ქართულ-ამერიკული უნივერსიტეტი
GEORGIAN AMERICAN UNIVERSITY

Business School

8 Merab Aleqsidze Street

Tbilisi, Georgia

PhD Thesis

Real Option Valuation of the Project with Robust Mean-variance Hedging

T. Uzunashvili
*S*upervised by Dr. R. Tevzadze

Acknowledgement

It would not have been possible to write this doctoral thesis without the help and support of the kind people around me, to only some of whom it is possible to give particular mention here. I would like to mention especially my PhD supervisor Dr. Revaz Tevzadze and Dean of Business School Dr. Teimuraz Toronjadze, whose valuable and permanent support predetermined success of current PhD thesis.

Contents

1 Project Valuation **6**

1 Introduction 6

2 Literature review of Real Options 9

3 Option Calculations 10

4 Limitations to Applying Financial Option Models to Real Options Analysis 13

5 Estimation of Input Variables 14

6 Types of Real Option. 15

(a) Simple Real Option. 15

(b) Compound Options. 19

7 Project valuation. 20

8 Hedging and Parallel Compound Real Option. 26

9 Parallel Compound Real Option Calculations. 30

2 Single Period Mean-Variance Hedging Model **35**

1 Introduction 35

2 The main results 37

3 Appendix A 50

4 Appendix B 54

3 Robust mean-variance pricing of contingent claims **57**

1 Robust mean-variance pricing of contingent claims 57

2 The main results 62

4 Robust utility maximization for diffusion market model **66**

1 Introduction 66

2 Generalized coefficients and the existence of a saddle point 74

3 Power and exponential utility cases 89
1 Appendix A 95
2 Appendix B 99

Bibliography **101**

List of Figures

1.1 Histogram 22

1.2 Real Option I 25

1.3 Gasoline Historical Spot Prices 27

1.4 Futures Prices 29

1.5 Real Option II 31

1.6 Decision Tree 32

Abstract

In the thesis considered project valuation analysis by using real option approach to decision making on project investment. Flexibility of managerial decision making is combined with project risk reduction which is achieved by robust mean-variance hedging. In the thesis is developed mathematical model of robust mean-variance hedging in a single period and continuous time for incomplete markets with misspecified parameters (appreciation rate and volatility). Solution of the hedging problem used in the project valuation example.

Chapter 1

Project Valuation

1 Introduction

Decision making under uncertainty is one of the challenging fields of study in contemporary project management. The traditional project management framework which based on discounted cash flow valuation doesn't provide managers with flexibility of decision making, and doesn't incorporate in project management valuation technique risk management tools.

Managers traditionally considering three mainstream approaches of assets valuation:

1. Market approach
2. Income approach
3. Cost approach

The market approach looks at comparable assets in the marketplace and their corresponding prices. The income approach looks at the future potential profit or free-cash-flow -generating potential of the asset and attempts to quantify, forecast, and discount these net free cash flows to a present value. The cost approach looks at the cost a firm would incur if it were to replace or reproduce the asset's future profitability potential. Traditional methods assume that the investment is an all-or-nothing strategy and do not account for managerial flexibility that exists such that management can alter the course of an investment over time when certain aspects of the project's uncertainty become know. There are additionally several potential problem areas in using a traditional discounted cash flow calculation on strategic investments:

1. Undervaluing an asset that currently produces little of no cash flow

2. The nonconstant nature of the weighted average cost of capital discount rate through time

3. The estimation of an asset's economic life

4. Forecast errors in creating the future cash flows

5. Insufficient tests for plausibility of the final results

Using the traditional, net present value approach we are giving the example of the project valuation with the following cash streams:

Investment - \$100

Payoff - \$120

Discount Rate - 0.15%

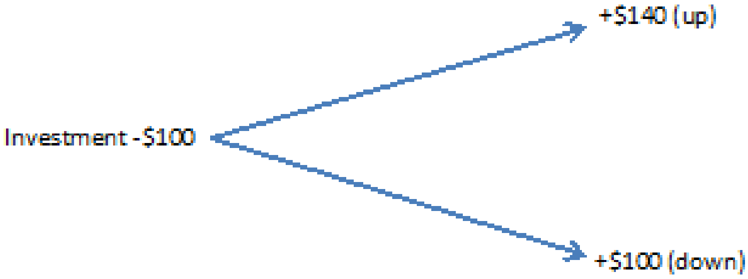
Risk Free Rate - 0.5%

Period - 1 year

$$NetPresentValue = \frac{120}{1.15^1} = \$4.3 \tag{1.1}$$

At time 1 future cash inflow can increase up to $140 or can decrease - $100

With the same expected cash flow $120



$$NetPresentValue_{(up)} = \frac{140}{1.15^1} = \$21.74 \tag{1.2}$$

$$NetPresentValue_{(down)} = \frac{100}{1.15^1} = -\$13.04 \tag{1.3}$$

The volatility of the net present value in above example shows inconsistency and uncertainty of the asset valuation of the project using traditional approach.

Consideration of real investments and associated risk lead us to two main approaches of risk management: through real option and financial hedging strategies. In most of publications these two risk management strategies considered as an alternative and mutually exclusive. In the thesis is considered project analysis by incorporating both approaches to risk management. Valuation of the project studied in real option framework which uses robust mean-variance hedging strategy approach to valuate real options.

Real option calculations based on assumption about completeness of market information, which is not true in real investments. In the thesis has been provided the project valuation with robust mean variance hedging in incomplete markets with uncertain appreciation rate and volatility.

2 Literature review of Real Options

Real Option approach for investment analysis is well summarized in Dixit and Pindyck (1994)[4] and Trigeorgis (1996)[5]. Preliminary works which led to the contemporary real option analysis of the projects had inspired by Mayers article (1977) "Determinants of Corporate Borrowings" [6]. Brennan and Schwarz (1985)[7] use an option pricing approach to analyze investment in natural resource. McDonald and Siegel (1986) [8] provided the standard continuous-time framework for analysis of a firm's investment in a single project. Majd and Pindyck (1987) [9] enrich the analysis with time-to-build feature. Dixit (1989) [10] uses the real option approach to examine entry and exit from a productive activity. Triantis and Hodder (1990)

[11] analyze manufacturing flexibility as an option. Titman (1985) [12] and Williams (1991) [13] use the real options approach to analyze real estate development.

3 Option Calculations

Multiple methodologies and approaches are used in financial options analysis to calculate an option's value:

1. Closed-form equations like the Black-Scholes model and it modifications,
2. Monte Carlo path-dependent simulation methods,
3. Lattices (e.g. binomial, trinomial, etc.)
4. Variance reduction
5. Partial-differential equations

For simplicity purpose in the PhD calculation of the real option done by using binomial lattices, however also presented closed-form equation, particularly Black-Scholes model. Lattices can solve all types of options, including American, Bermudan, European, and many types of exotic options. They are also highly flexible but require significant computing power and lattice steps to obtain good approximations. Results obtained through the use of binomial lattices tend to approach those derived from closed-form solutions.

Example:

Assume that stock price (S) and the strike price (X) are $100, the time to expiration (T) is one year, with a 5% risk-free rate (r_f) for the same duration, while the volatility (σ) of the underlying asset is 25% with no dividends (q). Let us look at a European Call Option (generalized Black-Scholes model):

$$\text{Call} = Se^{-q(T)}N\left[\frac{\ln(S/X)+(r_f-q+\sigma^2/2)T}{\sigma\sqrt{T}}\right]-Xe^{-r_f(T)}N\left[\frac{\ln(S/X+(r_f-q-\sigma^2/2)T}{\sigma\sqrt{T}}\right]=\$12.336$$

With binomial lattice we obtain the following results:

N = 10 steps \$12.0923

N = 20 steps \$12.2132

N = 50 steps \$12.2867

N = 100 steps \$12.3113

By increasing number of steps in binomial lattice will get the same results as with closed-form solution case. Throughout this course will be used binomial lattice approach.

For calculations of the option value by binomial approach is necessary to introduce so called risk-neutral probabilities. Instead of first taking the expectation and then adjusting for risk, one can first adjust the probabilities of future outcomes such that they incorporate the effects of risk, and then take the expectation under those different probabilities. Those adjusted, 'virtual' probabilities are called risk-neutral probabilities.

Irrespective of the option to be valued, the binomial lattice representing the underlying asset value has the same properties and can be described by the equations presented below:

$p = \frac{e^{\sqrt{\delta}t} - d}{u - d}$

The up and down factors, u and d, are a function of the volatility of the underlying asset and can be described as follows: and

$u = e^{\sigma\sqrt{\delta t}}$ and $d = e^{-\sigma\sqrt{\delta t}} = \frac{1}{u}$

Where

r_f - risk-free rate

σ - volatility of the natural logarithm of the underlying free cash flow returns

in

δt - stepping time

p - risk-neutral probability

u - up factor

d - down factor

b - continuous dividend outflows in

Example:

S (current asset value) = $160

X (strike price) = $200

σ (volatility) = 30

r_f (risk free rate) = 5

T (time to expiration) = 5 years

1. Black-Scholes Equation

$d_1 = \frac{ln(S/X)+\left(r_f+0.5\sigma^2\right)T}{\sigma\sqrt{T}} = 0.375$

$d_2 = d_1 - \sigma\sqrt{T} = -0.295$

$N\left(d_1\right) = 0.646$ Excel function = NORMSDIST(d_1)

$N\left(d_2\right) = 0.384$ Excel function = NORMSDIST(d_2)

$C = N\left(d_1\right)S - N\left(d_2\right)Xe^{-rT} = \$44$

2. Binomial Method

$$u = e^{\sigma\sqrt{\delta t}} = \exp\left(0.30 \times \sqrt{1}\right) = 1.350$$

$$d = 1/u = 1/1.350 = 0.741$$

$$p = \frac{e^{r_f \delta t} - d}{u - d} = \left[\exp(0.05 \times 1) - 0.741\right] / (1.350 - 0.741) = 0.510$$

Time	0	1	2	3	4	5
						717.45
					531.44	
				393.66		393.80
			291.60		291.70	
		216.00		216.08		216.15
S =	160.00		160.06		160.11	
		118.56		118.60		118.64
			87.85		87.88	
				65.10		65.12
					48.24	
						35.74
						517.45
					341.36	
				212.93		193.80
			127.96		101.54	
		74.87		52.91		16.15
C =	42.93		27.44		7.84	
		14.17		3.80		0.00
			1.84		0.00	
				0.00		0.00
					0.00	
						0.00

$$p\left(Su^5\right) + (1-p)\left(Su^4 d\right)\Big] \times \exp\left(-r_f \delta t\right)$$

$$\max(S - X, 0)$$

4 Limitations to Applying Financial Option Models to Real Options Analysis

There are two related concepts that form the foundation of the Black-Scholes formula and other option valuation models: arbitrage and replicating portfolios. One important assumption behind the options pricing models is that no "arbitrage" opportunity exists. Real assets are not as liquid as financial assets, and therefore option pricing models are inappropriate for real options valuation. Practitioners have used three different types of adjustment to overcome limitation of the model:

1. Use an interest rate that is slightly higher than the riskless rate in the option pricing model

2. Use a higher discount rate in calculating the discounted cash flow (DCF) value of the underlying asset

3. Apply an "illiquidity" discount factor to the final option value.

All three methods decrease the option value, making it more conservative.

The replicating portfolio approach is valid when the underlying asset is a traded security, because the risk related to the underlying asset value is strictly market risk and is captured in the traded security. Since most real options are not traded assets, application of financial option models to real options is questioned by some critics.

5 Estimation of Input Variables

Underlying Asset Value

1. In deriving the option pricing models, the value of the underlying asset is assumed to change in a continuous process without "jumps"

2. With real options, both dividend equivalent negative cash flows as well as positive cash flows can affect the underlying asset value, and these "leaks" must be

adjusted for accordingly.

Volatility

1. The volatility factor (σ) used in the options models is the volatility of the rates of return, which is measured as the standard deviation of the natural logarithm of cash flow returns - not the actual cash flows. The return for a given time period is the ratio of the current time period cash flow to the preceding one. In real options is more appropriate to use different formula

$X = ln\left(\frac{\sum_{i=1}^{n} PVCF_i}{\sum_{i=0}^{n} PVCF_i}\right)$. By using Monte Carlo simulation model of X values with standard normal distribution you can calculate standard deviation.

2. In any options model, the volatility factor used should be consistent with the time step used in the corresponding equations. The volatility factor based on one time frame can be converted to another using the following equation:

$\sigma\left(T_2\right) = \sigma\left(T_1\right)\sqrt{\frac{T_2}{T_2}}$

6 Types of Real Option.

(a) Simple Real Option.

Option to abandon

The option to abandon is embedded in virtually every project. This option is especially valuable where the net present value (NPV) is marginal but there is a great potential for losses. The losses can be minimized by selling off the project assets either on the spot or preferably by prearranged contracts. The contingent decision in this option is to abandon the project if the expected payoff (the underlying asset value) falls below the project salvage value, the strike price. This option therefore has the characteristics of a put option.

Example

S(DCF) = 150

$\sigma = 30\%$

$r_f = 5\%$

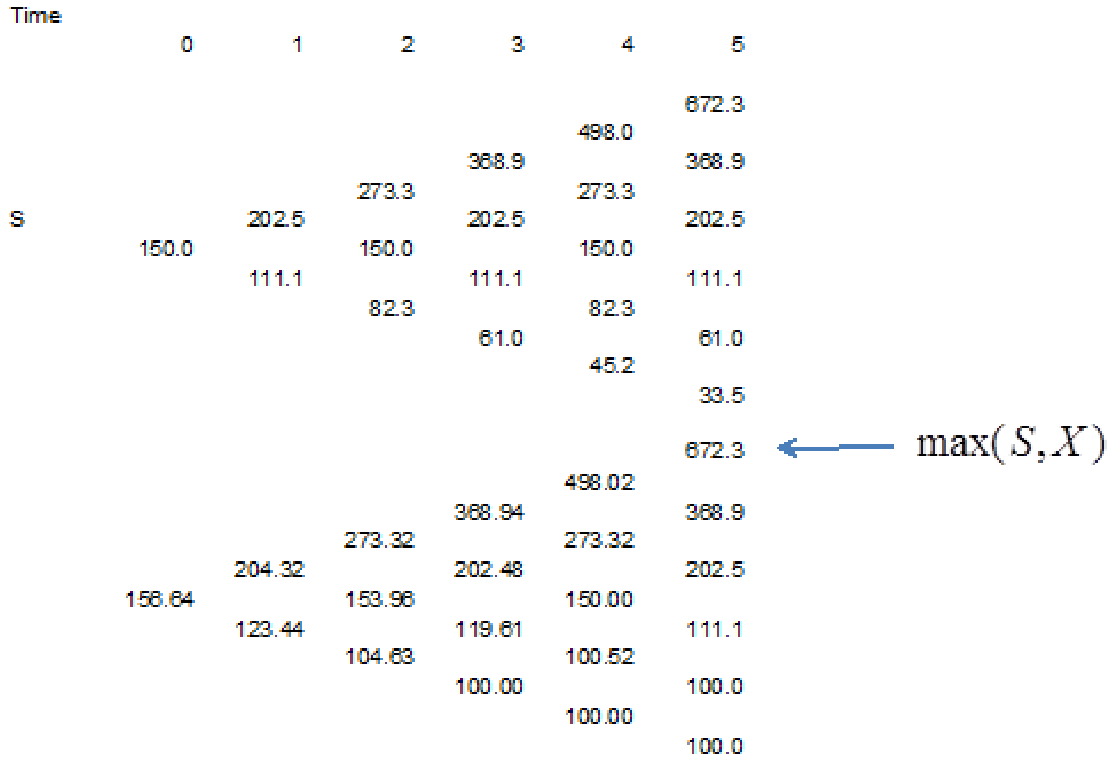
T = 5

Salvage Value (X) = 100

u = 1.3499

d = 0.7408

p = 0.5097



Option Value = 6.6412

Option to Expand

The option to expand is common in high-growth companies, especially during economic booms. For some projects, the initial NPV can be marginal or even negative, but when growth opportunities with high uncertainty exist, the option to expand can provide significant value. The option would be exercised if the expected payoff is greater than the strike price thereby making it a call option.

Option to Contract

The option to contract is significant in today's competitive marketplace, where companies need to downsize or outsource swiftly as external conditions change. The option to contract has the same characteristics as a put option, because the option value increases as the value of the underlying asset decreases.

Example

PV = \$1,000

Σ = 50%

r_f = 5%

Option to contract (C)= 50% of its current operations

Savings = \$400

T = 5.00

Δ = 0.20

Up factor = 1.64872

Down factor = 0.60653

p = 0.42674

Time	0	1	2	3	4	5
						12,182.49
					7,389.06	
				4,481.69		4,481.69
			2,718.28		2,718.28	
		1,648.72		1,648.72		1,648.72
S	$1,000		1,000.00		1,000.00	
		606.53		606.53		606.53
			367.88		367.88	
				223.13		223.13
					135.34	
						82.08
						12,182.49
					7,389.06	Open
				4,481.69	Open	4,481.69
			2,733.97	Open	2,718.28	Open
		1,702.86	Open	1,677.49	Open	1,648.72
	1,105.61	Open	1,087.60	Open	1,052.75	Open
	Open	759.91	Open	745.76	Open	703.27
		Open	583.94	Open	583.94	Contract
			Contract	511.57	Contract	511.57
				Contract	467.67	Contract
					Contract	441.04
						Contract
Contract - Option Value	$105.61					

$$\max(S, S \times C + Saving)$$

Option to Choose

The option to choose consists of multiple options combined as a single option. The multiple options are abandonment, expansion, and contraction. The main advantage with this option is the choice. This is a unique option in the sense that, depending upon the choice to be made, it can be considered a put (abandonment or contraction) or call (expansion) option.

Example

NPV = 100 million

$\sigma = 15\%$

$r_f = 5\%$

T = 5 years

$\delta = 0.20$

Contract = 10% of current operations

Savings = 25 million

Expansion increase = 30% of current operations

Investment for Expansion = 20 million

Abandonment Salvage Value = 100 million

Up factor = 1.1618

Down factor = 0.8607

p = 0.633

Time	0	1	2	3	4	5
						211.70
					182.21	
				156.83		156.83
			134.99		134.99	
		116.18		116.18		116.18
S_0	100		100.00		100.00	
		86.07		86.07		86.07
			74.08		74.08	
				63.76		63.76
					54.88	
						47.24

					255.21
				217.85	Expand
			185.78	Open	183.88
		158.76	Open	156.46	Expand
	136.52	Open	134.35	Open	131.04
119.03	Open	117.25	Open	115.00	Expand
Open	105.51	Open	104.15	Contract	102.46
	Open	100.00	Open	100.00	Contract
		Abandon	100.00	Abandon	100.00
			Abandon	100.00	Abandon
				Abandon	100.00
					Abandon

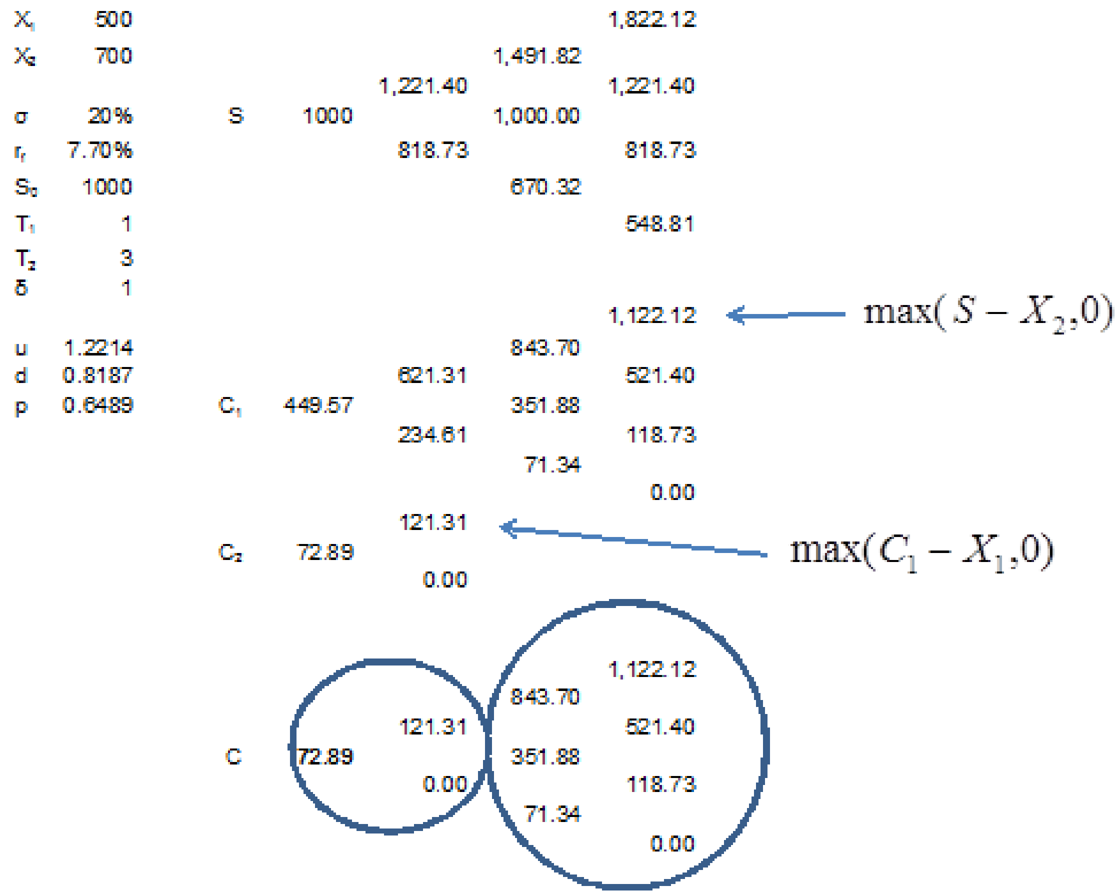
Option Value = 19.03

(b) Compound Options.

Many project initiatives are multistage project investments where management can decide to expand, scale bank, maintain the status quo, or abandon the project after gaining new information to resolve uncertainty. A compound option derives its value from another option - not from the underlying asset. The first investment creates the right but not the obligation to make a second investment, which in turn gives you the option to make a third investment, and so on.

A compound option can either be sequential or parallel, also known as simultaneous. For both sequential and compound options, valuation calculations are essentially the same except for minor differences.

Example - Sequential Option



Example - Parallel Option

Consider the project with two parallel options: the option to upgrade the network and the option to obtain a license. Both have the same lifetime, but the license must be obtained before the upgrade is completed for launch.

