characteristic of the binomial approach. The nonrecombining lattice also is used when the volatility factor for the underlying asset value changes within the option life, as shown in the following example.

Project Valuation using Real Options, Dr. Prasad Kodukula, PMP Chandra Papudesu; J. Ross Publishing, 2006, Chapter 8, pp. 162-168

99. Define and give an example of option with changing volatility

In most ROA calculations, the volatility of the project payoff is assumed to be relatively constant over the option life and is represented by a single aggregate factor. Therefore, a single volatility factor is used across the binomial tree to represent the option life. However, if the volatility is expected to change during the option life and is significant, it can be accounted for by modifying the binomial method. Start with the initial volatility factor, build the binomial tree, and calculate the asset values at each node of the tree using the corresponding up and down factors up to the point where the volatility changes. From that point on, calculate the asset values using the new up and down factors related to the new volatility factor, which will result in a nonrecombining lattice. The option value calculation method using backward induction will be the same for the entire tree. The following example for an option to wait demonstrates the calculations involved. (Although this is a simple option by definition, it requires an advanced form of the lattice method using the nonrecombining lattice and hence its inclusion in this chapter on advanced options.)

Example

EnviroTechno, an environmental technology company, is considering an investment in a new patented product that can help industries comply with an environmental regulation that is under consideration by the U.S. Congress. Whereas the regulation is expected to be passed two years from now, many multinational companies based in the United States are beginning to comply because of similar regulations already in place in Europe and many other countries. EnviroTechno forecasts the present value of the expected future cash flows and the investment (product development and launch) cost to be \$400 million each. The annual volatility of the logarithmic returns for the future cash flows is estimated to be 30% for the next two years and is expected to decrease to 20% at the point when the regulations go into effect. The annual continuous risk-free rate for the next four years is 5%. What is the value of this project using ROA?

Identify the input parameters

So = \$400 million

T = 4 years

σ = 30% (volatility during the first 2 years)

σ' = 20% (volatility after 2 years)

r = 5%

X = $400 million
δt = 1 year

Calculate the option parameters
Since the up and down factors depend on the volatility factor, which changes after two years, there will be two sets of up and down factors corresponding to the two volatility factors. They will be denoted by u, u′, d, and d′, respectively. There will also be two risk-neutral probability factors (p and p′) corresponding to the two sets of up and down factors.

$u = \exp(\sigma\sqrt{\delta t}) = \exp(0.30 * \sqrt{1}) = 1.350$
$d = 1/u = 1/1.350 = 0.741$
$u' = \exp(\sigma'\sqrt{\delta t}) = \exp(0.20 * \sqrt{1}) = 1.221$
$d' = 1/u' = 1/1.221 = 0.819$
$p = (\exp(r\delta t) - d)/(u - d) = [\exp(0.05 * 1) - 0.741]/(1.350 - 0.741) = 0.510$
$p' = (\exp(r\delta t) - d')/(u' - d') = [\exp(0.05 * 1) - 0.819]/(1.221 - 0.819) = 0.577$

Build a binomial tree, as shown in Figure 8-8, using one-year time intervals for four years to account for the change in the up and down factors after the first two years.

Start with S_0 and multiply it by the up factor and down factor to obtain S_0u and S_0d, respectively. For example, S_0u = $400 million * 1.350 = $540 million; S_0d = $400 million * 0.741 = $296 million. Continue in a similar fashion for every node of the binomial tree for the next time step. After two years, the volatility changes, and so do the up and down factors. Therefore, from this point on, the asset values are calculated using u′ and d′. Because of the change in these factors, the tree will no longer be recombining starting at the third year. For example, the value at node S_0u^2 is $729 million. There are two nodes in the third year that originate from this node. The upper node S_0u^2u' = $729 million * 1.221 = $890 million; the lower node S_0u^2d' = $729 million * 0.819 = $597 million. Continue in a similar fashion until the last time step, as shown in Figure 8-8, where the top values at each node represent the asset value.