Parallel option calculations are the same as with sequential option. First should be calculated the dependent option, network upgrade, which then become the underlying asset values for the option to obtain the license.

7 Project valuation.

In order to introduce the project valuation technique we are considering the case with gasoline station which require import of gasoline.

Time	0	1	2	3	4	5	6	7	8	9	10	11	12
Cash flow		8,153	8,153	8,153	8,153	8,153	8,153	8,153	8,153	8,153	8,153	8,153	8,153

Traditional valuation technique - NPV (net present value)- with (I) investment \$80,000 [1], monthly cash flows (C_n) \$8,153 during the 12 month (N=12), and discount rate recalculated for monthly data 0.8 %, shows us

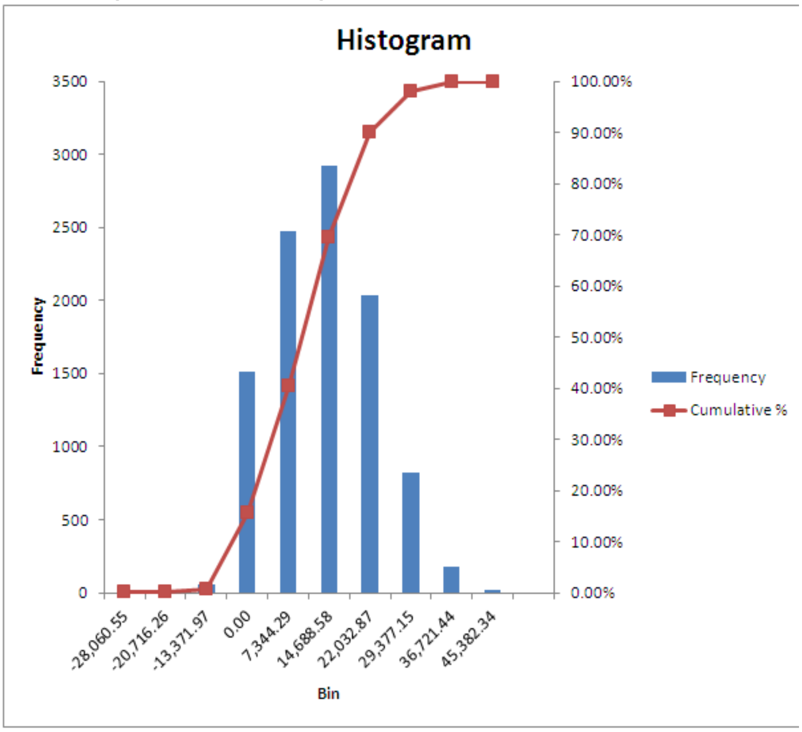
$$NPV = -I + \sum_{n=1}^{N} \frac{C_n}{(1+d)^n} = \$12,943 \tag{7.4}$$

which is slightly positive NPV. But cash flows in the project are random variable with some volatility. Adjusting the project for randomness of cash flows can be done via Monte Carlo simulation. Monte Carlo simulation in the thesis is done based on uniform distribution - uniformly simulating monthly quantity of gasoline sales between 9,510 and 19020 gallons. Data is collecting via market research - measurement done on different gasoline stations in Tbilisi. Taken 5 gasoline stations and measured sales during 10min in different period of day. Sold quantity accumulated for month and presented as data in current case. Constructing the histogram on simulated data (10000 of iterations) we can find that up to 15.71% of the iterations the project gives negative NPV.

[1] Investment amount as well as all the rest data are approximated after consultations with financial directors of some gasoline distribution firms in Georgia,)

Figure 1.1: Histogram

Bin	Frequency	Cumulative %
-28,060.55	1	0.01%
-20,716.26	3	0.04%
-13,371.97	59	0.63%
0.00	1508	15.71%
7,344.29	2471	40.42%
14,688.58	2918	69.60%
22,032.87	2032	89.92%
29,377.15	818	98.10%
36,721.44	175	99.85%
45,382.34	15	100.00%



Therefore we need to adjust the project to accommodate associated risk with uncertain sales of gasoline. It could be done by introducing real option analysis of project. In our case we are presenting the project as simple expansion real option - American call option (or if we will be more precise - Bermuda call option - combination of European and American options) with (N = 12):

$$PV = \sum_{n=1}^{N} \frac{C_n}{(1+d)^n} = \$92,943 \tag{7.5}$$

And investment as exercise I = $80,000. Calculations of this option can be done via Black-Scholes formula

$$Call = N(d_1)\, S_0 - N(d_2)\, X \exp(-rT) \tag{7.6}$$

with

$$d_1 = \frac{ln\left(S_0/X\right) + \left(r + 0.5\sigma^2\right)T}{\sigma\sqrt{T}} \tag{7.7}$$

and

$$d_2 = d_1 - \sigma\sqrt{T} \tag{7.8}$$

or using recombining binomial lattice which gives us good visualization of the project valuation and reaches the value provided by Black-Scholes formula as number of nodes of the binomial lattice increase and go to infinity. Binomial option pricing require us to introduce risk neutral probability concept, up factor, and down factor. Risk neutral probability depends on assumption of absence of arbitrage and is probability of future outcomes adjusted for risk. In mathematical finance, a risk-neutral measure, also called an equivalent martingale measure, is heavily used in the pricing of financial derivatives due to the fundamental theorem of asset pricing, which implies that in a complete market a derivative's price is the discounted expected value of the future payoff under the unique risk-neutral measure. Up (u) and Down (d) factors of asset value as well as risk neutral probability (p) based on Cox, Ross and Rubinstein (1979) model for our model presented below:

$$u = \exp\left(\sigma\sqrt{\delta t}\right) = 1.0107$$

$$d = \frac{1}{u} = 0.9894$$

$$p = \frac{\exp\left(r\delta t\right) - d}{u - d} = 0.73$$

Expected present value we can calculate from Monte Carlo simulation of present value of projected cash flows. Also we will require lognormal standard deviation.

Because we have in the project just 12 cash flows which are not enough to calculate lognormal standard deviation, we can use another method:

$$X = \ln\left(\frac{PV_1}{PV}\right)$$

where

$$PV = \sum_{t=0}^{T} \frac{C_t}{(1+d)^t} \text{ - calculated for t = 0 period}$$

and

$$PV_1 = \sum_{t=1}^{T} \frac{C_t}{(1+d)^t} \text{ - calculated for t = 1 period}$$

Applying Monte Carlo simulation to X we can calculate lognormal standard deviation (σ). Using call option formula $(S_n - I)^+$ and backward induction we can calculate option price, which in our case is \$7,291. Therefore our new project value is NPV + Option Value = \$12,943 + \$7,291 = \$20,234. Please see Figure 1.2.

Figure 1.2: Real Option I

Time	0	1	2	3	4	5	6	7	8	9	10	11	12
Option I	7,291.35	7,791.54	8,297.16	8,807.23	9,321.15	9,838.61	10,359.48	10,883.75	11,411.44	11,942.58	12,477.18	13,015.27	13,556.87
		5,969.23	6,455.70	6,950.18	7,450.76	7,956.04	8,465.17	8,977.74	9,493.67	10,012.97	10,535.66	11,061.76	11,591.30
			4,681.70	5,147.54	5,626.52	6,115.31	6,611.09	7,111.78	7,616.19	8,123.90	8,634.92	9,149.29	9,667.03
				3,447.23	3,878.75	4,332.29	4,803.19	5,286.62	5,778.15	6,274.51	6,774.12	7,277.01	7,783.18
					2,302.37	2,675.89	3,083.80	3,522.04	3,984.59	4,463.98	4,952.42	5,444.05	5,938.91
						1,310.40	1,592.80	1,920.53	2,294.61	2,712.97	3,168.98	3,649.61	4,133.39
							559.78	721.78	926.49	1,183.12	1,501.78	1,892.87	2,365.80
								128.89	177.33	243.97	335.66	461.80	635.35
									0.00	0.00	0.00	0.00	0.00
										0.00	0.00	0.00	0.00
											0.00	0.00	0.00
												0.00	0.00
													0.00

8 Hedging and Parallel Compound Real Option.

However, purchasing prices of the gasoline heavily depends on the gasoline spot prices on energy market which are highly volatile and can be barely considered as complete. In order to reduce the risk associated with volatile gasoline spot prices we will require hedging strategy. We should insure that our hedging strategy fully justify the risk of gasoline purchase price volatility (Please see Figures 1.3 and 1.4). As a solution of this problem in the thesis are presented robust mean-variance hedging strategy for discrete case (II and III parts) and continues (IV part) as well. Robust mean-variance hedging strategy is considered for incomplete markets where both parameters (appreciation rate and volatility) are misspecified.

Figure 1.3: Gasoline Historical Spot Prices

Month	Price	Change
May-08	3.1	-
Jun-08	3.29	6.26%
Jul-08	3.15	-4.37%
Aug-08	2.9	-7.97%
Sep-08	2.81	-3.18%
Oct-08	1.92	-31.55%
Nov-08	1.28	-33.18%
Dec-08	0.96	-25.49%
Jan-09	1.15	19.87%
Feb-09	1.22	6.11%
Mar-09	1.29	5.92%
Apr-09	1.38	7.07%
May-09	1.69	22.48%
Jun-09	1.91	12.85%
Jul-09	1.75	-8.18%
Aug-09	1.93	10.29%
Sep-09	1.78	-7.82%
Oct-09	1.92	8.15%
Nov-09	1.98	3.07%
Dec-09	1.92	-3.23%
Jan-10	2.04	6.31%
Feb-10	1.96	-3.77%
Mar-10	2.14	9.02%
Apr-10	2.23	4.07%
May-10	2.02	-9.34%
Jun-10	2.01	-0.25%
Jul-10	2	-0.94%
Aug-10	1.94	-2.81%
Sep-10	1.97	1.34%
Oct-10	2.16	9.90%
Nov-10	2.24	3.70%
Dec-10	2.39	6.46%

Month	Price	Change
Jan-11	2.45	2.47%
Feb-11	2.56	4.49%
Mar-11	2.84	10.95%
Apr-11	3.18	11.98%
May-11	3.02	-4.85%
Jun-11	2.84	-6.25%
Jul-11	3.02	6.56%
Aug-11	2.84	-6.16%
Sep-11	2.77	-2.36%
Oct-11	2.77	0.14%
Nov-11	2.63	-5.16%
Dec-11	2.63	0.19%
Jan-12	2.82	7.18%
Feb-12	3.04	7.83%
Mar-12	3.17	4.04%
Apr-12	3.21	1.23%
May-12	2.88	-10.26%
Jun-12	2.6	-9.56%
Jul-12	2.75	5.61%
Aug-12	3.02	10.01%
Sep-12	3.27	8.17%
Oct-12	2.98	-9.02%
Nov-12	2.82	-5.31%
Dec-12	2.73	-3.19%
Jan-13	2.85	4.58%
Feb-13	3.05	7.05%
Mar-13	2.91	-4.55%
Apr-13	2.71	-7.14%
May-13	2.74	1.33%

The results of the mathematical model of robust mean-variance hedging can be used in such type of the projects where one or more input parameters are highly dependent on price volatility, in our case, on energy market. To incorporate risk reduction from hedging into the project can be considered compound parallel real option, where one of the options will remain the same as in described above model - expansion (call) option with investment as the exercise price, and second - call option with contingent claim of hedging as the exercise price. In this case our model is becoming fine-tuned and adjusted for risk related to the input and output parameters - volatile purchase prices and demand of gasoline. As the underlying asset of the first real option in the compound parallel option is used expected present value of the future cash flows. For the second real option underlying asset will be expansion real option. So, parallel compound option is calculated as "option on option".

Figure 1.4: Futures Prices

RBOB Gasoline Futures

View Product List

Quotes | Contract Specifications | Performance Bonds / Margins | Product Calendar | Learn More

Quotes Time & Sales Volume Settlements

About this Report

Futures | Options

Trade Date

Thursday, June 20 2013

Daily Settlements for RBOB Gasoline Physcial Future Futures (FINAL)

Trade Date: 06/20/2013

Month	Open	High	Low	Last	Change	Settle	Estimated Volume	Prior Day Open Interest
JLY 13	2.8794	2.8794	2.7755	-	-.1052	2.7872	47,133	51,032
AUG 13	2.8699	2.8709	2.7657	-	-.1059	2.7775	62,828	69,706
SEP 13	2.8522	2.8522	2.7487	-	-.1050	2.7598	21,422	53,089
OCT 13	2.7236	2.7236	2.6189	-	-.1024	2.6299	12,795	33,594
NOV 13	2.6764	2.6764	2.5887	-	-.1011	2.5962	6,961	35,109
DEC 13	2.6685	2.6685	2.5621A	-	-.1000	2.5734	5,561	25,133
JAN 14	2.6216	2.6311	2.5540	-	-.0996	2.5654	2,377	10,548
FEB 14	2.5976	2.5976	2.5625	-	-.0989	2.5701	605	2,801
MAR 14	2.6596	2.6596	2.5775	-	-.0984	2.5838	311	4,696
APR 14	2.7403	2.7403	2.7403	-	-.0975	2.7447	131	2,183
MAY 14	2.7515	2.7515	2.7515	-	-.0970	2.7362	315	2,371
JUN 14	2.7900	2.7900	2.7019	-	-.0963	2.7169	241	1,394
JLY 14	2.7050	2.7050	2.6881A	-	-.0948	2.6909	2	440
AUG 14	-	-	2.6559A	-	-.0933	2.6569	0	181
SEP 14	-	-	-	-	-.0913	2.6159	0	347
OCT 14	-	-	-	-	-.0893	2.4759	0	106
NOV 14	-	-	-	-	-.0873	2.4444	0	35
DEC 14	2.4410	2.4410	2.4290	-	-.0848	2.4234	14	908
JAN 15	-	-	-	-	-.0848	2.4280	0	1
FEB 15	-	-	-	-	-.0848	2.4394	0	0
MAR 15	-	-	-	-	-.0848	2.4534	0	0
APR 15	-	-	-	-	-.0848	2.5834	0	0
MAY 15	-	-	-	-	-.0848	2.5859	0	0
JUN 15	-	-	-	-	-.0848	2.5709	0	0
JLY 15	-	-	-	-	-.0848	2.5529	0	0
AUG 15	-	-	-	-	-.0848	2.5339	0	0
SEP 15	-	-	-	-	-.0848	2.5109	0	0
OCT 15	-	-	-	-	-.0848	2.3909	0	0
NOV 15	-	-	-	-	-.0848	2.3609	0	0
DEC 15	-	-	-	-	-.0848	2.3409	0	0
JAN 16	-	-	-	-	-.0848	2.3409	0	0
FEB 16	-	-	-	-	-.0848	2.3429	0	0
MAR 16	-	-	-	-	-.0848	2.3479	0	0
APR 16	-	-	-	-	-.0848	2.4479	0	0
MAY 16	-	-	-	-	-.0848	2.4479	0	0
JUN 16	-	-	-	-	-.0848	2.4379	0	0
Total							160,696	293,674

Last Updated 06/20/2013 10:32 PM

Market data explanation/disclaimer

Get customized historical data with DataMine

9 Parallel Compound Real Option Calculations.

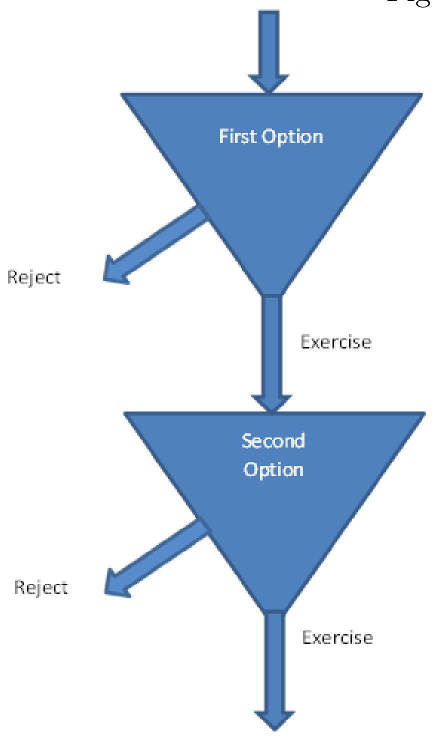
Using the results from II chapter (section 2, example 2) of the thesis we can find that cost of exercising the option is \$6,906. Knowing this we can calculate second option which gives us \$1,383.

Figure 1.5: Real Option II

Time	0	1	2	3	4	5	6	7	8	9	10	11	12
Option II	1,383.10	1,646.06	1,946.41	2,285.68	2,664.23	3,080.97	3,533.01	4,015.60	4,522.22	5,045.21	5,577.01	6,112.31	6,651.11
		684.39	848.09	1,045.18	1,280.25	1,557.63	1,880.89	2,252.22	2,671.47	3,135.21	3,635.49	4,158.80	4,685.53
			249.15	324.14	420.30	542.96	698.46	894.15	1,138.27	1,439.61	1,806.68	2,246.33	2,761.26
				49.68	68.35	94.04	129.38	178.00	244.89	336.93	463.54	637.75	877.41
					0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
						0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
							0.00	0.00	0.00	0.00	0.00	0.00	0.00
								0.00	0.00	0.00	0.00	0.00	0.00
									0.00	0.00	0.00	0.00	0.00
										0.00	0.00	0.00	0.00
											0.00	0.00	0.00
												0.00	0.00
													0.00

Therefore the risk adjusted value of the whole project will be \$12,943 + \$1,383 = \$14,326 which is less than calculated above value of the project (\$20,234), but in this case project not only are risk averse against volatile demand, but also are hedged against volatile purchasing prices of gasoline, and still are above the initial net present value of the project ($12,943). Considering the project as sequence of the decision we can draw the decision tree chart and denote the points where decision should be made and conditions which have influence on our decision (see Figure1.6).

Figure 1.6: Decision Tree



In first node of the tree can be three possibilities:

1. Investment is more than expended underlying asset (S_n), what will produce zero values of first option. In this case project cannot be implemented because - net present value significantly negative and future cash flows have small volatility and won't increase more than investment.

2. Investment is lower than expended underlying asset (S_n), what means that we cannot exercise the option. Possibly project net present value is significantly

positive and future cash flows have small volatility and won't decrease below investment.

3. Investment is between lower and upper values of expended underlying asset (S_n). It means that we are exercising the real option and project net present value is close to zero, and future cash flows volatility is large enough to exercise option.

In the second node of the tree also can be three possible outcomes which correspond to hedging strategy:

1. If hedging contingent claim is between of upper lower values of first option (t=12) - option will be exercised.

2. If more than upper value of the first option - hedging strategy producing the contingent claim which cannot be covered by expected present value of project cash flows and the project is very risky.

3. If less than lower value of the first option - purchasing price volatility and as a consequence the risk associated with this volatility is not significant and we do not need hedging strategy in the project.

Overall we can conclude that this model provide managers with flexible approach to decision making process on project investments and at the same time significantly reducing the risk caused by volatility of spot prices on the energy market and uncertain demand of gasoline. In addition in the thesis are provided mathematical tools of robust mean-variance hedging strategy with misspecified both parameters (drift and volatility) for incomplete markets (discrete and continuous cases) with explicit results which are used in the considered case of gasoline station in the PhD thesis. The mathematical model and its results can be used in all project valuation analysis which requires robust mean-variance approach for risk reduction in incomplete markets.

Chapter 2

Single Period Mean-Variance Hedging Model

1 Introduction

The study of mean variance hedging problem was initiated by H. Föllmer and D. Sondermann [20] and the solution of this problem for multiperiod model was given by H. Föllmer and M. Schweizer [19]. In this chapter we investigate the single period mean variance hedging problem of contingent claims in incomplete markets, when parameters of asset prices are not known with certainty. Usually such parameters may be appreciate rate (or drift) and volatility coefficients. In such models it is desirable to choose an optimal portfolio for the worst case of parameters. Such type problem one calls the robust hedging problem.

The majority of publications are concerned to the case when one of these parameters is known exactly. In the case of unknown drift coefficient the existence of saddle point of corresponding minimax problem has been established and characterization of the optimal strategy has been obtained (see [25],[29],[28]). For the case of unknown volatility coefficients the construction of hedging strategy were given in the works [17], [21], [18], [38].

The most difficult case is to characterize the optimal strategy of minimax (or maximin) problem under uncertainty of both drift and volatility terms. Talay and Zheng [43] applied the PDE-based approach to the maximin problem in the continuous time model and characterized the value as a viscosity solution of corresponding Bellman-Isaacs equation. However for robust hedging it is more convenient to consider the minimax problem. Such type of problem was studied for the single period model of financial market by Pinar [23], who consider the computational scheme to find the optimal strategy and optimal initial capital.

The purpose of the present chapter is to investigate the robust mean-variance hedging problem in the one-step model, when drift and volatility of the asset are not known exactly. We consider the minimax problem and construct the optimal strategy for some type of contingent claims. The main approach we develop is the randomization of the parameters and change the minimax problem by maximin one. This approach successfully works in the one period model and preliminary results show that it will be productive in multi-period and continuous time models. The other way based on the result of [46] leading to the less explicit solution is considered in the Appendix B..

The chapter is organized as follows. In section 2 we describe the market model and give a setting of the problem. Using randomization of parameters we argue the existence of saddle point. Further we construct explicit solution of obtained maximin problem. In examples 1-4 are considered the particular cases where the optimal strategy is expressed in the especial simple form.

2 The main results

We consider a financial market model with two assets. Let $S_t,\ t = 0, 1$ be the price of tradable asset and $\eta_t,\ t = 0, 1$ denotes the price of non-tradable asset. We suppose that

$$S_1 = S_0 + \mu + \sigma w,\ \eta_1 = \beta + \delta\bar{w}, \tag{2.1}$$

where $w, \bar{w}$ is random pair with $Ew = E\bar{w} = 0,\ Dw = D\bar{w} = 1,\ Cov(w, \bar{w}) \neq 0$ and $\mu, \sigma, \beta, \delta$ are constants. We suppose also that the appreciate rate μ and volatility σ of the asset price S_t are misspecified but stay in rectangle of uncertainty, i.e.

$$(\mu, \sigma) \in D = [\mu_-, \mu_+] \times [\sigma_-, \sigma_+].$$

Let β, δ be known exactly. We denote by π the number of stocks S bought at time $t = 0$ and by $x_0 = \pi S_0$ the initial capital. The wealth at time $t = 1$ is

$$X_1 = x_0 + \pi(S_1 - S_0) = x_0 + \pi\mu + \pi\sigma w.$$

The contingent claim $H(\eta)$ we assume depends on the asset η, which cannot be traded directly. The robust mean-variance hedging problem is

$$\min_{\pi} \max_{\mu,\sigma} E|H - x_0 - \pi\mu - \pi\sigma w|^2. \tag{2.2}$$

Let

$$H - x_0 = h_0 + h_1 w + H^{\perp} \tag{2.3}$$

be the decomposition of $H - x_0$ with $h_0 = E(H - x_0),\ h_1 = EwH,\ EwH^{\perp} = 0$. Then the problem can be rewritten as

$$\min_{\pi \in R} \max_{(\mu,\sigma)\in D} F(\pi, \mu, \sigma), \tag{2.4}$$

where

$$F(\pi, \mu, \sigma) = (h_0 - \pi\mu)^2 + (h_1 - \pi\sigma)^2. \tag{2.5}$$

The function $F(\pi, \cdot)$ can be continued on the space of probability measures on D as

$$F(\pi, \nu) = \int_D ((h_0 - \pi\mu)^2 + (h_1 - \pi\sigma)^2)\nu(d\mu d\sigma), \quad \text{for measure } \nu \text{ on } D \tag{2.6}$$

Hence we get

$$F(\pi, \nu) = \int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma) \left(\pi - \frac{\int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma)}{\int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma)}\right)^2 \tag{2.7}$$

$$+h_0^2 + h_1^2 - \frac{(\int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma))^2}{\int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma)} \tag{2.8}$$

and

$$\min_{\pi \in R} F(\pi, \nu) = h_0^2 + h_1^2 - \frac{(\int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma))^2}{\int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma)} \tag{2.9}$$

$$\pi^* = \frac{\int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma)}{\int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma)} \tag{2.10}$$

Since F is strictly convex in π by the Theorem Neumann at al. (see Theorem lX.4.1 of [45]) there exists a saddle point (π^*, ν^*), i.e.

$$F(\pi^*, \nu) \leq F(\pi^*, \nu^*) \leq F(\pi, \nu^*). \tag{2.11}$$

Since $\max_\nu F(\pi, \nu) = \max_{\mu,\sigma} F(\pi, \mu, \sigma)$ then we obtain

$$\min_\pi \max_{(\mu,\sigma)} F(\pi, \mu, \sigma) = \min_\pi \max_\nu F(\pi, \nu) = \max_\nu \min_\pi F(\pi, \nu). \tag{2.12}$$

Each pair of random variables (μ, σ) with the distribution ν may be realized on the probability space $([0.1], \mathcal{B}, P(d\omega) = d\omega)$ where $\mathcal{B}$ is the Borel $\sigma-$algebra on $[0, 1]$ and $d\omega$ the Lebesgue measure (see Proposition 26.6 of [40]). Hence the minimization problem

$$\min_\nu \frac{(\int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma))^2}{\int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma)} \tag{2.13}$$

can be written as

$$\min_{(\mu(\omega),\sigma(\omega))\in D} \frac{(\int_0^1 (h_0\mu(\omega) + h_1\sigma(\omega))d\omega)^2}{\int_0^1 (\mu^2(\omega) + \sigma^2(\omega))d\omega}. \tag{2.14}$$

To solve this problem we consider the deterministic control problem

$$\max_{(\mu(\omega),\sigma(\omega))\in D} \int_0^1 (\mu^2(\omega) + \sigma^2(\omega))d\omega, \tag{2.15}$$

$$\frac{dx(\omega)}{d\omega} = \mu(\omega) \quad \frac{dy(\omega)}{d\omega} = \sigma(\omega), \tag{2.16}$$

$$x(0) = 0, y(0) = 0, \quad x(1) = x, y(1) = y. \tag{2.17}$$

Lemma 2.1. *The solution of the problem (2.15) is of the form*

$$\mu^*(\omega) = \mu_-\chi_A(\omega) + \mu_+\chi_{A^c}(\omega), \quad \sigma^*(\omega) = \sigma_-\chi_B(\omega) + \sigma_+\chi_{B^c}(\omega), \tag{2.18}$$

with

$$P(A) = \frac{x - \mu_-}{\mu_+ - \mu_-}, \quad P(B) = \frac{y - \sigma_-}{\sigma_+ - \sigma_-} \tag{2.19}$$

and the maximal value is $2x\mu_M + 2y\sigma_M - \mu_-\mu_+ - \sigma_-\sigma_+$*, where* $\mu_M = \frac{\mu_+ + \mu_-}{2}$, $\sigma_M = \frac{\sigma_+ + \sigma_-}{2}$.

Proof. By the maximum principle (see [45]) we have

$$\mu^* = \arg \max_{\mu_- \leq \mu \leq \mu_+} (\mu^2 + p\mu), \; \sigma^* = \arg \max_{\sigma_- \leq \sigma \leq \sigma_+} (\sigma^2 + q\sigma),$$

where p, q are some constants maintaining the conditions (2.17). Hence the solution of the problem (2.15) is of the form (2.18). The relations

$$\int_0^1 \mu^*(\omega) d\omega = x, \; \int_0^1 \sigma^*(\omega) d\omega = y$$

uniquely determines the probabilities $P(A), P(B)$ by (2.19) and

$$\int_0^1 (\mu^{*2}(\omega) + \sigma^{*2}(\omega)) d\omega = 2x\mu_M + 2y\sigma_M - \mu_-\mu_+ - \sigma_-\sigma_+.$$

Corollary 2.1.

$$\min_{(\mu(\omega),\sigma(\omega)) \in D} \frac{(\int_0^1 (h_0\mu(\omega) + h_1\sigma(\omega)) d\omega)^2}{\int_0^1 (\mu^2(\omega) + \sigma^2(\omega)) d\omega} = \min_{(x,y) \in D} \frac{(h_0 x + h_1 y)^2}{2\mu_M x + 2\sigma_M y - \mu_-\mu_+ - \sigma_-\sigma_+}. \tag{2.20}$$

To characterize the minimum of the function

$$\psi(x, y) = \frac{(h_0 x + h_1 y)^2}{2\mu_M x + 2\sigma_M y - \mu_-\mu_+ - \sigma_-\sigma_+} \tag{2.21}$$

we use the following lemma.

Lemma 2.2. *The solution of the system*

$$\frac{\partial \psi}{\partial x}(x,y) = 0, \quad \frac{\partial \psi}{\partial y}(x,y) = 0 \tag{2.22}$$

satisfies the equation $h_0 x + h_1 y = 0$.

Proof. It is easy to see that

$$\frac{\partial \psi}{\partial x}(x,y) = 2(h_0 x + h_1 y)$$
$$\times \frac{h_0(2\mu_M x + 2\sigma_M y - \mu_-\mu_+ - \sigma_-\sigma_+) - h_0\mu_M x - h_1\mu_M y}{(2\mu_M x + 2\sigma_M y - \mu_-\mu_+ - \sigma_-\sigma_+)^2},$$
$$\frac{\partial \psi}{\partial y}(x,y) = 2(h_0 x + h_1 y)$$
$$\times \frac{h_1(2\mu_M x + 2\sigma_M y - \mu_-\mu_+ - \sigma_-\sigma_+) - h_0\sigma_M x - h_1\sigma_M y}{(2\mu_M x + 2\sigma_M y - \mu_-\mu_+ - \sigma_-\sigma_+)^2}.$$

Solving the system we obtain that either $h_0 x + h_1 y = 0$ or

$$h_0\mu_M x + (2h_0\sigma_M - h_1\mu_M)y = h_0(\mu_+\mu_+ + \sigma_-\sigma_+),$$

$$h_1\sigma_M y + (2h_1\mu_M - h_0\sigma_M)x = h_1(\mu_+\mu_+ + \sigma_-\sigma_+).$$

The latter system admits the unique solution

$$x = \frac{h_1}{2}\frac{\mu_-\mu_+ + \sigma_-\sigma_+}{h_1\mu_M - h_0\sigma_M},$$
$$y = -\frac{h_0}{2}\frac{\mu_-\mu_+ + \sigma_-\sigma_+}{h_1\mu_M - h_0\sigma_M},$$

which also satisfies the equation $h_0 x + h_1 y = 0$.

Corollary 2.2. *The minimum of* $\psi(x,y) \in D$ *is attained either on the line* $h_0 x + h_1 y = 0$ *or on the boundary of* D. *If there exists the pair* $(\bar{x},\bar{y}) \in D$ *such that* $h_0\bar{x} + h_1\bar{y} = 0$, *then*

$$\max_{\nu}\min_{\pi} F(\pi,\nu) = \min_{(x,y)\in D} \psi(x,y) = \psi(\bar{x},\bar{y}) = 0$$

and $\pi^* = 0$.

Proof. It is sufficient to take $(\mu^*, \sigma^*) = (\bar{x}, \bar{y})$ and to use (1).

The boundary ∂D of rectangle D consists from the sides $B_{--}, B_{-+}, B_{+-}, B_{++}$, where

$$B_{--} = \{(x, y) : x = \mu_-,\ \sigma_- \le y \le \sigma_+\},$$

$$B_{-+} = \{(x, y) : x = \mu_+,\ \sigma_- \le y \le \sigma_+\},$$

$$B_{+-} = \{(x, y) : y = \sigma_-,\ \mu_- \le x \le \mu_+\},$$

$$B_{++} = \{(x, y) : y = \sigma_+,\ \mu_- \le x \le \mu_+\}.$$

Obviously that functions defined on the sides

$$\varphi_{-a}(t) = \psi(\mu_a, \sigma_- + t(\sigma_+ - \sigma_-)),\ on\ \ B_{-a},\ \ a = -, +,$$

$$\varphi_{+b}(t) = \psi(\mu_- + t(\mu_+ - \mu_-), \sigma_b),\ on\ \ B_{+b},\ \ b = -, +,$$

coincide with functions of the Appendix A. It is easy to show that the $t_{ab} = \arg\min \varphi_{ab}(t),\ a = -, +,\ b = +, -$ can be computed as (see Appendix A)

$$t_{ab} = \begin{cases} 1, & \text{if } 1 \le \alpha_{ab} \text{ or } 1 \le 2\beta_{ab} - \alpha_{ab}, \\ 0, & \text{if } 2\beta_{ab} \le \alpha_{ab} \le 0 \\ \alpha_{ab}, & \text{if } 0 < \alpha_{ab} < 1 \\ 2\beta_{ab} - \alpha_{ab}, & \text{if } 0 < 2\beta_{ab} - \alpha_{ab} < 1. \end{cases} \tag{2.23}$$

Hence we have

Lemma 2.3. *The minimum of $\psi(x, y)$ is attained on the boundary ∂D, i.e.*

$$\min_{(x,y)\in D} \psi(x, y) = \min_{a=\pm, b=\pm} \varphi_{ab}(t_{ab}).$$

Moreover for $(x^*, y^*) = \arg\min_{(x,y)\in D} \psi(x,y)$ *we have* $(x^*, y^*) \in B_{a^*b^*}$*, where* $a^*b^* = \arg\min_{ab} \varphi_{ab}(t_{ab})$ *and* $t^* = t_{a^*b^*}$ *is the distance from the end of the side to* (x^*, y^*) *defined by (2.23) .*

The following Proposition allows us to exclude two sides of the rectangle. [1)]

Proposition 2.1.

$$\min_{(x,y)\in D} \psi(x,y) = \varphi_{-+}(t_{-+}) \wedge \varphi_{++}(t_{++}),$$

where

$$t_{-+} = \begin{cases} 1, & \text{if } 1 \le \alpha_{-+} \text{ or } 1 \le 2\beta_{-+} - \alpha_{-+}, \\ 0, & \text{if } 2\beta_{-+} < \alpha_{-+} \le 0, \\ \alpha_{-+}, & \text{if } 0 < \alpha_{-+} < 1 \\ 2\beta_{-+} - \alpha_{-+}, & \text{if } 0 < 2\beta_{-+} - \alpha_{-+} < 1, \end{cases} \tag{2.24}$$

$$t_{++} = \begin{cases} 1 & \text{if } 1 \le \alpha_{++} \text{ or } 1 \le 2\beta_{++} - \alpha_{++}, \\ 0, & \text{if } 2\beta_{++} < \alpha_{++} \le 0, \\ \alpha_{++} & \text{if } 0 < \alpha_{++} < 1 \\ 2\beta_{++} - \alpha_{++} & \text{if } 0 < 2\beta_{++} - \alpha_{++} < 1, \end{cases} \tag{2.25}$$

$$\alpha_{++} = -\frac{h_0\mu_- + h_1\sigma_+}{h_0(\mu_+ - \mu_-)}, \; \beta_{++} = -\frac{\mu_-^2 + \sigma_+^2}{\mu_+^2 - \mu_-^2}, \tag{2.26}$$

$$\alpha_{-+} = -\frac{h_0\mu_+ + h_1\sigma_-}{h_1(\sigma_+ - \sigma_-)}, \; \beta_{-+} = -\frac{\mu_+^2 + \sigma_-^2}{\sigma_+^2 - \sigma_-^2}. \tag{2.27}$$

Moreover for $(x^*, y^*) = \arg\min_{(x,y)\in D} \psi(x,y)$ *we have* $(x^*, y^*) \in B_{a^*+}$*, where*

$$a^* = \arg\min_{a\in\{-,+\}} \varphi_{a+}(t_{a+}).$$

Proof. The function $\psi(x,y)$ can be written as

$$\psi(x,y) = \frac{1}{g(x,y)} \left(\frac{h_0 x + h_1 y}{\sqrt{x^2 + y^2}} \right)^2,$$

[1)] $a \wedge b$ denotes the $\min(a,b)$

where $g(x,y)=\frac{2\mu_M x+2\sigma_M y-\mu_-\mu_+-\sigma_-\sigma_+}{\sqrt{x^2+y^2}}$. Since $g(x,px)=\frac{2\mu_M+2\sigma_M p}{\sqrt{1+p^2}}-\frac{\mu_-\mu_++\sigma_-\sigma_+}{x\sqrt{1+p^2}}$ the function $\frac{1}{g}$ is decreasing along the line $y=px$. Thus the function

$$\psi(x,px)=\frac{1}{g(x,px)}\left(\frac{h_0+h_1p}{\sqrt{1+p^2}}\right)^2,$$

is decreasing also and attains the minimum on the end of the piece $\{y=px\}\cap D$.

It remains to note that

$$\min_{(x,y)\in D}\psi(x,y)=\min_{\frac{\mu_-}{\sigma_+}\le p\le\frac{\mu_+}{\sigma_-}}\ \min_{(x,y)\in D,y=px}\psi(x,y).$$

Solving the inequality in (2.25),(2.24) with respect to $k=\frac{h_0}{h_1}$ we get

Corollary 2.3. *The equations (2.25),(2.24) are equivalent to*

$$t_{-+}=\begin{cases}1 & \text{if } k<-\frac{\sigma_+}{\mu_+},\\ -\frac{k\mu_+}{\sigma_+-\sigma_-}-\frac{\sigma_-}{\sigma_+-\sigma_-} & \text{if } -\frac{\sigma_+}{\mu_+}<k\le-\frac{\sigma_-}{\mu_+},\\ 0 & \text{if } -\frac{\sigma_-}{\mu_+}<k\le\frac{2\mu_+^2+\sigma_-^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)},\\ \frac{k\mu_+}{\sigma_+-\sigma_-}-\frac{2\mu_+^2+\sigma_-^2-\sigma_-\sigma_+}{\sigma_+^2-\sigma_-^2} & \text{if } \frac{2\mu_+^2+\sigma_-^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)}<k\le\frac{2\mu_+^2+\sigma_+^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)},\\ 1 & \text{if } \frac{2\mu_+^2+\sigma_+^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)}\le k,\end{cases}\tag{2.28}$$

$$t_{++}=\begin{cases}1 & \text{if } k^{-1}\le-\frac{\mu_+}{\sigma_+},\\ -\frac{k^{-1}\sigma_+}{\mu_+-\mu_-}-\frac{\mu_-}{\mu_+-\mu_-} & \text{if } -\frac{\mu_+}{\sigma_+}\le k^{-1}<-\frac{\mu_-}{\sigma_+},\\ 0 & \text{if } -\frac{\mu_-}{\sigma_+}<k^{-1}\le\frac{2\sigma_+^2+\mu_-^2-\mu_-\mu_+}{\sigma_+(\mu_++\mu_-)},\\ \frac{k^{-1}\sigma_+}{\mu_+-\mu_-}-\frac{2\sigma_+^2+\mu_-^2-\mu_-\mu_+}{\mu_+^2-\mu_-^2} & \text{if } \frac{2\sigma_+^2+\mu_-^2-\mu_-\mu_+}{\sigma_+(\mu_++\mu_-)}\le k^{-1}<\frac{2\sigma_+^2+\mu_+^2-\mu_-\mu_+}{\sigma_+(\mu_++\mu_-)},\\ 1 & \text{if } \frac{2\sigma_+^2+\mu_+^2-\mu_-\mu_+}{\sigma_+(\mu_++\mu_-)}\le k^{-1}.\end{cases}\tag{2.29}$$

Moreover t_{-+},t_{++} are continuous functions of the parameter of $k\in R$.

Proof. The solution of the equality $0<\alpha_{-+}<1$ from (2.24) for $\alpha_{-+}=-\frac{k\mu_+}{\sigma_+-\sigma_-}-\frac{\sigma_-}{\sigma_+-\sigma_-}$, with respect to k gives $-\frac{\sigma_+}{\mu_+}<k<-\frac{\sigma_-}{\mu_+}$. Similarly we can prove the rest inequalities. Since the values of quantities $t_{-+},\ t_{++}$ at the ends of intervals are coincides we get the continuous functions of the parameter of $k \in R$.

There exist 4 possibilities

$$
\begin{aligned}
&a) \quad 2\mu_+^2 + \sigma_-^2 - \sigma_-\sigma_+ > 0,\ 2\sigma_+^2 + \mu_-^2 - \mu_-\mu_+ > 0,\\
&b) \quad 2\mu_+^2 + \sigma_-^2 - \sigma_-\sigma_+ < 0,\ 2\sigma_+^2 + \mu_-^2 - \mu_-\mu_+ > 0,\\
&c) \quad 2\mu_+^2 + \sigma_-^2 - \sigma_-\sigma_+ > 0,\ 2\sigma_+^2 + \mu_-^2 - \mu_-\mu_+ < 0,\\
&d) \quad 2\mu_+^2 + \sigma_-^2 - \sigma_-\sigma_+ < 0,\ 2\sigma_+^2 + \mu_-^2 - \mu_-\mu_+ < 0.
\end{aligned}
$$

We restrict ourself to one of them.

Proposition 2.2. *Assume* $2\mu_+^2 + \sigma_-^2 - \sigma_-\sigma_+ > 0,\ 2\sigma_+^2 + \mu_-^2 - \mu_-\mu_+ > 0.$ *Then*

$$
t_{++} = \begin{cases}
0 & \text{if } k < -\frac{\sigma_+}{\mu_-},\\
\alpha_{++} & \text{if } -\frac{\sigma_+}{\mu_-} < k \le -\frac{\sigma_+}{\mu_+},\\
1 & \text{if } -\frac{\sigma_+}{\mu_+} < k \le \frac{\sigma_+(\mu_+ + \mu_-)}{2\sigma_+^2 + \mu_+^2 - \mu_-\mu_+},\\
2\beta_{++} - \alpha_{++} & \text{if } \frac{\sigma_+(\mu_+ + \mu_-)}{2\sigma_+^2 + \mu_+^2 - \mu_-\mu_+} < k \le \frac{\sigma_+(\mu_+ + \mu_-)}{2\sigma_+^2 + \mu_-^2 - \mu_-\mu_+}\\
0 & \text{if } \frac{\sigma_+(\mu_+ + \mu_-)}{2\sigma_+^2 + \mu_-^2 - \mu_-\mu_+} \le k
\end{cases}
\tag{2.30}
$$

If in addition $\frac{\sigma_+(\mu_+ + \mu_-)}{2\sigma_+^2 + \mu_-^2 - \mu_-\mu_+} \le \frac{2\mu_+^2 + \sigma_-^2 - \sigma_-\sigma_+}{\mu_+(\sigma_+ + \sigma_-)}$ *is satisfied then*

$$
(t_{-+}, t_{++}) = \begin{cases}
(1,0) & \text{if } k < -\frac{\sigma_+}{\mu_-},\\
(1,\alpha_{++}) & \text{if } -\frac{\sigma_+}{\mu_-} < k \le -\frac{\sigma_+}{\mu_+},\\
(\alpha_{-+},1) & \text{if } -\frac{\sigma_+}{\mu_-} < k \le -\frac{\sigma_-}{\mu_+},\\
(0,1) & \text{if } -\frac{\sigma_-}{\mu_+} < k \le \frac{\sigma_+(\mu_+ + \mu_-)}{2\sigma_+^2 + \mu_+^2 - \mu_-\mu_+},\\
(0, 2\beta_{++} - \alpha_{++}) & \text{if } \frac{\sigma_+(\mu_+ + \mu_-)}{2\sigma_+^2 + \mu_+^2 - \mu_-\mu_+} < k \le \frac{\sigma_+(\mu_+ + \mu_-)}{2\sigma_+^2 + \mu_-^2 - \mu_-\mu_+},\\
(0,0) & \text{if } \frac{\sigma_+(\mu_+ + \mu_-)}{2\sigma_+^2 + \mu_-^2 - \mu_-\mu_+} < k \le \frac{2\mu_+^2 + \sigma_-^2 - \sigma_-\sigma_+}{\mu_+(\sigma_+ + \sigma_-)},\\
(2\beta_{++} - \alpha_{++}, 0) & \text{if } \frac{2\mu_+^2 + \sigma_-^2 - \sigma_-\sigma_+}{\mu_+(\sigma_+ + \sigma_-)} < k \le \frac{2\mu_+^2 + \sigma_+^2 - \sigma_-\sigma_+}{\mu_+(\sigma_+ + \sigma_-)},\\
(1,0) & \text{if } \frac{2\mu_+^2 + \sigma_+^2 - \sigma_-\sigma_+}{\mu_+(\sigma_+ + \sigma_-)} < k,
\end{cases}
\tag{2.31}
$$

$$\varphi_{-+}(t_{-+})\wedge\varphi_{++}(t_{++}) = \begin{cases} \varphi_{-+}(1)\wedge\varphi_{++}(0) = \psi(\mu_-,\sigma_+) & \mathit{if}\ \ k < -\frac{\sigma_+}{\mu_-}, \\ \varphi_{-+}(1)\wedge\varphi_{++}(\alpha_{++}) = 0 & \mathit{if}\ \ -\frac{\sigma_+}{\mu_-} < k \le -\frac{\sigma_+}{\mu_+}, \\ \varphi_{-+}(\alpha_{-+})\wedge\varphi_{++}(1) = 0 & \mathit{if}\ \ -\frac{\sigma_+}{\mu_-} < k \le -\frac{\sigma_-}{\mu_+}, \\ \varphi_{-+}(0)\wedge\varphi_{++}(1) = \psi(\mu_+,\sigma_-) & \mathit{if}\ \ -\frac{\sigma_-}{\mu_+} < k \le \frac{\sigma_+(\mu_++\mu_-)}{2\sigma_+^2+\mu_+^2-\mu_-\mu_+}, \\ \varphi_{-+}(0)\wedge\varphi_{++}(2\beta_{++}-\alpha_{++}) & \mathit{if}\ \ \frac{\sigma_+(\mu_++\mu_-)}{2\sigma_+^2+\mu_+^2-\mu_-\mu_+} < k \le \frac{\sigma_+(\mu_++\mu_-)}{2\sigma_+^2+\mu_-^2-\mu_-\mu_+}, \\ \varphi_{-+}(0)\wedge\varphi_{++}(0) & \\ = \psi(\mu_+,\sigma_-)\wedge\psi(\mu_-,\sigma_+) & \mathit{if}\ \ \frac{\sigma_+(\mu_++\mu_-)}{2\sigma_+^2+\mu_-^2-\mu_-\mu_+} < k \le \frac{2\mu_+^2+\sigma_-^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)}, \\ \varphi_{-+}(2\beta_{-+}-\alpha_{-+})\wedge\varphi_{++}(0) & \mathit{if}\ \ \frac{2\mu_+^2+\sigma_-^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)} < k \le \frac{2\mu_+^2+\sigma_+^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)}, \\ \varphi_{-+}(1)\wedge\varphi_{++}(0) = \psi(\mu_-,\sigma_+) & \mathit{if}\ \ \frac{2\mu_+^2+\sigma_+^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)} < k. \end{cases} \tag{2.32}$$

Proof. (2.30) follows from (2.29). The assumptions of Proposition give the following arrangement of points

$$-\frac{\sigma_+}{\mu_-} \le -\frac{\sigma_+}{\mu_+} < -\frac{\sigma_-}{\mu_+} \le 0 < \frac{\sigma_+(\mu_++\mu_-)}{2\sigma_+^2+\mu_+^2-\mu_-\mu_+}$$
$$\le \frac{\sigma_+(\mu_++\mu_-)}{2\sigma_+^2+\mu_-^2-\mu_-\mu_+} \le \frac{2\mu_+^2+\sigma_-^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)} \le \frac{2\mu_+^2+\sigma_+^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)}.$$

Hence intersections of intervals including in (2.28),(2.30) give intervals for k in (2.31).