Calculate the option values at each node of the binomial tree using backward induction. These are the bottom italicized numbers in Figure 8-8. Each terminal node represents the value maximization of investing at a cost of $400 million versus letting the option expire. At every node in the prior years, you have a choice to either keep the option open for the future or exercise it by committing the investment.

1. Start with the terminal nodes that represent the last time step. At node Sou2u′2, the expected asset value is $1,087 million. Since this is greater

than the investment value of $400 million, you will exercise the option, and the option value would be $1,087 million – $400 million = $687 million.

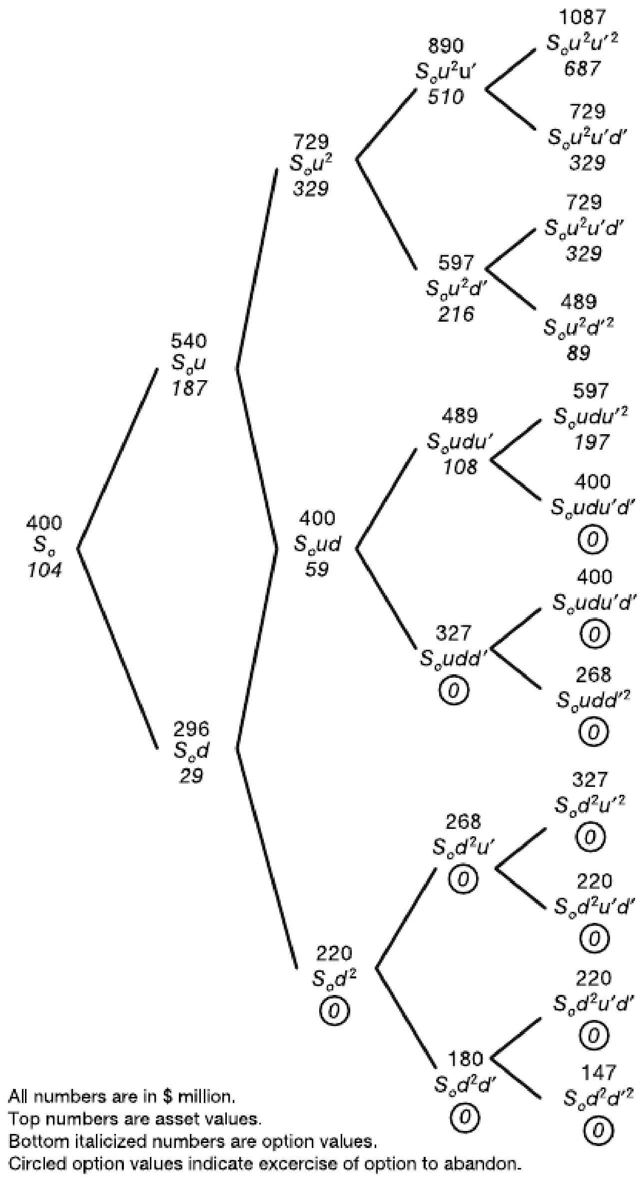


Figure 8-8. Nonrecombining Binomial Tree for the Option to Wait with Changing Volatilities

2. At node Sod 2d′ 2, the expected asset value is $147 million. Since this is less than the exercise cost of $400 million, you will let the option expire worthless. Thus the option value at this node is $0.
3. Next, move on to the intermediate nodes, one step away from the last time step. Starting at the top, at node Sou2u′, calculate the expected value for keeping the option open and accounting for the downstream optimal decisions. This is simply the discounted (at the risk-free rate) weighted average of potential future option values using the risk-neutral probability. That value, for example, at this node is:

$$[p'(S_0u^2u'^2) + (1-p')(S_0u^2u'd')]e^{-r\delta t}$$
$$= [0.577(\$687m) + (1-0.577)(\$329\ million)]e^{(-0.05)(1)}$$
$$= \$510m$$