To prove $\varphi_{-+}(1)\wedge\varphi_{++}(0) = \psi(\mu_-,\sigma_+)$ we note that by (2.30) $k < -\frac{\sigma_+}{\mu_-}$ implies $t_{++} = 1$ and $\psi(\mu_+,\sigma_+) = \varphi_{++}(1) > \varphi_{++}(0) = \psi(\mu_-,\sigma_+)$. Thus

$$\varphi_{-+}(1)\wedge\varphi_{++}(0) = \psi(\mu_+,\sigma_+)\wedge\psi(\mu_-,\sigma_+) = \psi(\mu_-,\sigma_+).$$

Similarly can be proved the all equalities in (2.32).

Remark . Analogously we can consider the other arrangements of

$$\frac{\sigma_+(\mu_++\mu_-)}{2\sigma_+^2+\mu_+^2-\mu_-\mu_+}, \frac{\sigma_+(\mu_++\mu_-)}{2\sigma_+^2+\mu_-^2-\mu_-\mu_+}, \frac{2\mu_+^2+\sigma_-^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)}, \frac{2\mu_+^2+\sigma_+^2-\sigma_-\sigma_+}{\mu_+(\sigma_++\sigma_-)}.$$

Proposition 2.3. *Let* (x^*, y^*) *be the minimizer of* $\psi(x, y)$. *Then the solution of the optimization problem (1.7) is of the form*

$$(\mu^*, \sigma^*) = \begin{cases} (\mu_+, \sigma_-\chi_B + \sigma_+\chi_{B^c}), \ \textit{if } (x^*, y^*) \in B_{-+}, \\ (\mu_-\chi_A + \mu_+\chi_{A^c}, \sigma_+), \ \textit{if } (x^*, y^*) \in B_{++}. \end{cases} \tag{2.33}$$

Proof. By Proposition (2.1) (x^*, y^*) belongs on a side. Hence the Bernoulli pair (μ^*, σ^*) such that

$$P(\mu^* = \mu_-) = \frac{\mu_+ - x^*}{\mu_+ - \mu_-}, \ P(\sigma^* = \sigma_-) = \frac{\sigma_+ - y^*}{\sigma_+ - \sigma_-}$$

is the optimal pair.

Finally we obtain

Theorem 2.1. *The solution of the robust mean-variance hedging problem (1.5),* π^* *is given by the formula*

$$\pi^* = \frac{h_0 x^* + h_1 y^*}{2\mu_M x^* + 2\sigma_M y^* - \mu_-\mu_+ - \sigma_-\sigma_+}, \tag{2.34}$$

where $(x^*, y^*) = \arg\min \psi(x, y)$ *is computed by Proposition 2.3. Moreover the saddle point* (π^*, μ^*, σ^*) *of the minimax problem (2.12) is given by*

$$P(\mu^* = \mu_-) = \frac{\mu_+ - x^*}{\mu_+ - \mu_-}, \ P(\sigma^* = \sigma_-) = \frac{\sigma_+ - y^*}{\sigma_+ - \sigma_-},$$

$$P(\mu^* = \mu_+) = \frac{x^* - \mu_-}{\mu_+ - \mu_-}, \ P(\sigma^* = \sigma_+) = \frac{y^* - \sigma_-}{\sigma_+ - \sigma_-}$$

and (2.38).

Example 1. Suppose $\mu_- = \sigma_- = 1, \mu_+ = \sigma_+ = 2$. Then

$$\psi(x, y) = \frac{(h_0 x + h_1 y)^2}{3x + 3y - 4}, \ (x, y) \in [1, 2]^2. \tag{2.35}$$

Let $k = \frac{h_0}{h_1}$ and $\varphi(x,y) = \frac{(kx+y)^2}{3x+3y-4}$. From (2.32) we get

$$\frac{1}{h_1^2}\min_{(x,y)\in D}\psi(x,y) = \begin{cases} \varphi(1,2) = \frac{1}{5}(k+2)^2 & \text{if } k \in (-\infty,-2], \\ 0, & \text{if } k \in (-2,-\frac{1}{2}], \\ \varphi(2,1) = \frac{1}{5}(2k+1)^2 & \text{if } k \in (-\frac{1}{2},\ \frac{3}{5}], \\ \frac{1}{5}(2k+1)^2 \wedge \frac{8}{9}k(3-k) & \text{if } k \in (\frac{3}{5},\ \frac{6}{7}], \\ \frac{1}{5}(k+2)^2 \wedge \frac{1}{5}(2k+1)^2 & \text{if } k \in (\frac{6}{7},\ \frac{7}{6}], \\ \frac{8}{9}(3k-1) \wedge \frac{1}{5}(k+2)^2 & \text{if } k \in (\frac{7}{6},\ \frac{5}{3}], \\ \varphi(1,2) = \frac{1}{5}(k+2)^2 & \text{if } k \in (\frac{5}{3},\ \infty). \end{cases}$$

$$= \begin{cases} \varphi(1,2) = \frac{1}{5}(k+2)^2 & \text{if } k \in (-\infty,-2], \\ 0, & \text{if } k \in (-2,-\frac{1}{2}], \\ \varphi(2,1) = \frac{1}{5}(2k+1)^2 & \text{if } k \in (-\frac{1}{2},\ 1], \\ \varphi(1,2) = \frac{1}{5}(k+2)^2 & \text{if } k \in (1,\ \infty). \end{cases} \tag{2.36}$$

Hence

$$(x^*,y^*) = \begin{cases} (1,2) & \text{if } \frac{h_0}{h_1} \in (-\infty,-2], \\ (x,-kx),\ x \in [-\frac{1}{k},-\frac{2}{k}] \cap [1,2] & \text{if } \frac{h_0}{h_1} \in (-2,-\frac{1}{2}], \\ (2,1) & \text{if } \frac{h_0}{h_1} \in (-\frac{1}{2},\ 1], \\ (1,2) & \text{if } \frac{h_0}{h_1} \in (1,\ \infty) \end{cases} \tag{2.37}$$

and

$$\begin{aligned} \pi^* &= \frac{h_0 x^* + h_1 y^*}{3x^* + 3y^* - 4} \\ &= \frac{h_0 + 2h_1}{5}\chi\left(\frac{h_0}{h_1} \le -2\right) \\ &+ \frac{2h_0 + h_1}{5}\chi\left(-\frac{1}{2} < \frac{h_0}{h_1} \le 1\right) \\ &+ \frac{h_0 + 2h_1}{5}\chi\left(1 < \frac{h_0}{h_1}\right). \end{aligned}$$

Example 2. Let H be a constant. i.e. $h_1 = 0$. It is evident

$$\min_{(x,y)\in D} \psi(x,y) = \min_{\mu_- \le x \le \mu_+} \frac{h_0^2 x^2}{2\mu_M x - \mu_-\mu_+ + \sigma_+^2} = \min_{0\le t\le 1} \frac{h_0^2(\mu_- + t(\mu_+ - \mu_-))^2}{\mu_-^2 + t(\mu_+^2 - \mu_-^2) + \sigma_+^2}.$$

Hence $(x^*, y^*) \in B_{++}$ and we must find $t_{++} = \arg\min \varphi_{++}(t)$. From (3.53) we have

$$\alpha_{++} = -\frac{\mu_-}{\mu_+ - \mu_-} < 0,\ \beta_{++} = -\frac{\mu_-^2 + \sigma_+^2}{\mu_+^2 - \mu_-^2},\ \gamma_+ = \frac{h_0^2(\Delta\mu)^2}{\mu_+^2 - \mu_-^2} = h_0^2 \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-} > 0. \tag{2.38}$$

Moreover

$$2\beta_{++} - \alpha_{++} = -\frac{\mu_-^2 - \mu_-\mu_+ + 2\sigma_+^2}{\mu_+^2 - \mu_-^2} < \frac{\mu_-}{\mu_+ + \mu_-} < 1. \tag{2.39}$$

Thus

$$t_{++} = \begin{cases} 0, & \text{if } 2\beta_{++} - \alpha_{++} \le 0 \\ \frac{\mu_-\mu_+ - \mu_-^2 - 2\sigma_+^2}{\mu_+^2 - \mu_-^2}, & \text{if } 0 < 2\beta_{++} - \alpha_{++}. \end{cases} \tag{2.40}$$

Simplifying we obtain

$$t_{++} = \begin{cases} 0, & \text{if } \mu_+\mu_- - 2\sigma_+^2 \le \mu_-^2 \\ \frac{\mu_-\mu_+ - \mu_-^2 - 2\sigma_+^2}{\mu_+^2 - \mu_-^2}, & \text{if } \mu_-^2 < \mu_+\mu_- - 2\sigma_+^2 \end{cases} \tag{2.41}$$

and

$$\min_\nu \frac{(\int_D h_0\mu\nu(d\mu d\sigma))^2}{\int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma)} = \varphi_{++}(t_{++}) = \begin{cases} \frac{h_0^2\mu_-^2}{\mu_-^2 + \sigma_+^2}, & \text{if } \mu_+\mu_- - 2\sigma_+^2 \le \mu_-^2 \\ \frac{\mu_+\mu_- - \sigma_+^2}{\mu_M^2} h_0^2, & \text{if } \mu_-^2 < \mu_+\mu_- - 2\sigma_+^2. \end{cases} \tag{2.42}$$

By (1) the optimal strategy is

$$\pi^* = h_0 \frac{\mu_- + t_{++}(\mu_+ - \mu_-)}{\mu_-^2 + t_{++}(\mu_+^2 - \mu_-^2) + \sigma_+^2} = \begin{cases} \frac{h_0\mu_-}{\mu_-^2 + \sigma_+^2}, & \text{if } \mu_+\mu_- - 2\sigma_+^2 \le \mu_-^2 \\ \frac{h_0}{\mu_M}, & \text{if } \mu_-^2 < \mu_+\mu_- - 2\sigma_+^2. \end{cases} \tag{2.43}$$

Example 3. Analogously we can consider the case $h_0 = 0$. Then $(x^*, y^*) = (\mu_+, \sigma_- + t_{-+}(\sigma_+ - \sigma_-)) \in B_{-+}$,

$$t_{-+} = \begin{cases} 0, & \text{if } \sigma_+\sigma_- - 2\mu_+^2 \le \sigma_-^2 \\ \frac{\sigma_-\sigma_+ - \sigma_-^2 - 2\mu_+^2}{\mu_+^2 - \mu_-^2}, & \text{if } \sigma_-^2 < \sigma_+\sigma_- - 2\mu_+^2, \end{cases} \tag{2.44}$$

$$\pi^* = \begin{cases} \frac{h_1\sigma_-}{\sigma_-^2+\mu_+^2}, & \text{if } \sigma_+\sigma_- - 2\mu_+^2 \le \sigma_-^2 \\ \frac{h_1}{\sigma_M}, & \text{if } \sigma_-^2 < \sigma_+\sigma_- - 2\mu_+^2. \end{cases} \tag{2.45}$$

Example 4. let $\mu_- = \mu_+ = 0$. Then $\psi(x,y) = \frac{h_1^2 y^2}{2\sigma_M y - \sigma_-\sigma_+}$ and

$$t^* = \arg\min_{0\le t\le 1} \frac{h_1^2(\sigma_- + t(\sigma_+ - \sigma_-))^2}{\sigma_-^2 + t(\sigma_+^2 - \sigma_-^2)} = \frac{\sigma_-}{\sigma_+ + \sigma_-}.$$

Therefore $\pi^* = \frac{h_1}{\sigma_M}$.

Remark 2.1. The quantity

$$\max_\nu \min_\pi F(\pi, \nu). \tag{2.46}$$

is a function of initial capital x_0. Minimizing this expression by x_0 we find x_0^* and further construct the optimal (π^*, μ^*, σ^*) assuming $h_0 = EH - x_0^*$. Therefore we find the solution of the problem

$$\min_{x_0,\pi} \max_{(\mu,\sigma)} F(\pi, \mu, \sigma). \tag{2.47}$$

3 Appendix A

Our aim is to find the measure $\nu_t = t\delta_{(\mu_a,\sigma_b)} + (1-t)\delta_{(\mu_c,\sigma_d)},\ t \in [0,1]$ minimizing the expression

$$\min_\nu \frac{(\int_D (h_0\mu + h_1\sigma)\nu(d\mu d\sigma))^2}{\int_D (\mu^2+\sigma^2)\nu(d\mu d\sigma)} \tag{3.48}$$

for $(a,b),(c,d) \in \{-,+\} \times \{-,+\}$. We consider only the case $0 < \mu_- < \mu_+,\ 0 < \sigma_- < \sigma_+$.

Let

$$\varphi(t) = \frac{(\int_D (h_0\mu + h_1\sigma)\nu_t(d\mu d\sigma))^2}{\int_D (\mu^2+\sigma^2)\nu_t(d\mu d\sigma)} \tag{3.49}$$

$$\equiv \frac{(h_0\mu_a + h_1\sigma_b + t(h_0\Delta\mu + h_1\Delta\sigma))^2}{(\mu_a^2 + \sigma_b^2 + t(\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_c^2))}. \tag{3.50}$$

When $\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2 = 0$ and $h_0\Delta\mu + h_1\Delta\sigma = 0$, then $\varphi(t) = const.$

If $\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2 = 0$ and $h_0\Delta\mu + h_1\Delta\sigma \neq 0$ then

$$\varphi(t) = \frac{1}{\mu_a^2 + \sigma_b^2}(h_0\mu_a + h_1\sigma_b + t(h_0\Delta\mu + h_1\Delta\sigma))^2. \tag{3.51}$$

If $\mu_b^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2 \neq 0$ and $h_0\Delta\mu - h_1\Delta\sigma \neq 0$ then

$$\varphi(t) = \gamma\frac{(t-\alpha)^2}{t-\beta} \tag{3.52}$$

where $\Delta\mu = \mu_c - \mu_a,\ \Delta\sigma = \sigma_d - \sigma_b$

$$\alpha = -\frac{h_0\mu_a + h_1\sigma_b}{h_0\Delta\mu + h_1\Delta\sigma},\ \beta = -\frac{\mu_a^2 + \sigma_b^2}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2},\ \gamma = \frac{(h_0\Delta\mu + h_1\Delta\sigma)^2}{\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2} \tag{3.53}$$

Proposition 3.1. *Let $t^* = \arg\min_{t\in[0,1]}\varphi(t)$ and $\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2 \neq 0$ be satisfied. Then for the case $\gamma < 0$*

$$t^* = \begin{cases} 1 & \text{if } 1 \le \alpha \le \beta \ \text{ or } 1 \le 2\beta - \alpha < \beta, \\ 0 & \text{if } 2\beta < \alpha \le 0 \\ \alpha & \text{if } 0 < \alpha < 1 \\ 2\beta - \alpha & \text{if } 0 < 2\beta - \alpha < 1 \end{cases} \tag{3.54}$$

and for the case $\gamma > 0$

$$t^* = \begin{cases} 1 & \text{if } 1 \le \alpha \ \text{ or } 1 \le 2\beta - \alpha, \\ 0 & \text{if } 2\beta < \alpha \le 0 \\ \alpha & \text{if } 0 < \alpha < 1 \\ 2\beta - \alpha & \text{if } 0 < 2\beta - \alpha < 1 \end{cases} \tag{3.55}$$

Proof. Obviously that

$$\varphi(t) = \gamma\left(t - \beta - 2(\alpha - \beta) + \frac{(\alpha - \beta)^2}{t - \beta}\right) \tag{3.56}$$

and

$$\varphi'(t) = \gamma\frac{(t - 2\beta + \alpha)(t - \alpha)}{(t - \beta)^2}. \tag{3.57}$$

The case $\alpha = \beta$ is trivial.

I) Set $\gamma > 0$ and $\alpha \neq \beta$. Then $\beta < -1$ and

$$\lim_{t\uparrow\beta}\varphi(t) = -\infty,\ \lim_{t\to-\infty}\varphi(t) = -\infty, \tag{3.58}$$

$$\lim_{t\downarrow\beta}\varphi(t) = \infty,\ \lim_{t\to\infty}\varphi(t) = \infty \tag{3.59}$$

From (3.53) follows that $\beta < -1$. Hence $[0, 1] \subset (\beta, \infty)$. Thus if $\alpha > \beta$ as follows from the Fermat Theorem and (3.57), α is the minimizer of $\varphi(t)$ on (β, ∞), and if $\alpha < \beta$ then $2\beta - \alpha > \beta$ is satisfied and $t = 2\beta - \alpha$ is the the minimizer of $\varphi(t)$ on (β, ∞). Since we seek the minimum of $\varphi(t)$ on $[0, 1]$ we have

$$t^* = \begin{cases} 1 & \text{if } 1 \le \alpha \\ 0 & \text{if } \beta < \alpha \le 0 \\ \alpha & \text{if } 0 < \alpha < 1 \end{cases}$$

for the case $\alpha > \beta$ and

$$t^* = \begin{cases} 1 & \text{if } 1 \le 2\beta - \alpha, \\ 0 & \text{if } \beta < 2\beta - \alpha \le 0 \\ 2\beta - \alpha & \text{if } 0 < 2\beta - \alpha < 1 \end{cases}$$

for the case $\alpha < \beta$. Combining of two cases we get

$$t^* = \begin{cases} 1 & \text{if } 1 \le \alpha \ \text{ or } \ 1 \le 2\beta - \alpha, \\ 0 & \text{if } \beta < \alpha \le 0 \ \text{ or } \ \beta < 2\beta - \alpha \le 0 \\ \alpha & \text{if } 0 < \alpha < 1 \\ 2\beta - \alpha & \text{if } 0 < 2\beta - \alpha < 1. \end{cases}$$

It remains to note that $\beta < \alpha \le 0$ or $\beta < 2\beta - \alpha \le 0$ is equivalent to $2\beta < \alpha \le 0$.

II) Set $\gamma < 0$ and $\alpha \ne \beta$. Then $\beta > 1$ and

$$\lim_{t \uparrow \beta} \varphi(t) = \infty, \lim_{t \to -\infty} \varphi(t) = \infty, \tag{3.60}$$

$$\lim_{t \downarrow \beta} \varphi(t) = -\infty, \lim_{t \to \infty} \varphi(t) = -\infty \tag{3.61}$$

Hence $\varphi(t)$ has a minimum on $(-\infty, \beta)$ and has a maximum on (β, ∞). Thus if $\alpha < \beta$ then as follows from (3.57) the minimum is attained at $t = \alpha$, and if $\alpha > \beta$ then $2\beta - \alpha < \beta$ and the minimum is attained at $t = 2\beta - \alpha$. The minimizer of $\varphi(t)$ on $[0, 1]$ is defined as in the case I).