If the option is exercised to invest by spending $400 million, the expected option value would be:

$$(\$890\ million) - \$400\ million\ =\ \$490\ million$$

Since this value is less than the $510 million corresponding to the alternative to continue, you would not exercise the option, and the option value at this node would be $510 million.
4. Similarly, at node S_0u^2d', the expected asset value for keeping the option open, taking into account the downstream optimal decisions, is:

$$[0.577(\$329m) + (1-0.577)(\$89m)]e^{(-0.05)(1)} = \$216m$$

If, on the other hand, you exercise the option to invest at a cost of $400 million, the expected asset value would be:

$$(\$597\ million) - \$400\ million\ =\ \$197\ million$$

Maximizing $216 million versus $197 million, you would not exercise the option. Therefore, the option value at node S_0u^2d' would be $216 million.
5. Complete the option valuation binomial tree all the way to time = 0. You should use the value of p′ as the risk-neutral probability for nodes in year 3 and year 2. For the nodes in year 1 and year 0, however, you should use the value of p as the risk-neutral probability. For example, at node Sou, the expected value of keeping the option open, taking into account the downstream optimal decisions, is:

$$[0.510(\$329m) + (1-0.510)(\$59m)]e^{(-0.05)(1)} = \$187m$$

If, on the other hand, you exercise the option to invest at a cost of $400 million, the expected asset value would be:

$$(\$540\ million) - \$400\ million\ =\ \$140\ million$$

Maximizing $187 million versus $140 million, you would not exercise the option. Therefore, the option valuation at node S_0u would be $187 million.

Let us first compare the value of this project based on DCF versus ROA. The present value of the cash flows for the current operation based on the risk-adjusted DCF method is $400 million, and with the same amount of investment the NPV is:

$$\$400\ million - \$400\ million\ =\ \$0\ million$$

Based on the DCF analysis alone, EnviroTechno would decide not to proceed with this project. However, ROA suggests that the option to wait to undertake this project is worth $104 million. Comparing this with the baseline NPV of $0 million for the project, the additional value provided by the option, therefore, is:

$$\$104\ million - \$0\ million\ =\ \$104\ million$$

The value added to the project because of real options is substantial. Therefore, EnviroTechno may decide to keep the option to invest in this project open at this time and exercise it when the uncertainty clears and conditions become favorable.

The volatility of the underlying asset is expected to remain the same during the option lifetime in most cases but may change due to special circumstances, such as those encountered by EnviroTechno. When new regulations are established, the volatility is likely to decrease as the market tries to comply with them. If the volatility is assumed to be constant (30%) for the entire option life in the EnviroTechno example, the option value would be $112 million, whereas with 20% volatility after two years it decreases to $104 million. Further sensitivity analysis shows the following results:

25%	$108 million
15%	$100 million
10%	$97 million

The ROV in this example increases by about $8 million for a 10% increase in volatility. Consideration of volatility changes during the option life in the ROA calculations provides a more accurate estimate of the ROV of the project.

Project Valuation using Real Options, Dr. Prasad Kodukula, PMP Chandra Papudesu; J. Ross Publishing, 2006, Chapter 8, pp. 162-168

100. Case Study: Investment Projects Valuation and Real Options and Games Theory

Consider an investor willing to build a hotel with initial investment $25 mln. Given the negative relationship between occupancy of the hotel and the price per night per room, it was estimated that maximum profits were obtained at price $140, which would produce 35% occupancy level. Given these data, future revenues were estimated using Monte Carlo simulation. NPV was estimated to be $-3.75 mln. As this number is close to zero, it is not clear to decide whether to invest or not. Additional analysis needs to be conducted in order to better estimate fair value of the hotel business.

While operating a hotel, manager might face several options such as Expansion, Contract, and Abandonment. Expansion involves building one more, similar class hotel. This decision may double value of the company, but requires additional investment of $20mln. This sort of decision may be made at time when there is high demand and value of business increases. Contract option involves renting part of the hotel; thus, value of the company is decreased by contraction rate, however this decision generates some cost saving. This sort of decision may be made at time when the demand has slightly decreased. Abandonment option involves sale of company's assets and in this manner obtaining higher value then by operating a business. This sort of decision may be made when demand has significantly decreased and investor is willing to leave market. Investor would choose option that generates maximum value of these three, i.e. Chooser Option. However, remember that starting business is itself a call option, thus we are having Sequential Option. Sequential Option gives the right to start the business which automatically gives us chooser option.