Denote by $\varphi_{--}(t), \varphi_{-+}(t), \varphi_{+-}(t), \varphi_{++}(t)$ the function $\varphi(t)$ for the cases $(a, b, c, d) = (-, -, -, +), (+, -, +, +), (-, -, +, -), (-, +, +, +)$ respectively. We may say that they are functions defined on sides of the rectangle D. Then (3.53) takes the form

$$\alpha_{-\pm} = -\frac{h_0 \mu_\pm + h_1 \sigma_-}{h_1 \Delta\sigma}, \; \beta_{-\pm} = -\frac{\mu_\pm^2 + \sigma_-^2}{\sigma_+^2 - \sigma_-^2}, \; \gamma_- = \frac{(h_1 \Delta\sigma)^2}{\sigma_+^2 - \sigma_-^2} \tag{3.62}$$

$$\alpha_{+\pm} = -\frac{h_0 \mu_- + h_1 \sigma_\pm}{h_0 \Delta\mu}, \; \beta_{+\pm} = -\frac{\mu_-^2 + \sigma_\pm^2}{\mu_+^2 - \mu_-^2}, \; \gamma_+ = \frac{(h_0 \Delta\mu)^2}{\mu_+^2 - \mu_-^2} \tag{3.63}$$

Obviously that $\mu_c^2 - \mu_a^2 + \sigma_d^2 - \sigma_b^2 \ne 0$ and $\gamma > 0$ for this cases. Hence from the previous Proposition we obtain

Proposition 3.2. *Let* $t_{ab} = \arg\min_{t \in [0,1]} \varphi_{ab}(t)$, *for* $(a, b) \in \{-, +\}^2$. *Then*

$$t_{ab} = \begin{cases} 1, & \text{if } 1 \le \alpha_{ab} \text{ or } 1 \le 2\beta_{ab} - \alpha_{ab}, \\ 0, & \text{if } 2\beta_{ab} \le \alpha_{ab} \le 0 \\ \alpha_{ab}, & \text{if } 0 < \alpha_{ab} < 1 \\ 2\beta_{ab} - \alpha_{ab}, & \text{if } 0 < 2\beta_{ab} - \alpha_{ab} < 1. \end{cases} \tag{3.64}$$

4 Appendix B

The purpose of the appendix is to show what kind result can be obtained by the approach of [46]. For two distinct pairs $(a,b),(c,d) \in \{+,-\} \times \{+,-\}$ such that $\mu_a^2 + \sigma_b^2 \le \mu_c^2 + \sigma_d^2$ we define the functions $f_{abcd}(\pi) = \max(f_{ab}(\pi), f_{cd}(\pi))$, where $f_{ab}(\pi) = F(\pi, \mu_a, \sigma_b)$. Obviously that

$$f_{abcd}(\pi) \le \max_{(a,b)\in\{+,-\}^2}(f_{ab}(\pi)) \text{ and } \min_{\pi} f_{abcd}(\pi) \le \min_{\pi} \max_{(a,b)\in\{+,-\}^2}(f_{ab}(\pi))$$

Hence by Theorem 3.3 of [46] (Chapter Vl p.197)

$$\min_{\pi} \max_{(\mu,\sigma)\in D} F(\pi,\mu,\sigma) = \min_{\pi} \max_{(a,b)\in\{+,-\}^2}(f_{ab}(\pi)) = \max_{(abcd)} \min_{\pi} f_{abcd}(\pi).$$

Lemma 4.1. *For $\pi_{abcd} = \arg\min_{\pi} f_{abcd}(\pi)$ we have*

$$\pi_{abcd} = \begin{cases} 0, & \text{if } (h_0\mu_a + h_1\sigma_b)(h_0\mu_c + h_1\sigma_d) \le 0, \\ \dfrac{h_0\mu_a + h_1\sigma_b}{\mu_a^2 + \sigma_b^2}, & \text{if } \frac{h_0\mu_a+h_1\sigma_b}{\mu_a^2+\sigma_b^2}, \frac{h_0\mu_c+h_1\sigma_d}{\mu_c^2+\sigma_d^2} > 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2} \\ \dfrac{h_0\mu_c + h_1\sigma_d}{\mu_c^2 + \sigma_d^2}, & \text{if } \frac{h_0\mu_a+h_1\sigma_b}{\mu_a^2+\sigma_b^2}, \frac{h_0\mu_c+h_1\sigma_d}{\mu_c^2+\sigma_d^2} \le 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2} \\ 2\dfrac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2}, & \text{if } \frac{h_0\mu_a+h_1\sigma_b}{\mu_a^2+\sigma_b^2} \le 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2} \le \frac{h_0\mu_c+h_1\sigma_d}{\mu_c^2+\sigma_d^2} \\ & \text{or } \frac{h_0\mu_c+h_1\sigma_d}{\mu_c^2+\sigma_d^2} \le 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2} \le \frac{h_0\mu_a+h_1\sigma_b}{\mu_a^2+\sigma_b^2}. \end{cases} \tag{4.65}$$

Moreover

$$f_{abcd}(\pi_{abcd}) = \begin{cases} h_0^2 + h_1^2, & if\ (h_0\mu_a + h_1\sigma_b)(h_0\mu_c + h_1\sigma_d) \le 0, \\ h_0^2 + h_1^2 - \frac{(h_0\mu_a+h_1\sigma_b)^2}{\mu_a^2+\sigma_b^2}, & if\ \frac{h_0\mu_a+h_1\sigma_b}{\mu_a^2+\sigma_b^2}, \frac{h_0\mu_c+h_1\sigma_d}{\mu_c^2+\sigma_d^2} > 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2} \\ h_0^2 + h_1^2 - \frac{(h_0\mu_c+h_1\sigma_d)^2}{\mu_c^2+\sigma_d^2}, & if\ \frac{h_0\mu_a+h_1\sigma_b}{\mu_a^2+\sigma_b^2}, \frac{h_0\mu_c+h_1\sigma_d}{\mu_c^2+\sigma_d^2} \le 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2} \\ h_0^2 + h_1^2 - \frac{(h_0\mu_a+h_1\sigma_b)^2}{\mu_a^2+\sigma_b^2} + (\mu_a^2+\sigma_b^2)\times & \\ (2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2} - \frac{h_0\mu_a+h_1\sigma_b}{\mu_a^2+\sigma_b^2})^2, & if\ \frac{h_0\mu_a+h_1\sigma_b}{\mu_a^2+\sigma_b^2} \le 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2} \\ & \le \frac{h_0\mu_c+h_1\sigma_d}{\mu_c^2+\sigma_d^2} \\ & \ or\ \frac{h_0\mu_c+h_1\sigma_d}{\mu_c^2+\sigma_d^2} \le 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2} \\ & \le \frac{h_0\mu_a+h_1\sigma_b}{\mu_a^2+\sigma_b^2} \end{cases} \tag{4.66}$$

Proof. The minimal value of $f_{ab}(\pi), f_{cd}(\pi)$ are achieved at $\frac{h_0\mu_a+h_1\sigma_b}{\mu_a^2+\sigma_b^2}$ and $\frac{h_0\mu_c+h_1\sigma_d}{\mu_c^2+\sigma_d^2}$ respectively. If $(h_0\mu_a + h_1\sigma_b)(h_0\mu_c + h_1\sigma_d) \le 0$ then by continuity of $h_0x + h_1y$ there exists $(x, y) \in D$ such that $h_0x + h_1y = 0$ and $\pi^* = 0$. If $(h_0\mu_a + h_1\sigma_b)(h_0\mu_c + h_1\sigma_d) > 0$ then we assume $h_0\mu_a + h_1\sigma_b > 0,\ h_0\mu_c + h_1\sigma_d > 0$. The roots of the equation $f_{ab}(\pi) = f_{cd}(\pi)$ are $\pi = 0$ and $\pi = 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2}$. There exists three possibilities:

$$\begin{aligned} &1) \quad \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2+\sigma_b^2}, \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2+\sigma_d^2} > 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2}, \\ &2) \quad \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2+\sigma_b^2}, \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2+\sigma_d^2} \le 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2}, \\ &3) \quad \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2+\sigma_b^2} \le 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2} \le \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2+\sigma_d^2} \\ &\quad \text{or } \frac{h_0\mu_c + h_1\sigma_d}{\mu_c^2+\sigma_d^2} \le 2\frac{h_0(\mu_c-\mu_a)+h_1(\sigma_d-\sigma_b)}{\mu_c^2-\mu_a^2+\sigma_d^2-\sigma_b^2} \le \frac{h_0\mu_a + h_1\sigma_b}{\mu_a^2+\sigma_b^2}. \end{aligned}$$

In each cases the corresponding minimal value calculated by the equation (4.66).

Corollary 4.1. *The solution of minmax problem (1.5) can be given as* $\pi^* = \pi_{a^*b^*c^*d^*}$, *where*

$$a^*b^*c^*d^* = \arg\max_{(abcd)} f_{abcd}(\pi_{abcd}).$$

Chapter 3

Robust mean-variance pricing of contingent claims

1 Robust mean-variance pricing of contingent claims

We consider a financial market model consisting with one riskless asset and two risky assets. Let $S_t, \eta_t,\ t = 0, 1$ be the prices of the tradable risky assets and non-tradable risky asset respectively and assume that the price of riskless asset is equal to one. We suppose that

$$S_1 = S_0 + \mu + \sigma w,\ \eta_1 = \beta + \bar{\sigma}\bar{w}, \tag{1.1}$$

where $w, \bar{w}$ is random pair with $Ew = E\bar{w} = 0,\ Var(w) = Var(\bar{w}) = 1,\ Cov(w, \bar{w}) \neq 0$ and $\mu, \sigma, \beta, \bar{\sigma}$ are constants. We denote by π the number of stocks S_t bought at time $t = 0$ and by $q_t,\ t = 0, 1$ the amount invested in the riskless asset. The value of the portfolio at time t is given by $x_t = \pi S_t + q_t$. The contingent claim $H(\eta_1)$ we assume depends on the asset price η_1, which cannot be traded directly. Such type asset and contingent claim may be the market index and the option on the index respectively. The problem is to choose values of π and q_t to minimize the cost of the hedging strategy. Since one can always set $x_1 = H$, this gives $q_1 = H - \pi S_1$ which implies that the hedging strategy is non-self-financing. Let us denote the cumulative cost of the strategy by C_t . Then we have

$$C_0 = x_0 = \pi S_0 + q_0.$$

The additional cost involved in passing from time period 0 to time period 1 is given by

$$C_t - C_0 = q_1 - q_0 = H - x_0 - \pi(S_1 - S_0).$$

The problem is now to determine the values x_0 and π so as to minimize the expected quadratic additional cost of the trading strategy

$$\min_{(x_0,\pi)\in R^2} E|H - x_0 - \pi\mu - \pi\sigma w|^2. \tag{1.2}$$

If (x_0^*, π^*) is the optimal pair then π^* and x_0^* one calls the mean-variance hedging strategy and mean-variance price of contingent claim respectively. The formulae for (x_0^*, π^*) is given in [19]

Suppose now that the appreciate rate μ and volatility σ of the asset price S_t are misspecified but stay in rectangle of uncertainty, i.e.

$$(\mu, \sigma) \in D = [\mu_-, \mu_+] \times [\sigma_-, \sigma_+].$$

Let $\beta, \bar{\sigma}$ be known exactly. It is natural to consider in this case the robust quadratic hedging error

$$\max_{(\mu,\sigma)\in D} E|H - x_0 - \pi\mu - \pi\sigma w|^2$$

as the objective function and try to minimize it, i.e. study the robust mean-variance hedging problem

$$\min_{(x_0,\pi)\in R^2} \max_{(\mu,\sigma)\in D} E|H - x_0 - \pi\mu - \pi\sigma w|^2. \tag{1.3}$$

Let

$$H = h_0 + h_1 w + H^{\perp} \tag{1.4}$$

be the decomposition of H with $EH^{\perp} = 0,\ EH = h_0,\ E(wH^{\perp}) = 0,\ E(wH) = h_1$. Then the problem can be rewritten as

$$\min_{(x_0,\pi)\in R^2} \max_{(\mu,\sigma)\in D} F(x_0, \pi, \mu, \sigma), \tag{1.5}$$

where

$$F(x_0, \pi, \mu, \sigma) = (h_0 - x_0 - \pi\mu)^2 + (h_1 - \pi\sigma)^2.$$

The function $F(x_0, \pi, \cdot)$ can be continued on the space of probability measures on D as

$$F(x_0, \pi, \nu) = \int_D ((h_0 - x_0 - \pi\mu)^2 + (h_1 - \pi\sigma)^2)\nu(d\mu d\sigma), \ \text{ for measure } \nu \text{ on } D.$$

Hence we get

$$\begin{aligned} F(x_0, \pi, \nu) = \int_D (\mu^2 + \sigma^2)\nu(d\mu d\sigma) \left(\pi - \frac{\int_D((h_0 - x_0)\mu + h_1\sigma)\nu(d\mu d\sigma)}{\int_D(\mu^2 + \sigma^2)\nu(d\mu d\sigma)}\right)^2 \\ +(h_0 - x_0)^2 + h_1^2 - \frac{(\int_D((h_0 - x_0)\mu + h_1\sigma)\nu(d\mu d\sigma))^2}{\int_D(\mu^2 + \sigma^2)\nu(d\mu d\sigma)} \end{aligned}$$

and

$$\pi^* = h_1 \frac{\int_D \sigma\nu(d\mu d\sigma)}{\int_D(\mu^2 + \sigma^2)\nu(d\mu d\sigma) - (\int_D \mu\nu(d\mu d\sigma))^2},$$
$$x_0^* = h_0 - \pi^* \int_D \mu\nu(d\mu d\sigma).$$

$$\min_{(x_0,\pi)\in R^2} F(x_0,\pi,\nu) = \min_{\pi\in R}\min_{x_0\in R} F(x_0,\pi,\nu)$$
$$= h_1^2 - h_1^2 \frac{(\int_D \sigma\nu(d\mu d\sigma))^2}{\int_D(\mu^2+\sigma^2)\nu(d\mu d\sigma) - (\int_D \mu\nu(d\mu d\sigma))^2},$$

Since F is strictly convex in (x_0,π) by the Theorem Neumann at al. (see Theorem lX.4.1 of [45]) there exists a saddle point (x_0^*,π^*,ν^*), i.e.

$$F(x_0^*,\pi^*,\nu) \le F(x_0^*,\pi^*,\nu^*) \le F(x_0,\pi,\nu^*).$$

On the other hand

$$\max_{\nu} F(x_0,\pi,\nu) = \max_{\mu,\sigma} F(x_0,\pi,\mu,\sigma)$$

Then value of minimax problem (see [46])

$$\min_{x_0,\pi}\max_{\nu} F(x_0,\pi,\nu) = \max_{\nu}\min_{x_0,\pi} F(x_0,\pi,\nu) \tag{1.6}$$

is equal to

$$V = \min_{x_0,\pi}\max_{(\mu,\sigma)\in D} F(x_0,\pi,\mu,\sigma) = F(x_0^*,\pi^*,\nu^*)$$

and the robust strategy of the problem (1.3) is (x_0^*,π^*). If we denote by (μ,σ) the pair of random variables with the distribution ν the minimization problem

$$\min_{\nu} \frac{(\int_D \sigma\nu(d\mu d\sigma))^2}{\int_D(\mu^2+\sigma^2)\nu(d\mu d\sigma) - (\int_D \mu\nu(d\mu d\sigma))^2}$$

can be written as

$$\min_{(\mu(\omega),\sigma(\omega))\in D} \frac{E^2\sigma}{E(\mu^2+\sigma^2)-E^2\mu}. \tag{1.7}$$

In summary we can say that if (μ^*, σ^*) is the optimal pair of the problem (1.7) then robust hedging strategy is $\pi^* = h_1 \frac{E\sigma^*}{E(\mu^{*2}+\sigma^{*2})-E^2\mu^*}$ and robust mean-variance price of the contingent claims will equal to $x_0^* = h_0 - \pi^* E\mu^*$.

2 The main results

Lemma 2.1. *Let Φ be the continuous function on $[a^2, b^2] \times [a, b]$ such that $\Phi(\cdot, y)$ is the increasing for each y. Then the maximization problem*

$$\max_{a \le \eta \le b} \Phi(E\eta^2, E\eta) \tag{2.8}$$

admits a solution of the form $\eta = a\chi_A + b\chi_{A^c}$.

Proof. Obviously that

$$\max_{a \le \eta \le b} \Phi(E\eta^2, E\eta) = \max_{a \le y \le b} \max_{\eta, E\eta = y} \Phi(E\eta^2, y) = \max_{\eta, E\eta = y^*} \Phi(E\eta^2, y^*), \tag{2.9}$$

where y^* is the maximizer of the continuous function $\max_{\eta, E\eta=y} \Phi(E\eta^2, y)$. Since $\Phi(\cdot, y)$ is increasing it is sufficient to show that the solution η^* of the problem

$$\max_{\eta, E\eta = y} E\eta^2, \tag{2.10}$$

is of the form $\eta^* = a\chi_A + b\chi_{A^c}$ for some event A. Find the A such that $E\eta^* = y$. Hence $E\eta^{*2} = (a+b)y - ab$ and η^* is the maximizer of the problem (2.10). Indeed, for any η, with $E\eta = y$ we have

$$\begin{aligned} E\eta^2 &= E\left(\eta - \frac{a+b}{2}\right)^2 + 2\frac{a+b}{2}y - \left(\frac{a+b}{2}\right)^2 \\ &\le \left(\frac{b-a}{2}\right)^2 + (a+b)y - \left(\frac{a+b}{2}\right)^2 \\ &= (a+b)y - ab = E\eta^{*2}. \end{aligned}$$

We will use the notation $\mu_M = \frac{\mu_+ + \mu_-}{2}$, $\sigma_M = \frac{\sigma_+ + \sigma_-}{2}$, $\delta\mu = \frac{\mu_+ - \mu_-}{2}$.

Theorem 2.1. *The saddle point and value of the problem (1.6),* $(x_0^*, \pi^*, \mu^*, \sigma^*)$ *is given by the following formulas*

a)

$$(\mu^*, \sigma^*) = (\mu_-\chi_A + \mu_+\chi_{A^c}, \sigma_-\chi_B + \sigma_+\chi_{B^c})$$

$$\text{with } P(A) = \frac{1}{2}, \quad P(B) = \frac{\sigma_-\sigma_+ + \delta\mu^2}{\sigma_M(\sigma_+ + \sigma_-)},$$

$$\pi^* = \frac{h_1}{\sigma_M}, \quad x_0^* = h_0 - h_1\frac{\mu_M}{\sigma_M},$$

$$V = h_1^2\left(1 - \frac{(\sigma_-\sigma_+ - \delta\mu^2)^2}{\sigma_M^2(\sigma_-\sigma_+ - \delta\mu^2)}\right)$$

in the case $\sigma_- \leq \frac{\sigma_-\sigma_+ - \delta\mu^2}{\sigma_M}$ *,*

b)

$$(\mu^*, \sigma^*) = (\mu_-\chi_A + \mu_+\chi_{A^c}, \sigma_-), \text{ with } P(A) = \frac{1}{2},$$

$$\pi^* = \frac{h_1\sigma_-}{\sigma_-^2 + \delta\mu^2}, \quad x_0^* = h_0 - h_1\frac{\mu_M\sigma_-}{\sigma_-^2 + \delta\mu^2},$$

$$V = h_1^2\frac{\delta\mu^2}{\sigma_-^2 + \delta\mu^2}$$

in the case $\sigma_- > \frac{\sigma_-\sigma_+ - \delta\mu^2}{\sigma_M}$.

Proof. By the Lemma $E\mu^2 - E^2\mu$ attains its maximum on the Bernoulli random variable. Thus $\mu^* = \mu_-\chi_A + \mu_+\chi_{A^c}$, with $P(A) = \frac{1}{2}$ is maximizer and $E\mu^{*2} - E^2\mu^* = \delta\mu^2$. Since the function $\frac{E^2\sigma}{E\sigma^2 + y}$ is decreasing for each σ then

$$
\begin{aligned}
&\min_{(\mu(\omega),\sigma(\omega))\in D} \frac{E^2\sigma}{E(\mu^2+\sigma^2)-E^2\mu} \\
&= \min_{\sigma(\omega)\in[\sigma_-,\sigma_+]} \min_{\mu(\omega)\in[\mu_-,\mu_+]} \frac{E^2\sigma}{E(\mu^2+\sigma^2)-E^2\mu} \\
&= \min_{\sigma(\omega)\in[\sigma_-,\sigma_+]} \frac{E^2\sigma}{E\sigma^2+\delta\mu^2}.
\end{aligned} \tag{2.11}
$$

Using the Lemma for the function $\Phi(x,y) = \frac{x+\delta\mu^2}{y^2}$ we conclude that the minimizer σ^* in (2.11) is of the form $\sigma_-\chi_B + \sigma_+\chi_{B^c}$. If $E\sigma^* = y^*$ then $E\sigma^{*2} = 2\sigma_M y^* - \sigma_-\sigma_+$. The minimizer y^* of the function $\psi(y) = \frac{y^2}{2\sigma_M y - \sigma_-\sigma_+ + \delta\mu^2}$ is

$$
y^* = \begin{cases} \frac{\sigma_-\sigma_+-\delta\mu^2}{\sigma_M}, \text{ if } \sigma_- \le \frac{\sigma_-\sigma_+-\delta\mu^2}{\sigma_M}, \\ \sigma_-, \text{ if } \sigma_- > \frac{\sigma_-\sigma_+-\delta\mu^2}{\sigma_M}. \end{cases} \tag{2.12}
$$

The corresponding value is

$$
\psi(y^*) = \begin{cases} \frac{(\sigma_-\sigma_+-\delta\mu^2)^2}{\sigma_M^2(\sigma_-\sigma_+-\delta\mu^2)}, \text{ if } \sigma_- \le \frac{\sigma_-\sigma_+-\delta\mu^2}{\sigma_M}, \\ \frac{\sigma_-^2}{\sigma_-^2+\delta\mu^2}, \text{ if } \sigma_- > \frac{\sigma_-\sigma_+-\delta\mu^2}{\sigma_M}. \end{cases} \tag{2.13}
$$

It remains insert $E\sigma^*$, $E\sigma^{*2}$ in the formulas

$$
\pi^* = h_1 \frac{E^2\sigma^*}{E\sigma^{*2}+\delta\mu^2},\ x_0^* = h_0 - \pi^*\mu_M,\ V = h_1^2 - h_1^2\frac{E^2\sigma^*}{E\sigma^{*2}+\delta\mu^2}.
$$

Corollary 2.1. *The robust strategy (x_0^*, π^*) and hedging error V of the problem (1.3), are given by the following formulae*

a)

$$
\pi^* = \frac{h_1}{\sigma_M},\quad x_0^* = h_0 - h_1\frac{\mu_M}{\sigma_M},
$$
$$
V = h_1^2\left(1 - \frac{(\sigma_-\sigma_+ - \delta\mu^2)^2}{\sigma_M^2(\sigma_-\sigma_+ - \delta\mu^2)}\right)
$$

in the case $\sigma_- \le \frac{\sigma_-\sigma_+-\delta\mu^2}{\sigma_M}$, *b)*

$$\pi^* = \frac{h_1\sigma_-}{\sigma_-^2 + \delta\mu^2}, \quad x_0^* = h_0 - h_1\frac{\mu_M\sigma_-}{\sigma_-^2 + \delta\mu^2},$$
$$V = h_1^2\frac{\delta\mu^2}{\sigma_-^2 + \delta\mu^2}$$

in the case $\sigma_- > \frac{\sigma_-\sigma_+ - \delta\mu^2}{\sigma_M}$.

Chapter 4

Robust utility maximization for diffusion market model

1 Introduction

The purpose of the current chapter of the thesis is to study the robust maximization of terminal wealth utility in a diffusion financial market model where the trend and volatility of an asset price are uncertain.

The concept of robustness was introduced by P. Huber (see [32]) in the context of statistical estimation of an unknown distribution parameter. The essence of our approach is as follows. Suppose we need to estimate the mean of some symmetric distribution. If the estimation is based on "pure" observations, then the effective estimate is the sample mean. But if observations are contaminated by outliers, then the situation completely changes. Huber introduced the so-called gross error model (the contaminated neighborhood of a true distribution) and showed that an optimal estimate is a maximum likelihood estimate constructed for the so-called least favorable distribution. Analytically, this means that we need to solve a minimax problem analogous to the problem given by formula (2.6) below with the asymptotic mean square error as a risk function. In some limiting cases, an optimal estimate is a median,but not a sample mean. In mathematical finance, for most approaches and settings it is implicitly supposed that the underlying asset model is fully specified: the parameters (trend and volatility) of the model are known.Actually, we have all the same to estimate these parameters and construct, say, confidence intervals for them. Hence we only know that a pair (μ, σ) belongs with high probability to the rectangle $[\mu_-, \mu_+] \times [\sigma_-, \sigma_+]$. In that case there arises a problem of construction of robust trading strategies where an optimal strategy is the best strategy against the worst state of Nature. If the risk function of the problem is the expected terminal wealth utility, then our definition of the optimization problem (2.6) is an exact one.

In 1999, Chen and Epstein introduced a continuous time intertemporal version of a multiple-priors utility function for Brownian filtration. In that case, beliefs are represented by a set $\mathcal{P}$ of probability measures and the utility is defined as a minimum of the expected utilities over the set $\mathcal{P}$. Independently, Cvitanic (2000) and Cvitanic and Karatzas (1999) studied, for a given option, the hedging strategies which minimize the expected "shortfall", i.e. the difference between the payoff and the terminal wealth. They considered the problem of determining the "worst-case" model $\tilde{Q}$, i.e. the model which maximizes a minimal shortfall risk over all possible priors $Q \in \mathcal{P}$. It was shown that under certain assumptions their maximin problem could be written as a minimax problem. In 2004, Quenez studied the problem of utility maximization in an incomplete multiple-priors model, where asset prices are semimartingales. This problem corresponds to a maximin problem where the maximum is taken over the set of feasible wealth X (or portfolios) and where the minimum is taken over the set of priors $\mathcal{P}$. The author showed that, under suitable conditions, there exists a saddle point for this problem. Moreover, Quenez developed the dual approach which consists in solving a dual minimization problem over the set of priors and supermartingale measures and showed how the solution of the dual problem leads to a solution of the primal problem.

The above maximin problems can also be called robust optimization problems since optimization involves an entire class $\mathcal{P}$ of possible probabilistic models and thus takes into account the model risk. Optimal investment problems for such robust utility functionals were considered in particular by Talay and Zheng (2002), Korn and Wilmott (2002), Quenez (2004), Schied (2005),(2008), Korn and Menkens (2005), Gundel (2005),Bordigoni at al. (2007), Schied and Wu (2005), Föllmer and Gundel (2006), Hernández-Hernández and Schied (2006, 2007).

The majority of the relevant published works are concerned with the case where one of the parameters is known exactly. For the unknown drift coefficient, the existence of a saddle point of the corresponding minimax problem was established and the characterization of an optimal strategy obtained in [25], [29], [28]. For the unknown volatility coefficients, the hedging strategy was constructed in [17], [21], [18], [24], [38], [26].

The most difficult case is to characterize the optimal strategy of the maximin problem under the uncertainty of both drift and volatility terms.

Talay and Zheng [43] applied the PDE-based approach to the minimax problem and characterized the value as a viscosity solution of the corresponding Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation. In general, such a problem does not contain a saddle point. Moreover, in robust maximization problems, the maximin should be taken instead of the minimax used by Talay and Zheng.

In this chapter of the thesis, we consider the incomplete diffusion financial market model which resembles the model considered by Schied (2008), Hernández-Hernández and Schied (2006, 2007). We suppose that the market consists of a risk-free asset, a risky tradable asset with misspecified trend and volatility and a non-tradable asset with known parameters. As different from the approach of Quenez (2004) and Schied (2008), we solve the maximin problem using the HJBI equation which corresponds to the primal problem. When the trend and volatility coefficients are uncertain, such a maximin problem has no saddle point in general. We extend the set of model coefficients, i.e. carry out some "randomization" and obtain as a result a minimax problem with a saddle point. This makes it possible to replace the maximin problem by a minimax problem which is easier to study using the HJBI equation properties. In particular, we have found such a form of this equation that coincides with the equation derived by Hernández-Hernández and Schied (2006) when the volatility is assumed to be known. We establish the solvability of the obtained equation in the classical sense and solve the HJBI equation explicitly for the specific drift coefficient. The saddle point (an optimal portfolio and optimal coefficients) of the considered maximin problem has been found as well.

To illustrate our approach, we present a simple quadratic hedging problem. Let $(B, B^{\perp})$ be the 2-dimensional Brownian motion and $F^B = (\mathcal{F}_t^B)_{t\in[0,T]}$, $F^{B,B^{\perp}} = (\mathcal{F}_t^{B,B^{\perp}})_{t\in[0,T]}$ denote the augmented filtrations generated by B and $(B, B^{\perp})$, respectively. We consider the filtration $F = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions and $F^B \subset F \subset F^{B,B^{\perp}}$. Let H be square integrable $\mathcal{F}_T^B$-measurable random variables. Denote by Π^2 the set of square-integrable predictable processes with respect to the filtration F. Let $\mathcal{P}([\sigma_-, \sigma_+])$ be the set of probability measures on $[\sigma_-, \sigma_+]$ and $\mathcal{U}$, $\tilde{\mathcal{U}}$ denote the set of predictable processes with respect to the filtration F with values in $[\sigma_-, \sigma_+]$ and $\mathcal{P}$,respectively. We use the notation $f \cdot \nu$ for $\int_{\sigma_-}^{\sigma_+} f(\sigma)d\nu(\sigma)$, $f \in C[\sigma_-, \sigma_+]$, $\nu \in \mathcal{P}([\sigma_-, \sigma_+])$. The wealth process corresponding to a portfolio process $\pi \in \Pi$ and volatility $\sigma \in \mathcal{U}$ is defined as

$$X_t(\pi, \sigma) = c + \int_0^t \pi_s \sigma_s dB_s. \tag{1.1}$$

The problem is to find $\pi^* \in \Pi^2$ minimizing the worst case mean-variance hedging error

$$\max_{\sigma\in\mathcal{U}} E|H - X_T(\pi^*, \sigma)|^2 = \min_{\pi\in\Pi} \max_{\sigma\in\mathcal{U}} E|H - X_T(\pi, \sigma)|^2, \tag{1.2}$$

Such π^* is called a robust hedging strategy.

Let us extend problem (1.2) as follows. For each $\nu \in \tilde{\mathcal{U}}$ we define the processes

$$W_t^{\nu} = \int_0^t \frac{p \cdot \nu_s}{\sqrt{p^2 \cdot \nu_s}} dB_s + \int_0^t \sqrt{1 - \frac{(p \cdot \nu_s)^2}{p^2 \cdot \nu_s}} dB_s^{\perp},$$
$$W_t^{\nu,\perp} = \int_0^t \sqrt{1 - \frac{(p \cdot \nu_s)^2}{p^2 \cdot \nu_s}} dB_s - \int_0^t \frac{p \cdot \nu_s}{\sqrt{p^2 \cdot \nu_s}} dB_s^{\perp},$$

where $p,\ p^2$ are the functions $p(\sigma) = \sigma,\ p^2(\sigma) = \sigma^2$ respectively. One can easily check that $(W^{\nu}, W^{\nu,\perp})$ is also 2-dimensional Brownian motion and the equation

$$B_t = \int_0^t \frac{p \cdot \nu_s}{\sqrt{p^2 \cdot \nu_s}} dW_s^{\nu} + \int_0^t \sqrt{1 - \frac{(p \cdot \nu_s)^2}{p^2 \cdot \nu_s}} dW_s^{\nu,\perp} \tag{1.3}$$

is satisfied.

For each $\pi \in \Pi^2,\ \nu \in \tilde{\mathcal{U}}$ we define

$$X_t(\pi, \nu) = c + \int_0^t \pi_s \sqrt{p^2 \cdot \nu_s} dW_s^{\nu}. \tag{1.4}$$

It is clear that $\mathcal{U} \subset \tilde{\mathcal{U}}$ and for $\nu \in \mathcal{U}$, $W^{\nu} = B$ and (1.1) coincides with (1.4). Hence we can consider the minimax problem

$$\min_{\pi \in \Pi} \max_{\nu \in \tilde{\mathcal{U}}} E|H - X_T(\pi, \nu)|^2, \tag{1.5}$$

which is the extension of problem (1.2).

For the sake of simplicity, it is assumed that $c = EH$ and, using the stochastic integral representation

$$H = EH + \int_0^T h_t dB_t$$
$$= EH + \int_0^T h_t \frac{p \cdot \nu_t}{\sqrt{p^2 \cdot \nu_t}} dW_t^{\nu} + \int_0^T h_t \sqrt{1 - \frac{(p \cdot \nu_t)^2}{p^2 \cdot \nu_t}} dW_t^{\nu,\perp},$$

(1.5) is rewritten as

$$\min_{\pi\in\Pi}\max_{\nu\in\tilde{\mathcal{U}}}\Bigg[E\int_0^T |h_t\frac{p\cdot\nu_t}{\sqrt{p^2\cdot\nu_t}}-\pi_t\sqrt{p^2\cdot\nu_t}|^2dt$$

$$+E\int_0^T h_t^2\left(1-\frac{(p\cdot\nu_t)^2}{p^2\cdot\nu_t}\right)dt\Bigg]$$

$$=\min_{\pi\in\Pi}\max_{\nu\in\tilde{\mathcal{U}}}E\int_0^T[\pi_t^2(p^2\cdot\nu_t)-2h_t\pi_t(p\cdot\nu_t)+h_t^2]dt.$$

Since for each $\pi\in\Pi$

$$\max_{\nu\in\tilde{\mathcal{U}}}E\int_0^T[\pi_t^2(p^2\cdot\nu_t)-2h_t\pi_t(p\cdot\nu_t)+h_t^2]dt$$

$$=\max_{\sigma\in\mathcal{U}}E\int_0^T[\pi_t^2\sigma_t^2-2h_t\pi_t\sigma_t+h_t^2]dt,$$

we have

$$\min_{\pi\in\Pi}\max_{\sigma\in\mathcal{U}}E|H-X_T(\pi,\sigma)|^2$$

$$=\min_{\pi\in\Pi}\max_{\nu\in\tilde{\mathcal{U}}}E|H-X_T(\pi,\nu)|^2.$$

We will see below that this expression is positive. Moreover,

$$\max_{\sigma\in\mathcal{U}}\min_{\pi\in\Pi}E|H-X_T(\pi,\sigma)|^2=\max_{\sigma\in\mathcal{U}}\min_{\pi\in\Pi}E\int_0^T|h_t-\pi_t\sigma_t|^2dt=0.$$

This means that the saddle point does not exist for the problem (1.2).

On the other hand, the function G defined on $\Pi\times\tilde{\mathcal{U}}$ by

$$G(\pi,\nu)=E\int_0^T[\pi_t^2(p^2\cdot\nu_t)-2h_t\pi_t(p\cdot\nu_t)+h_t^2]dt.$$

is convex in π and linear in ν. Then by the Neumann theorem (see Theorem 8 of

[16], Chapt. 6) there exists a saddle point $(\pi^*, \sigma^*) \in \Pi \times \tilde{\mathcal{U}}$. Therefore we have

$$0 = \max_{\sigma \in \mathcal{U}} \min_{\pi \in \Pi} E|H - X_T(\pi, \sigma)|^2$$

$$< \min_{\pi \in \Pi} \max_{\sigma \in \mathcal{U}} E|H - X_T(\pi, \sigma)|^2 = \min_{\pi \in \Pi} \max_{\nu \in \tilde{\mathcal{U}}} E|H - X_T(\pi, \nu)|^2$$

$$= G(\pi^*, \nu^*) = \max_{\nu \in \tilde{\mathcal{U}}} \min_{\pi \in \Pi} E|H - X_T(\pi, \nu)|^2$$

$$= \max_{\nu \in \tilde{\mathcal{U}}} \min_{\pi \in \Pi} \left[E \int_0^T \left| h_t \frac{p \cdot \nu_t}{\sqrt{p^2 \cdot \nu_t}} - \pi_t \sqrt{p^2 \cdot \nu_t} \right|^2 dt + E \int_0^T h_t^2 \left(1 - \frac{(p \cdot \nu_t)^2}{p^2 \cdot \nu_t} \right) dt \right]$$

$$= \max_{\nu \in \tilde{\mathcal{U}}} E \int_0^T h_t^2 \left(1 - \frac{(p \cdot \nu_t)^2}{p^2 \cdot \nu_t} \right) dt.$$

It is easy to see that the saddle point is [1)]

$$\nu_t^* = \frac{\sigma_-}{\sigma_+ + \sigma_-} \delta_{\sigma_+} + \frac{\sigma_+}{\sigma_+ + \sigma_-} \delta_{\sigma_-},$$

$$\pi_t^* = h_t \frac{p \cdot \nu_t^*}{p^2 \cdot \nu_t^*} = \frac{2h_t}{\sigma_- + \sigma_+}.$$

Thus

$$\min_{\pi \in \Pi} \max_{\sigma \in \mathcal{U}} E|H - X_T(\pi, \sigma)|^2 = F(\pi^*, \nu^*)$$

$$= \left(\frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+} \right)^2 E \int_0^T h_t^2 dt.$$

As we see, the extension of the problem allows us to find the robust strategy and the worst case mean-variance hedging error for the original problem (1.2). In Section 2 we will obtain this result by means of the HJBI equation in the case of a terminal contingent claim $H(B_T)$.

Notice, that the problem (1.2) can be solved also directly, but in more general cases (e.g. for the models with nonzero drift) such "explicit computations" are

[1)] δ_a denotes the measure with support at a point a

complicated and in our knowledge does not exist in the literature. The aim of this work is to show that the existence of a saddle point in the extended problem simplifies solving the original problem and enables us to find "explicit solutions".

The chapter is organized as follows. In Section 2, we describe the model and consider the misspecified coefficients as generalized controls. Furthermore, we show the existence of a saddle point of the generalized maximin problem and derive the HJBI equation for the value function. Some examples are also discussed. In Section 3, we prove the solvability in the classical sense of obtained PDE in the case of power and exponential utility and give an explicit PDE-characterization of the robust maximization problem.

2 Generalized coefficients and the existence of a saddle point

Suppose that the financial market consists of a risk-free asset

$$dS_t^0 = r(Y_t)S_t^0 dt \tag{2.6}$$

with $r(y) \geq 0$ and a risky financial assets whose prices are defined through the stochastic differential equation (SDE)

$$\frac{dS_t}{S_t} = (\tilde{b}(Y_t) + \mu_t)dt + \sigma_t dW_t. \tag{2.7}$$

Here W_t is a standard Brownian motion and Y_t denotes a return of non-traded asset modeled by the SDE

$$dY_t = \beta(Y_t)dt + \left(\rho dW_t + \sqrt{1-\rho^2}dW_t^{\perp}\right), \tag{2.8}$$

for some correlation factor $\rho \in [0,1]$ and standard Brownian motion $W^{\perp}$ which is independent of W. Let $(\mathcal{F}_t)_{t\in[0,T]}$ denote the augmented filtration generated by $W, W^{\perp}$. Denote $b = \tilde{b} - r$ and assume that

A1) $b(y),\ \beta(y),\ r(y)$ belong to $C_b^1(\mathbb{R})$,

A2) $b'(y),\ r'(y)$ belong to $C_0(\mathbb{R})$,

where $C_b^1(\mathbb{R})$ is the class of bounded continuous functions with bounded derivatives and $C_0(\mathbb{R})$ denotes the class of continuous functions with compact support.

Introduce the set $\mathcal{P}(K)$ of probability distributions with support on $K = [\mu_-, \mu_+] \times [\sigma_-, \sigma_+]$ ($\mathcal{P}(K)$ is a compact metric space in a weak topology, see [40]), where $0 \le \mu_- \le \mu_+,\ 0 < \sigma_- \le \sigma_+$. Let $\tilde{\mathcal{U}}_K$ be the set of predictable $\mathcal{P}(K)$-valued processes with respect to filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Such type process usually called the generalized control in control theory [45]. We identify the set of predictable K-valued processes $\mathcal{U}_K$ to the subset of $\tilde{\mathcal{U}}_K$ assigning to each (μ_t, σ_t) from $\mathcal{U}_K$ the $\mathcal{P}(K)$-valued process $\delta_{(\mu_t,\sigma_t)}$.

By $\tilde{\Pi}_x$ we denote the set of predictable processes such that $\int_0^T \tilde{\pi}_t^2 dt < \infty,\ P-a.s.$ and corresponding wealth process, defined as a solution of SDE

$$dX_t = X_t(1-\tilde{\pi}_t)\frac{dS_t^0}{S_t^0} + X_t\tilde{\pi}_t\frac{dS_t}{S_t}, \tag{2.9}$$

$$X_0 = x, \tag{2.10}$$

satisfies the condition $X_t(\tilde{\pi}) \ge 0$. Let Π_x be the set of predictable processes $\{\pi_t\}$ where $\pi_t = \tilde{\pi}_t X_t$.

The objective of economic agent is to find the optimal robust strategy of the problem

$$\max_{\pi \in \Pi_x} \min_{(\mu,\sigma) \in \mathcal{U}_K} EU(X_T^{\mu,\sigma}(\pi), Y_T), \tag{2.11}$$

with

$$\begin{aligned} dX_t &= r(Y_t)X_t dt + \pi_t(b(Y_t) + \mu_t)dt + \pi_t \sigma_t dW_t, \quad X_0 = x, \\ dY_t &= \beta(Y_t)dt + \rho dW_t + \sqrt{1-\rho^2} dW_t^{\perp}, \quad Y_0 = y, \end{aligned} \tag{2.12}$$

where $U(x, y)$ is a continuous function satisfying the linear growth condition.

If we denote by $f \cdot \nu_t$ the integral $\int_K f(\mu, \sigma)\nu_t(d\mu d\sigma)$, where $f(\mu, \sigma)$ is an arbitrary continuous function, and by $p_\mu,\ p_\sigma$ the functions $p_\mu(\mu, \sigma) = \mu,\ p_\sigma(\mu, \sigma) = \sigma$, respectively, we can consider the following extended maximin problem

$$\max_{\pi \in \Pi_x} \min_{\nu \in \tilde{\mathcal{U}}_K} EU(X_T^{\nu}(\pi), Y_T^{\nu}), \tag{2.13}$$

$$\begin{aligned} dX_t &= r(Y_t)X_t dt + \pi_t(b(Y_t) + p_\mu \cdot \nu_t)dt + \pi_t \sqrt{p_\sigma^2 \cdot \nu_t} dW_t, \quad X_0 = x, \\ dY_t &= \beta(Y_t)dt + \rho \frac{p_\sigma \cdot \nu_t}{\sqrt{p_\sigma^2 \cdot \nu_t}} dW_t + \sqrt{1 - \rho^2 \frac{(p_\sigma \cdot \nu_t)^2}{p_\sigma^2 \cdot \nu_t}} dW_t^{\perp}, \quad Y_0 = y. \end{aligned} \tag{2.14}$$

Notice that for $(\mu, \sigma) \in \mathcal{U}_K$ the equation (2.14) coincides with (2.12). Our aim is to show that

$$\max_{\pi \in \Pi_x} \min_{(\mu,\sigma) \in \mathcal{U}_K} EU(X_T^{\mu,\sigma}(\pi), Y_T) = \max_{\pi \in \Pi_x} \min_{\nu \in \tilde{\mathcal{U}}_K} EU(X_T^{\nu}(\pi), Y_T^{\nu}) \tag{2.15}$$

and the latter problem admits a saddle point (π^*, ν^*). It is clear that then π^* will be an optimal robust strategy of the initial problem (2.11),(2.12).

The link between problems (2.11),(2.12) and (2.13),(2.14) will be discussed in Theorem 1 below.