Taking into consideration these opportunities that investor may use, increased NPV to the level of $36.77 mln. However, as stated before, real option analysis doesn't take into consideration competition and competitive analysis. In order to incorporate competition into real options, Games Theory was used. To describe Game, Cournot competition model was used, which states that, if number of market participants increase, market output goes to competitive level and prices converges to marginal cost. Game can be illustrated by the following matrix

Value / Occ. rate	31%	25.61%
NPV	-3.75	-25
NPV + options	36.77	**-18.25**

Matrix shows the value of the company with and without options at given level of occupancy level and competitive occupancy level. Equilibrium of this game is $-18.25 mln, stating that it is not good time for the investor to enter market.

Investment Project Valuation with Real Options and Game Theory,
Bachlor Thesis, Levan Gachechiladze, Irakli Chelidze,2010

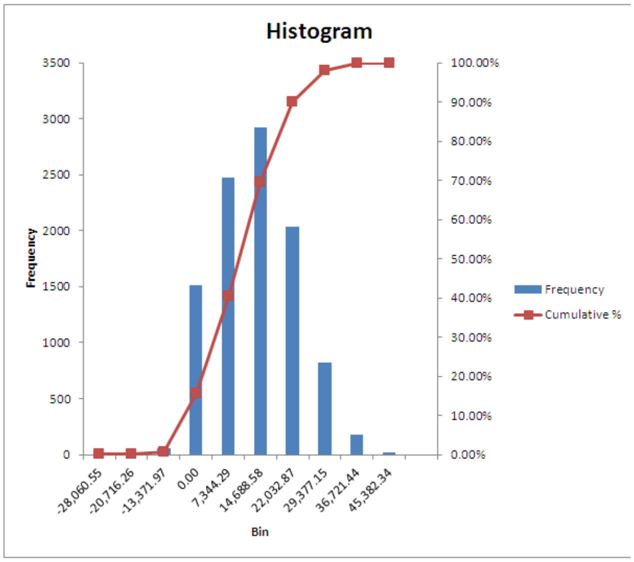
101. Case Study: Real Option Valuation of the Project

In order to introduce the project valuation technique, we are considering the case with gasoline station which require import of gasoline.

Traditional valuation technique - NPV (net present value)- with (I) investment$80,000 monthly cash flows (C_n) $8,153 during the 12 month (N=12), and discount rate recalculated for monthly data 0.8 %, shows us that NPV = 12,943. This is slightly positive NPV. But cash flows in the project are random variable with some volatility. Adjusting the project for randomness of cash flows can be done via Monte Carlo simulation. Monte Carlo simulation in the thesis is done based on uniform distribution - uniformly simulating monthly quantity of gasoline sales between 9,510 and 19020 gallons. Constructing the histogram on simulated data (10000 of iterations) we can find that up to 15.71% of the iterations the project gives negative NPV.

Therefore, we need to adjust the project to accommodate associated risk with uncertain sales of gasoline. It could be done by introducing real option analysis of project. In our case we are presenting the project as simple expansion real option - American call option (or if we will be more precise - Bermuda call option - combination of European and American options). Calculations of this option can be done via Black-Scholes formula or using recombining binomial lattice which gives us good visualization of the project valuation and reaches the value provided by Black-Scholes formula as number of nodes of the binomial lattice increase and go to infinity.

Bin	*Frequency*	*Cumulative %*
-28,060.55	1	0.01%
-20,716.26	3	0.04%
-13,371.97	59	0.63%
0.00	1508	15.71%
7,344.29	2471	40.42%
14,688.58	2918	69.60%
22,032.87	2032	89.92%
29,377.15	818	98.10%
36,721.44	175	99.85%
45,382.34	15	100.00%



Using call option formula $(S_n - I)^+$ and backward induction we can calculate option price, which in our case is $7,291. Therefore, our new project value is NPV + Option Value = $12,943 + $7,291 = $20,234.

Real Option Valuation of the Project with RobustMean-variance Hedging, PHD Thesis, Tamaz Uzunashvili,201

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