Remark 2.1. Let $\mathcal{B}[0,T]$ be the Borel $\sigma-$algebra on $[0,T]$ and $\tilde{\mathcal{F}}$ be some $\sigma-$algebra with $\mathcal{F}_T \subset \tilde{\mathcal{F}}$. Then the $\mathcal{B}[0,T] \bigotimes \tilde{\mathcal{F}}$-measurable process (μ_t, σ_t) (not necessarily adapted to $(\mathcal{F}_t)_{t\in[0,T]}$) with values in the set K, defines the element $\nu \in \tilde{\mathcal{U}_K}$ by the formula $P((\mu_t,\sigma_t) \in B|\mathcal{F}_t) = \nu_t(B)$. More precisely, denoting ${}^pY$ the predictable projection of a process Y (see [39]), we have the equalities ${}^p\mu_t = \int_K \mu\nu_t(d\mu d\sigma)$, ${}^p\sigma_t = \int_K \sigma\nu_t(d\mu d\sigma)$. Hence instead of (2.14) we can write

$$\begin{aligned} dX_t &= r(Y_t)X_t dt + \pi_t(b(Y_t) + {}^p\mu_t)dt + \pi_t\sqrt{{}^p\sigma_t^2}dW_t, \quad X_0 = x, \\ dY_t &= \beta(Y_t)dt + \rho\frac{{}^p\sigma_t}{\sqrt{{}^p\sigma_t^2}}dW_t + \sqrt{1-\rho^2\frac{({}^p\sigma_t)^2}{{}^p\sigma_t^2}}dW_t^{\perp}, \quad Y_0 = y. \end{aligned} \tag{2.16}$$

□

Since

$$\begin{pmatrix} \pi_t\sqrt{p_\sigma^2\cdot\nu_t} & 0 \\ \rho\frac{p_\sigma\cdot\nu_t}{\sqrt{p_\sigma^2\cdot\nu_t}} & \sqrt{1-\rho^2\frac{(p_\sigma\cdot\nu_t)^2}{p_\sigma^2\cdot\nu_t}} \end{pmatrix} \begin{pmatrix} \pi_t\sqrt{p_\sigma^2\cdot\nu_t} & \rho\frac{p_\sigma\cdot\nu_t}{\sqrt{p_\sigma^2\cdot\nu_t}} \\ 0 & \sqrt{1-\rho^2\frac{(p_\sigma\cdot\nu_t)^2}{p_\sigma^2\cdot\nu_t}} \end{pmatrix} = \begin{pmatrix} (p_\sigma^2\cdot\nu_t)\pi_t^2 & \rho(p_\sigma\cdot\nu_t)\pi_t \\ \rho(p_\sigma\cdot\nu_t)\pi_t & 1 \end{pmatrix} \tag{2.17}$$

the generator of the process (X_t, Y_t) can be given by the function

$$\frac{1}{2}\pi^2(p_\sigma^2\cdot\nu)q_{11} + \rho\pi(p_\sigma\cdot\nu)q_{12} + \frac{1}{2}q_{22} + xr(y)p_1 + \pi b(y)p_1 + \pi(p_\mu\cdot\nu)p_1 + \beta(y)p_2.$$

For all $\nu \in \mathcal{P}(K), \pi \in \mathbb{R}, (\mu,\sigma) \in K$ and $(x,y,p,q) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^3$ we set

$$\begin{aligned} &\mathcal{H}^{\pi,\mu,\sigma}(x,y,p,q) \\ &\quad = \frac{1}{2}\pi^2\sigma^2 q_{11} + \rho\pi\sigma q_{12} + \frac{1}{2}q_{22} + xr(y)p_1 + \pi b(y)p_1 + \pi\mu p_1 + \beta(y)p_2. \end{aligned} \tag{2.18}$$

$$\mathcal{H}^{\pi,\nu}(x,y,p,q) = \mathcal{H}^{\pi,\cdot,\cdot}(x,y,p,q)\cdot\nu \tag{2.19}$$

and

$$\mathcal{H}(x,y,p,q)=\max_{\pi\in\mathbb{R}}\min_{\nu\in\mathcal{P}(K)}\mathcal{H}^{\pi,\nu}(x,y,p,q). \tag{2.20}$$

Proposition 2.1. *For each fixed* $(x,y,p,q)\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^3$*, with* $q_{11}<0$ *the function* $(\pi,\nu)\to\mathcal{H}^{\pi,\nu}(x,y,p,q)$ *admits a saddle point* (π^*,ν^*)*, i.e.*

$$\mathcal{H}^{\pi^*,\nu^*}(x,y,p,q)=\max_{\pi\in\mathbb{R}}\min_{\nu\in\mathcal{P}(K)}\mathcal{H}^{\pi,\nu}(x,y,p,q)=\min_{\nu\in\mathcal{P}(K)}\max_{\pi\in\mathbb{R}}\mathcal{H}^{\pi,\nu}(x,y,p,q). \tag{2.21}$$

Moreover,

$$\max_{\pi\in\mathbb{R}}\min_{\nu\in\mathcal{P}(K)}\mathcal{H}^{\pi,\nu}(x,y,p,q)=\max_{\pi\in\mathbb{R}}\min_{(\mu,\sigma)\in K}\mathcal{H}^{\pi,\mu,\sigma}(x,y,p,q). \tag{2.22}$$

Proof. By the Neumann theorem (see Theorem 8 of [16],Chapt.6) for each fixed point (x,y,p,q) the function of $\pi\in R$ and $\nu\in\mathcal{P}(K)$

$$(\pi,\nu)\to\mathcal{H}^{\lambda,\nu}(x,y,p,q)$$

admits a saddle point (π^*,ν^*), i.e.

$$\max_{\pi\in\mathbb{R}}\min_{\nu\in\mathcal{P}(K)}\mathcal{H}^{\pi,\nu}(x,y,p,q)=\min_{\nu\in\mathcal{P}(K)}\max_{\pi\in\mathbb{R}}\mathcal{H}^{\pi,\nu}(x,y,p,q)=\mathcal{H}^{\pi^*,\nu^*}(x,y,p,q). \tag{2.23}$$

It is obvious that

$$\pi^*=-\frac{b(y)p_1+(p_\mu\cdot\nu^*)p_1+(p_\sigma\cdot\nu^*)\rho q_{12}}{(p_\sigma^2\cdot\nu)q_{11}}.$$

Moreover, for each continuous function f on K

$$\min_{\nu\in\mathcal{P}(K)}f\cdot\nu=\min_{(\mu,\sigma)\in K}f(\mu,\sigma),$$

since for $\nu^* = \arg\min_\nu f \cdot \nu$ we have $\operatorname{supp} \nu^* \subseteq \{(\mu^*, \sigma^*) | f(\mu^*, \sigma^*) = \min f(\mu, \sigma)\}$.

Hence

$$\min_{\nu \in \mathcal{P}(K)} \mathcal{H}^{\pi,\nu}(x, y, p, q) = \min_{(\mu,\sigma) \in K} \mathcal{H}^{\pi,\mu,\sigma}(x, y, p, q)$$

and equality (2.22) is satisfied.

Now we define the value functions

$$\begin{aligned} v^-(t, x, y) &= \max_{\pi \in \Pi_x} \min_{(\mu,\sigma) \in \mathcal{U}_K} EU(X_T^{t,x,y}, Y_T^{t,x,y}), \\ v^+(t, x, y) &= \min_{(\mu,\sigma) \in \tilde{\mathcal{U}}_K} \max_{\pi \in \Pi_x} EU(X_T^{t,x,y}, Y_T^{t,x,y}). \end{aligned} \tag{2.24}$$

Since the Isaacs condition is satisfied (by virtue of Proposition 2.1), there exists, as we will see below, a value of the differential game $v \equiv v^+ = v^-$, which will be a solution of the HJBI equation

$$\frac{\partial}{\partial t} v(t, x, y) + \mathcal{H}(x, y, v_x(t, x, y), v_y(t, x, y), v_{xx}(t, x, y), v_{xy}(t, x, y), v_{yy}(t, x, y)) = 0, \tag{2.25}$$

$$v(T, x, y) = U(x, y). \tag{2.26}$$

The latter equation can be rewritten as

$$\begin{aligned} &\frac{\partial}{\partial t} v(t, x, y) + \frac{1}{2} v_{yy}(t, x, y) + \beta(y) v_y(t, x, y) + x r(y) v_x(t, x, y) \\ &\quad + \min_{\nu \in \mathcal{P}(K)} \max_{\pi \in R} \Big[\frac{1}{2} (p_\sigma^2 \cdot \nu) v_{xx}(t, x, y) \pi^2 + (p_\sigma \cdot \nu) \rho v_{xy}(t, x, y) \pi \\ &\quad + (b(y) + p_\mu \cdot \nu) v_x(t, x, y) \pi \Big] = 0, \end{aligned} \tag{2.27}$$

$$v(T, x, y) = U(x, y). \tag{2.28}$$

Simplifying (2.22) we get

$$
\begin{aligned}
&\min_{\nu\in\mathcal{P}(K)}\max_{\pi\in\mathbb{R}}\left[\frac{1}{2}(p_\sigma^2\cdot\nu)q_{11}\pi^2+(p_\sigma\cdot\nu)\rho q_{12}\pi+b(y)p_1\pi+(p_\mu\cdot\nu)p_1\pi\right]\\
&=\min_{\nu\in\mathcal{P}(K)}\left[\frac{((p_\sigma\cdot\nu)\rho q_{12}+(b(y)+p_\mu\cdot\nu)p_1)^2}{-2(p_\sigma^2\cdot\nu)q_{11}}\right]\\
&=\begin{cases}-\dfrac{p_1^2}{2q_{11}}\displaystyle\min_{\nu\in\mathcal{P}(K)}\left[\frac{((p_\sigma\cdot\nu)\kappa+b(y)+p_\mu\cdot\nu)^2}{p_\sigma^2\cdot\nu}\right], & \text{if } p_1\neq 0,\\ -\dfrac{\rho^2 q_{12}^2}{2\sigma_M}, & \text{if } p_1=0,\end{cases}
\end{aligned}
\tag{2.29}
$$

where we suppose that $q_{11}<0$ and use the notation $\kappa=\frac{\rho q_{12}}{p_1}$, $\sigma_M=\frac{\sigma_-+\sigma_+}{2}$.

For the sake of simplicity we assume in addition that

A3) $b(y)+\mu_-\geq 0$, for all $y\in\mathbb{R}$.

Proposition 2.2. *There exists $\nu^*\in\mathcal{P}(K)$ of the form $\nu^*=\alpha\delta_{\mu_a,\sigma_-}+(1-\alpha)\delta_{\mu_a,\sigma_+}$, with some $(\alpha,a)\in[0,1]\times\{-,+\}$, such that*

$$
\min_{\nu\in\mathcal{P}(K)}\left[\frac{(b(y)+p_\mu\cdot\nu+\kappa p_\sigma\cdot\nu)^2}{p_\sigma^2\cdot\nu}\right]=\frac{(b(y)+p_\mu\cdot\nu^*+\kappa p_\sigma\cdot\nu^*)^2}{p_\sigma^2\cdot\nu^*},
\tag{2.30}
$$

$$
(p_\mu\cdot\nu^*,p_\sigma\cdot\nu^*)=\begin{cases}
(\mu_+,\frac{\mu_+}{\kappa}+\frac{\sigma_-\sigma_+}{\sigma_M}) & \text{if } \kappa\in\left(-\infty,\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_--\sigma_+\sigma_-}\right],\\
(\mu_+,\sigma_-) & \text{if } \kappa\in\left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_--\sigma_+\sigma_-},-\dfrac{\mu_+}{\sigma_-}\right],\\
(\kappa,-1)\text{ constant} & \text{if } \kappa\in\left(-\dfrac{\mu_+}{\sigma_-},-\dfrac{\mu_-}{\sigma_+}\right],\\
(\mu_-,\sigma_+) & \text{if } \kappa\in\left(-\dfrac{\mu_-}{\sigma_+},\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+-\sigma_+\sigma_-}\right],\\
(\mu_-,\frac{\mu_-}{\kappa}+\frac{\sigma_-\sigma_+}{\sigma_M}) & \text{if } \kappa\in\left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+-\sigma_+\sigma_-},\infty\right),
\end{cases}
\tag{2.31}
$$

and

$$\frac{(b(y)+p_\mu\cdot\nu^*+\kappa p_\sigma\cdot\nu^*)^2}{p_\sigma^2\cdot\nu^*}$$
$$=\begin{cases}\dfrac{\kappa(2(b(y)+\mu_+)\sigma_M+\kappa\sigma_-\sigma_+)}{\sigma_M^2} & \text{if } \kappa\in\left(-\infty,\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- -\sigma_+\sigma_-}\right],\\ \dfrac{(b(y)+\mu_+ +\kappa\sigma_-)^2}{\sigma_-^2} & \text{if } \kappa\in\left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- -\sigma_+\sigma_-},-\dfrac{\mu_+}{\sigma_-}\right],\\ 0 & \text{if } \kappa\in\left(-\dfrac{\mu_+}{\sigma_-},-\dfrac{\mu_-}{\sigma_+}\right],\\ \dfrac{(b(y)+\mu_- +\kappa\sigma_+)^2}{\sigma_+^2} & \text{if } \kappa\in\left(-\dfrac{\mu_-}{\sigma_+},\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ -\sigma_+\sigma_-}\right],\\ \dfrac{\kappa(2(b(y)+\mu_-)\sigma_M+\kappa\sigma_-\sigma_+)}{\sigma_M^2}, & \text{if } \kappa\in\left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ -\sigma_+\sigma_-},\infty\right).\end{cases} \tag{2.32}$$

Moreover if $\varphi(\kappa)$ *be the linear function of* $\kappa\in\left[-\frac{\mu_+}{\sigma_-},-\frac{\mu_-}{\sigma_+}\right]$ *with* $\varphi(-\frac{\mu_+}{\sigma_-})=\sigma_-,\varphi(-\frac{\mu_-}{\sigma_+})=\sigma_+$ *and*

$$(p_\mu\cdot\nu^*,p_\sigma\cdot\nu^*)=\begin{cases}(\mu_+,\frac{\mu_+}{\kappa}+\frac{\sigma_-\sigma_+}{\sigma_M}) & \text{if } \kappa\in\left(-\infty,\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- -\sigma_+\sigma_-}\right],\\ (\mu_+,\sigma_-) & \text{if } \kappa\in\left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- -\sigma_+\sigma_-},-\dfrac{\mu_+}{\sigma_-}\right],\\ (-\kappa\varphi(\kappa),\varphi(\kappa)) & \text{if } \kappa\in\left(-\dfrac{\mu_+}{\sigma_-},-\dfrac{\mu_-}{\sigma_+}\right],\\ (\mu_-,\sigma_+) & \text{if } \kappa\in\left(-\dfrac{\mu_-}{\sigma_+},\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ -\sigma_+\sigma_-}\right],\\ (\mu_-,\frac{\mu_-}{\kappa}+\frac{\sigma_-\sigma_+}{\sigma_M}) & \text{if } \kappa\in\left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ -\sigma_+\sigma_-},\infty\right),\end{cases} \tag{2.33}$$

then $(p_\mu\cdot\nu^*,p_\sigma\cdot\nu^*)$ *is a continuous, piecewise smooth function of* $\kappa\in(-\infty,\infty)$*.*

The proof is given in Appendix.

Corollary 2.1.

$$
\begin{aligned}
&\min_{\nu\in\mathcal{P}(K)}\left[\frac{(b(y)+p_\mu\cdot\nu)p_1+(p_\sigma\cdot\nu)\rho q_{12})^2}{-2p_\sigma^2\cdot\nu q_{11}}\right]=\min_{(\mu,\sigma)\in K}\left[\frac{(b(y)p_1+\mu p_1+\sigma\rho q_{12})^2}{-2(2\sigma_M\sigma-\sigma_-\sigma_+)q_{11}}\right]\\
&=-\frac{\rho q_{12}(2p_1(b(y)+\mu_+)\sigma_M+\rho q_{12}\sigma_-\sigma_+)}{2q_{11}\sigma_M^2}\chi\left(\frac{\rho q_{12}}{p_1}\in\left(-\infty,\frac{\mu_+\sigma_M}{\sigma_M\sigma_--\sigma_+\sigma_-}\right]\right)\\
&\qquad-\frac{(p_1(b(y)+\mu_+)+\rho q_{12}\sigma_-)^2}{2q_{11}\sigma_-^2}\chi\left(\frac{\rho q_{12}}{p_1}\in\left(\frac{\mu_+\sigma_M}{\sigma_M\sigma_--\sigma_+\sigma_-},-\frac{\mu_+}{\sigma_-}\right]\right)\\
&\qquad-\frac{(p_1(b(y)+\mu_-)+\rho q_{12}\sigma_+)^2}{2q_{11}\sigma_+^2}\chi\left(\frac{\rho q_{12}}{p_1}\in\left(-\frac{\mu_-}{\sigma_+},\frac{\mu_-\sigma_M}{\sigma_M\sigma_+-\sigma_+\sigma_-}\right]\right)\\
&\quad-\frac{\rho q_{12}(2p_1(b(y)+\mu_-)\sigma_M+\rho q_{12}\sigma_-\sigma_+)}{2q_{11}\sigma_M^2}\chi\left(\frac{\rho q_{12}}{p_1}\in\left(\frac{\mu_-\sigma_M}{\sigma_M\sigma_+-\sigma_+\sigma_-},\infty\right)\right)\\
&\qquad\qquad-\frac{\rho^2q_{12}^2}{2\sigma_M}\chi(p_1=0).
\end{aligned}
\tag{2.34}
$$

Proof. It is sufficient to verify that for $\nu_\pm^*=\alpha\delta_{\mu_\pm,\sigma_-}+(1-\alpha)\delta_{\mu_\pm,\sigma_+},\ 0\le\alpha\le1$ we get $p_\sigma^2\cdot\nu_\pm^*=2\sigma_M(p_\sigma\cdot\nu_\pm^*)-\sigma_-\sigma_+$.

From this Corollary we obtain that the HJBI equation has the form

$$
\begin{aligned}
&\frac{\partial}{\partial t}v(t,x,y)+\frac{1}{2}v_{yy}(t,x,y)+\beta(y)v_y(t,x,y)+xr(y)v_x(t,x,y)\\
&\quad+\min_{(\mu,\sigma)\in K}\frac{(b(y)v_x(t,x,y)+\mu v_x(t,x,y)+\rho\sigma v_{xy}(t,x,y))^2}{-2(2\sigma_M\sigma-\sigma_-\sigma_+)v_{xx}(t,x,y)}=0,
\end{aligned}
\tag{2.35}
$$

$$
v(T,x,y)=U(x,y). \tag{2.36}
$$

Theorem 2.1 (Verification Theorem)**.** *Let $v(t,x,y)$ be a classical solution of* (2.27), (2.28) *such that $v_{xx}(t,x,y)<0$ and*

$$
\begin{aligned}
&|v(t,x,y)|\le L(1+|x|+|y|)^p,\ \left|\frac{v_x(t,x,y)}{v_{xx}(t,x,y)}\right|\le L(1+|x|+|y|),\\
&\left|\frac{v_{xy}(t,x,y)}{v_{xx}(t,x,y)}\right|\le L(1+|x|+|y|),
\end{aligned}
\tag{2.37}
$$

holds for some constants $L>0,\ p\ge1$. Suppose also that the triplet $(\pi^*(t,x,y),p_\mu\cdot\nu^*(t,x,y),p_\sigma\cdot\nu^*(t,x,y))$ satisfies the Lipschitz condition on each compact subsets of

$[0,T] \times \mathbb{R}_+ \times \mathbb{R}$, *where*

$$\pi^*(t,x,y) = -\frac{(b(y) + p_\mu \cdot \nu^*(t,x,y))v_x(t,x,y) + p_\sigma \cdot \nu^*(t,x,y)\rho v_{xy}(t,x,y)}{(2\sigma_M p_\sigma \cdot \nu^*(t,x,y) - \sigma_-\sigma_+)v_{xx}(t,x,y)}, \quad (2.38)$$

and

$$(p_\mu \cdot \nu^*(t,x,y), p_\sigma \cdot \nu^*(t,x,y))$$
$$= \begin{cases} \left(\mu_+, \frac{\mu_+ v_x(t,x,y)}{\rho v_{xy}(t,x,y)} + \frac{\sigma_-\sigma_+}{\sigma_M}\right) & \text{if } \frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)} \in \left(-\infty, \frac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}\right], \\ (\mu_+, \sigma_-), & \text{if } \frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)} \in \left(\frac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}, -\frac{\mu_+}{\sigma_-}\right], \\ \left(-\frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\varphi\left(\frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\right), \varphi\left(\frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\right)\right) & \text{if } \frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)} \in \left(-\frac{\mu_+}{\sigma_-}, -\frac{\mu_-}{\sigma_+}\right], \\ (\mu_-, \sigma_+) & \text{if } \frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)} \in \left(-\frac{\mu_-}{\sigma_+}, \frac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right], \\ \left(\mu_-, \frac{\mu_+ v_x(t,x,y)}{\rho v_{xy}(t,x,y)} + \frac{\sigma_-\sigma_+}{\sigma_M}\right) & \text{if } \frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)} \in \left(\frac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}, \infty\right). \end{cases} \quad (2.39)$$

Then $(\pi^*(t,x,y), p_\mu \cdot \nu^*(t,x,y), p_\sigma \cdot \nu^*(t,x,y))$ *is a saddle point of problem (2.13),(2.14) and*

$$\max_{\pi\in\Pi_x} \min_{(\mu,\sigma)\in\mathcal{U}_K} EU(X_T^{\mu,\sigma}(\pi), Y_T)$$
$$= \max_{\pi\in\Pi_x} \min_{\nu\in\tilde{\mathcal{U}}_K} EU(X_T^\nu(\pi), Y_T^\nu) = \min_{\nu\in\tilde{\mathcal{U}}_K} \max_{\pi\in\Pi_x} EU(X_T^\nu(\pi), Y_T^\nu).$$

Proof. By the definition of (2.38),(2.39) the pair $(\pi^*(t,x,y), \nu^*(t,x,y))$ is a saddle point of the function

$$f(t,x,y,\pi,\nu) = \frac{1}{2}(p_\sigma^2 \cdot \nu)v_{xx}(t,x,y)\pi^2 + (p_\sigma \cdot \nu)\rho v_{xy}(t,x,y)\pi + (b(y) + p_\mu \cdot \nu)v_x(t,x,y)\pi$$

for each (t,x,y). It is easy to see that this pair is a continuous, piecewise-smooth function of variables $(t,x,y) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}$. By the definition, the triplet of functions $(\pi^*(t,x,y), p_\mu \cdot \nu^*(t,x,y), p_\sigma \cdot \nu^*(t,x,y))$ consists of Lipschitz functions on the each compact subset.

Thus the stochastic differential equation (SDE)

$$dX_t^* = r(Y_t^*)X_t^* dt + \pi(t, X_t^*, Y_t^*)(b(Y_t^*) + p_\mu \cdot \nu(t, X^*, Y^*))dt$$

$$+\pi(t, X_t^*, Y_t^*)\sqrt{p_\sigma^2 \cdot \nu(t, X_t^*, Y_t^*)}dW_t,$$

$$X_0 = x,$$

$$dY_t^* = \beta(Y_t^*)dt + \rho\frac{p_\sigma \cdot \nu(t, X_t^*, Y_t^*)}{\sqrt{p_\sigma^2 \cdot \nu(t, X_t^*, Y_t^*)}}dW_t + \sqrt{1 - \rho^2\frac{p_\sigma \cdot \nu(t, X_t^*, Y_t^*)^2}{p_\sigma^2 \cdot \nu(t, X_t^*, Y_t^*)}}dW_t^\perp,$$

$$Y_0 = y,$$

defining an optimal wealth process has the coefficients which are Lipschitz functions on each $\{(t, x, y) : \ |x| \leq R, |y| \leq R\}$ and satisfy the linear growth condition (see (2.37)). Thus there exists unique strong solution of SDE with $E\sup_{t\leq T}|X_t^*|^2 < \infty$, $E\sup_{t\leq T}|Y_t^*|^2 < \infty$. For each control pair $(\pi_t, \nu_t) \in \Pi_x \times \tilde{\mathcal{U}}_K$ we denote by $(X_t(\pi^*, \nu), Y_t(\pi^*, \nu))$, $(X_t(\pi, \nu^*), Y_t(\pi, \nu^*))$ the solutions of system (2.14) corresponding to π_t^*, ν_t and π_t, ν_t^*. respectively.

Let $\tau_R = T \wedge \inf\{t : |X_t^*| \geq R, |Y_t^*| \geq R\}$. Since

$$\begin{aligned}
&\frac{\partial}{\partial t}v + \mathcal{H}^{\pi^*,\nu^*}(x, y, v_x, v_y, v_{xx}, v_{xy}, v_{yy}) \\
\equiv\ &\frac{\partial}{\partial t}v + \frac{1}{2}v_{yy} + \beta(y)v_y + xr(y)v_x + f(t, x, y, \pi^*, \cdot) \cdot \nu^* \\
&= 0
\end{aligned}$$

and $v_x\pi^*$, v_y are the continuous bounded functions on each ball, we can apply Ito's formula to $v(t, X_t^*, Y_t^*)$ and get $v(t, x, y) = Ev(X_{\tau_R}^{*t,x,y}, Y_{\tau_R}^{*t,x,y})$. Passing to the limit as $R \to \infty$ we obtain $v(t, x, y) = EU(X_T^{*t,x,y}, Y_T^{*t,x,y})$.

Similarly, using Ito's formula for the processes $v(t, X_t(\pi^*, \nu), Y_t(\pi^*, \nu))$, $v(t, X_t(\pi, \nu^*), Y_t(\pi, \nu^*))$ and taking into account the inequalities

$$f(t,x,y,\pi,\cdot)\cdot\nu^*(t,x,y) \leq f(t,x,y,\pi^*(t,x,y),\cdot)\cdot\nu^*(t,x,y) \leq f(t,x,y,\pi^*(t,x,y),\cdot)\cdot\nu$$

we get $EU(X_T^{t,x,y}(\pi,\nu^*), Y_T^{t,x,y}(\pi,\nu^*)) \leq v(t,x,y) \leq EU(X_T^{t,x,y}(\pi^*,\nu), Y_T^{t,x,y}(\pi^*,\nu))$. Finally, we obtain

$$EU(X_T^{t,x,y}(\pi,\nu^*), Y_T^{t,x,y}(\pi,\nu^*)) \leq EU(X_T^{*t,x,y}, Y_T^{*t,x,y})) \leq EU(X_T^{t,x,y}(\pi^*,\nu), Y_T^{t,x,y}(\pi^*,\nu)).$$

This means that the pair (π^*, ν^*) is a saddle point of problem (2.13).

Since $v(t,x,y) = \inf_{\nu\in\tilde{\mathcal{U}}_K} EU(X_T^{t,x,y}(\pi^*,\nu), Y_T^{t,x,y}(\pi^*,\nu))$ satisfies the HJB equation of the stochastic control problem and

$$\min_{\nu\in\mathcal{P}(K)} f(t,x,y,\pi^*(t,x,y),\cdot)\cdot\nu = \min_{(\mu,\sigma)\in K} f(t,x,y,\pi^*(t,x,y),\mu,\sigma),$$

we conclude that

$$v(t,x,y) = \min_{\nu\in\tilde{\mathcal{U}}_K} EU(X_T^{t,x,y}(\pi^*,\nu), Y_T^{t,x,y}(\pi^*,\nu)) = \min_{(\mu,\sigma)\in\mathcal{U}_K} EU(X_T^{t,x,y}(\pi^*,\mu,\sigma), Y_T^{t,x,y}).$$

Thus

$$\begin{aligned}
&\min_{\nu\in\tilde{\mathcal{U}}_K}\max_{\pi\in\Pi} EU(X_T^{t,x,y}(\pi,\nu), Y_T^{t,x,y}(\pi,\nu))) \\
&\leq \max_{\pi\in\Pi} EU(X_T^{t,x,y}(\pi,\nu^*), Y_T^{t,x,y}(\pi,\nu^*)) \leq v(t,x,y) \\
&= \min_{(\mu,\sigma)\in\mathcal{U}_K} EU(X_T^{t,x,y}(\pi^*,\mu,\sigma), Y_T^{t,x,y}) \leq \max_{\pi\in\Pi}\min_{(\mu,\sigma)\in\mathcal{U}_K} EU(X_T^{t,x,y}(\pi,\mu,\sigma), Y_T^{t,x,y}).
\end{aligned}$$

On the other hand,

$$\max_{\pi\in\Pi}\min_{(\mu,\sigma)\in\mathcal{U}_K} EU(X_T^{t,x,y}(\pi,\mu,\sigma), Y_T^{t,x,y}) \leq \min_{\nu\in\tilde{\mathcal{U}}_K}\max_{\pi\in\Pi} EU(X_T^{t,x,y}(\pi,\nu), Y_T^{t,x,y}(\pi,\nu)).$$

Therefore we get that the values of problems (2.11),(2.12) and (2.13),(2.14) are equal to

$$\min_{\nu\in\tilde{\mathcal{U}}_K}\max_{\pi\in\Pi} EU(X_T^{t,x,y}(\pi,\nu),Y_T^{t,x,y}(\pi,\nu)) = \max_{\pi\in\Pi}\min_{(\mu,\sigma)\in\mathcal{U}_K} EU(X_T^{t,x,y}(\pi,\mu,\sigma),Y_T^{t,x,y}).$$

Corollary 2.2. *The optimal strategy of the robust utility maximization problem (2.11),(2.12) is given by*

$$\pi^*(t,x,y) = -\frac{(b(y)+\alpha(t,x,y))v_x(t,x,y)+\beta(t,x,y)\rho v_{xy}(t,x,y)}{(2\sigma_M\alpha(t,x,y)-\sigma_-\sigma_+)v_{xx}(t,x,y)}, \tag{2.40}$$

where

$$\begin{aligned}&(\alpha(t,x,y),\beta(t,x,y))\\ &=\begin{cases}\left(\mu_+,\frac{\mu_+v_x(t,x,y)}{\rho v_{xy}(t,x,y)}+\frac{\sigma_-\sigma_+}{\sigma_M}\right) & \text{if } \frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\in\left(-\infty,\frac{\mu_+\sigma_M}{\sigma_M\sigma_--\sigma_+\sigma_-}\right],\\ (\mu_+,\sigma_-) & \text{if } \frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\in\left(\frac{\mu_+\sigma_M}{\sigma_M\sigma_--\sigma_+\sigma_-},-\frac{\mu_+}{\sigma_-}\right],\\ \left(-\frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\varphi\left(\frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\right),\varphi\left(\frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\right)\right) & \text{if } \frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\in\left(-\frac{\mu_+}{\sigma_-},-\frac{\mu_-}{\sigma_+}\right],\\ (\mu_-,\sigma_+) & \text{if } \frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\in\left(-\frac{\mu_-}{\sigma_+},\frac{\mu_-\sigma_M}{\sigma_M\sigma_+-\sigma_+\sigma_-}\right],\\ \left(\mu_-,\frac{\mu_+v_x(t,x,y)}{\rho v_{xy}(t,x,y)}+\frac{\sigma_-\sigma_+}{\sigma_M}\right) & \text{if } \frac{\rho v_{xy}(t,x,y)}{v_x(t,x,y)}\in\left(\frac{\mu_-\sigma_M}{\sigma_M\sigma_+-\sigma_+\sigma_-},\infty\right)\end{cases}\end{aligned} \tag{2.41}$$

and $v(t,x,y)$ *is a solution of* (2.35), (2.36).

Example 2.1. Consider the robust mean-variance hedging problem with zero drift and unknown volatility

$$\min_{\pi\in\Pi_x}\max_{\sigma_t\in[\sigma_-,\sigma_+]} E(H(Y_T)-X_T(\pi,\sigma))^2, \tag{2.42}$$

$$\begin{aligned}dX_t &= rX_tdt+\pi_t\sigma_tdW_t,\quad X_0=x,\\ dY_t &= \beta(Y_t)dt+\rho dW_t+\sqrt{1-\rho^2}dW_t^{\perp},\quad Y_0=y.\end{aligned} \tag{2.43}$$

Therefore we have $U(x,y) = -(x - H(y))^2,\ \mu_- = \mu_+ = 0,\ r'(y) = 0$. By equation (2.39) we get

$$(p_\mu \cdot \nu^*(t,x,y), p_\sigma \cdot \nu^*(t,x,y)) = \left(0,\ \frac{\sigma_-\sigma_+}{\sigma_M}\right)$$

since $p_\sigma \cdot \nu(t,x,y) = \frac{2\sigma_-\sigma_+}{\sigma_+ + \sigma_-} = \frac{\sigma_-\sigma_+}{\sigma_M}$ (this means $\nu^*(t,x,y) = \frac{\sigma_-}{\sigma_+ + \sigma_-}\delta_{(0,\sigma_+)} + \frac{\sigma_+}{\sigma_+ + \sigma_-}\delta_{(0,\sigma_-)}$).

Thus

$$\arg \min_{\sigma \in [\sigma_-, \sigma_+]} \frac{\rho^2\sigma^2}{-2(2\sigma_M\sigma - \sigma_-\sigma_+)} = \frac{\sigma_-\sigma_+}{\sigma_M}$$

and from (2.35) follows

$$\begin{aligned} &\frac{\partial}{\partial t}v(t,x,y) + \frac{1}{2}v_{yy}(t,x,y) + \beta(y)v_y(t,x,y) + xr(y)v_x(t,x,y) \\ &\qquad + \min_{\sigma \in [\sigma_-, \sigma_+]} \frac{\rho^2\sigma^2 v_{xy}^2(t,x,y)}{-2(2\sigma_M\sigma - \sigma_-\sigma_+)v_{xx}(t,x,y)} \\ \equiv\ &\frac{\partial}{\partial t}v(t,x,y) + \frac{1}{2}v_{yy}(t,x,y) + \beta(y)v_y(t,x,y) + xr(y)v_x(t,x,y) \\ &\qquad - \rho^2\frac{\sigma_-\sigma_+}{2\sigma_M^2}\frac{v_{xy}^2(t,x,y)}{v_{xx}(t,x,y)} = 0, \end{aligned} \tag{2.44}$$

$$v(T,x,y) = -(x - H(y))^2. \tag{2.45}$$

The solution of (2.44),(2.45) can be given as a quadratic polynomial in x

$$v(t,x,y) = -A(t,y)x^2 + 2B(t,y)x - C(t,y),$$

where the triplet (A, B, C) satisfies the system of PDEs

$$\frac{\partial}{\partial t}A(t,y) + \frac{1}{2}A_{yy}(t,y) + \beta(y)A_y(t,y) + 2rA(t,y) + \rho^2\frac{\sigma_-\sigma_+}{2\sigma_M^2}\frac{A_y^2(t,y)}{A(t,y)} = 0,$$

$$A(T,y) = 1,$$

$$\frac{\partial}{\partial t}B(t,y) + \frac{1}{2}B_{yy}(t,y) + \beta(y)B_y(t,y) + 2rB(t,y) + \rho^2\frac{\sigma_-\sigma_+}{2\sigma_M^2}\frac{A_y(t,y)B_y(t,y)}{A(t,y)} = 0,$$

$$B(T,y) = H(y),$$

$$\frac{\partial}{\partial t}C(t,y) + \frac{1}{2}C_{yy}(t,y) + \beta(y)C_y(t,y) + \rho^2\frac{\sigma_-\sigma_+}{2\sigma_M^2}\frac{B_y^2(t,y)}{A(t,y)} = 0,$$

$$C(T,y) = H^2(y).$$

The system admits an explicit solution

$$A(t,y) = e^{2r(T-t)},\ B(t,y) = e^{2r(T-t)}EH(Y_T^{t,y}),$$

$$C(t,y) = \rho^2\frac{\sigma_-\sigma_+}{2\sigma_M^2}e^{2r(T-t)}\int_t^T EB_y^2(s, Y_s^{t,y})ds + EH^2(Y_T^{t,y})$$

(notice that $B_y(t,y) = e^{2r(T-t)}EH_y(Y_T^{t,y})e^{\int_t^T \beta_y(Y_s^{t,y})ds}$, when H is differentiable). It is clear that

$$|v(t,x,y)| \le L(1+|x|^2),\ |\frac{v_x(t,x,y)}{v_{xx}(t,x,y)}| \le L(1+|x|),\ |\frac{v_{xy}(t,x,y)}{v_{xx}(t,x,y)}| \le L(1+|x|)$$

for some $L > 0$. The optimal strategy then takes the form

$$\begin{aligned}\pi^*(t,x,y) &= -\frac{\rho\frac{\sigma_-\sigma_+}{\sigma_M}v_{xy}(t,x,y)}{(2\sigma_M\frac{\sigma_-\sigma_+}{\sigma_M} - \sigma_-\sigma_+)v_{xx}(t,x,y)} \\ &= -\frac{\rho}{\sigma_M}\frac{v_{xy}(t,x,y)}{v_{xx}(t,x,y)} = -\frac{\rho}{\sigma_M}\frac{B_y(t,y) - xA_y(t,y)}{-A(t,y)} \\ &= \frac{\rho}{\sigma_M}\frac{B_y(t,y)}{A(t,y)} = \frac{\rho}{\sigma_M}e^{-2r(T-t)}B_y(t,y)\end{aligned}$$

and the pair $(\pi^*(t,x,y), \nu^*(t,y))$ satisfies all the conditions of Theorem 1.

The case $\rho = 1,\ r = 0, \beta \equiv 0$ is discussed in the introduction. In this case the second equation in (2.14) the defines the Brownian motion $Y_t = B_t$ for all non-anticipating strategies $\nu_t(Y) \equiv \nu_t(B)$ and (2.14) coincides with (1.3).

3 Power and exponential utility cases

Now let us consider the robust utility maximization problem with power utility $U(x) = \frac{1}{q}x^q$, with $q < 1,\ q \neq 0$

$$\max_{\pi \in \Pi_x} \min_{(\mu,\sigma) \in \mathcal{U}_K} \frac{1}{q} E(X_T^{\mu,\sigma}(\pi))^q, \tag{3.46}$$

subject to

$$\begin{aligned} dX_t &= r(Y_t)X_t dt + \pi_t(b(Y_t) + \mu_t)dt + \pi_t \sigma_t dW_t, \quad X_0 = x, \\ dY_t &= \beta(Y_t)dt + \rho dW_t + \sqrt{1-\rho^2} dW_t^{\perp}, \quad Y_0 = y. \end{aligned} \tag{3.47}$$

In this case, the HJBI equation (2.35),(2.36) gets

$$\begin{aligned} \frac{\partial}{\partial t} v(t,x,y) + \frac{1}{2} v_{yy}(t,x,y) + \beta(y) v_y(t,x,y) + x r(y) v_x(t,x,y) \\ + \min_{(\mu,\sigma) \in K} \frac{((b(y)+\mu)v_x(t,x,y) + \rho\sigma v_{xy}(t,x,y))^2}{-2(2\sigma_M \sigma - \sigma_- \sigma_+) v_{xx}(t,x,y)} = 0, \end{aligned} \tag{3.48}$$

$$v(T,x,y) = \frac{1}{q} x^q. \tag{3.49}$$

A solution of this equation is of the form $v(t,x,y) = \frac{1}{q} x^q e^{u(t,y)}$, where u satisfies the equation

$$\frac{\partial}{\partial t}u(t,y) + \frac{1}{2}u_{yy}(t,y) + \beta(y)u_y(t,y) + \frac{1}{2}u_y^2(t,y) + qr(y)$$
$$-\frac{q}{2(q-1)} \min_{(\mu,\sigma)\in K} \frac{(b(y)+\mu+\rho\sigma u_y(t,y))^2}{2\sigma_M\sigma - \sigma_-\sigma_+} = 0, \tag{3.50}$$

$$u(T,y) = 0. \tag{3.51}$$

Relations (2.39) take the form

$$(p_\mu \cdot \nu^*(t,y), p_\sigma \cdot \nu^*(t,y))$$
$$= \begin{cases} \left(\mu_+, \dfrac{\mu_+}{\rho u_y(t,y)} + \dfrac{\sigma_-\sigma_+}{\sigma_M}\right) & \text{if } \rho u_y(t,y) \in \left(-\infty, \dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}\right], \\ (\mu_+, \sigma_-) & \text{if } \rho u_y(t,y) \in \left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}, -\dfrac{\mu_+}{\sigma_-}\right], \\ (-\rho u_y(t,y)\varphi(\rho u_y(t,y)), \varphi(\rho u_y(t,y))) & \text{if } \rho u_y(t,y) \in \left(-\dfrac{\mu_+}{\sigma_-}, -\dfrac{\mu_-}{\sigma_+}\right], \\ (\mu_-, \sigma_+) & \text{if } \rho u_y(t,y) \in \left(-\dfrac{\mu_-}{\sigma_+}, \dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right], \\ (\mu_-, \dfrac{\mu_+}{\rho u_y(t,y)} + \dfrac{\sigma_-\sigma_+}{\sigma_M}) & \text{if } \rho u_y(t,y) \in \left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}, \infty\right). \end{cases} \tag{3.52}$$

Remark 3.1. By Corollary 2.1 and (2.39), equation (3.50) can be written as

$$\frac{\partial}{\partial t}u(t,y) + \frac{1}{2}u_{yy}(t,y) + \beta(y)u_y(t,y) + \frac{1}{2}u_y^2(t,y) + qr(y)$$
$$- \frac{q\rho u_y(t,y)}{2(q-1)\sigma_M^2}(2(b(y)+\mu_+)\sigma_M + \sigma_-\sigma_+\rho u_y(t,y))\chi\left(\rho u_y(t,y) \le \frac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}\right)$$
$$- \frac{q}{2(q-1)\sigma_-^2}(b(y)+\mu_+ + \rho\sigma_- u_y(t,y))^2\chi\left(\frac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-} < \rho u_y(t,y) \le -\frac{\mu_+}{\sigma_-}\right)$$
$$- \frac{q}{2(q-1)\sigma_+^2}(b(y)+\mu_- + \rho\sigma_+ u_y(t,y))^2\chi\left(-\frac{\mu_-}{\sigma_+} < \rho u_y(t,y) \le \frac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right)$$
$$- \frac{q\rho u_y(t,y)}{2(q-1)\sigma_M^2}(2(b(y)+\mu_-)\sigma_M + \sigma_-\sigma_+\rho u_y(t,y))\chi\left(\rho u_y(t,y) > \frac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right) = 0, \tag{3.53}$$
$$u(T,y) = 0, \tag{3.54}$$

where $\chi(A)$ denotes the indicator of a set A.

Theorem 3.1. *Under conditions A1)-A3) the Cauchy problem* (3.50), (3.51) *admits a classical solution with bounded* $u_y(t,y)$ *and a saddle point* $(\nu^*(t,y), \pi^*(t,x,y))$ *of the problem* (2.13), (2.14) *is defined by equation* (3.52) *and by the formula*

$$\pi^*(t,x,y) = \frac{x}{1-q}\left(\frac{b(y) + p_\mu \cdot \nu^*(t,y)}{p_\sigma^2 \cdot \nu^*(t,y)} + \rho \frac{p_\sigma \cdot \nu^*(t,y)}{p_\sigma^2 \cdot \nu^*(t,y)} u_y(t,y)\right). \tag{3.55}$$

Moreover, $\pi^*(t,x,y)$ *is the optimal strategy of robust utility maximization problem* (3.46), (3.47).

Proof. By the Proposition of Appendix B there exists a classical solution of (3.50),(3.51) with bounded $u_y(t,y)$. By the Lemma 1.1 the pair $(p_\mu \cdot \nu^*(t,y), p_\sigma \cdot \nu^*(t,y))$, where $\nu^*(t,y)$ defined by (3.52), is the Lipschitz function of variables y. Hence

$$\begin{aligned}
\pi^*(t,x,y) &= -\frac{1}{q-1}\frac{b(y) + p_\mu \cdot \nu^*(t,y) + p_\sigma \cdot \nu^*(t,y)\rho u_y(t,y)}{2p_\sigma \cdot \nu^*(t,y)\sigma_M - \sigma_-\sigma_+} x \\
&= \frac{1}{1-q}\left(\frac{b(y) + p_\mu \cdot \nu^*(t,y)}{p_\sigma^2 \cdot \nu^*(t,y)} + \rho \frac{p_\sigma \cdot \nu^*(t,y)}{p_\sigma^2 \cdot \nu^*(t,y)} u_y(t,y)\right) x,
\end{aligned} \tag{3.56}$$

following from (2.38) is also the Lipschitz function. It is obvious that $v_{xx}(t,x,y) = (q-1)x^{q-2}e^{u(t,y)} < 0$ and all the conditions of Theorem 2.1 are satisfied. Therefore we can conclude that $(\pi^*(t,x,y), \nu^*(t,y))$ is the saddle point of the problem (2.11), (2.16).

Corollary 3.1. *If* $b = 0,\ r = 0$ *then*

$$u(t,y) = -\frac{q}{2(q-1)}(T-t) \min_{(\mu,\sigma)\in K} \frac{\mu^2}{2\sigma_M\sigma - \sigma_-\sigma_+} = -\frac{q}{2(q-1)}(T-t)\frac{\mu_-^2}{\sigma_+^2}$$

is a solution of (3.50) *and a saddle point of the maximin problem can be given* *explicitly*

$$(\mu_t^*, \sigma_t^*) = (\mu_-, \sigma_+), \;\; \pi^*(t,x,y) = -\frac{\mu_-}{2(q-1)\sigma_+^2}x.$$

Example 3.1. When $\sigma_- = \sigma_+ = \sigma_M$ we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}u(t,y) + \frac{1}{2}u_{yy}(t,y) + \beta(y)u_y(t,y) + \frac{1}{2}u_y^2(t,y) \\
&\quad - \frac{q}{2(q-1)\sigma_M^2}\min_{\mu_-\le\mu\le\mu_+}(b(y)+\mu+\rho\sigma_M u_y(t,y))^2 \\
&\quad \equiv \frac{\partial}{\partial t}u(t,y) + \frac{1}{2}u_{yy}(t,y) + (2\rho\sigma_M b(y) + \beta(y))u_y(t,y) + \frac{1}{2}\left(1 - \frac{q\rho^2\sigma_M}{q-1}\right)u_y^2(t,y) \\
&\quad - \frac{q}{2(q-1)\sigma_M^2}\min_{\mu_-\le\mu\le\mu_+}((b(y)+\mu)^2 + 2\mu\rho\sigma_M u_y(t,y)) = 0, \\
&\qquad u(T,y) = 0.
\end{aligned}$$

Applications of such type equations in finance and the existence of a classical solution are discussed in [29].

Remark 3.2. Instead of PDE (3.50) we can use the BSDE with quadratic growth

$$\begin{aligned}
dV_t &= -\Bigg(\frac{1}{2}Z_t^2 + qr(Y_t) \\
&\quad - \frac{q}{2(q-1)}\min_{(\mu,\sigma)\in K}\frac{(b(Y_t)+\mu+\rho\sigma Z_t)^2}{2\sigma_M\sigma - \sigma_-\sigma_+}\Bigg)dt + Z_t dW_t + Z_t^{\perp}dW_t^{\perp}, \\
V_T &= 0.
\end{aligned}$$

solvability of which follows from the results of [34], [44]. The solution of the BSDE can be constructed using the solution of PDE (3.50) by the formulas

$$V_t = u(t,Y_t), \; Z_t = \rho u_y(t,Y_t), \; Z_t^{\perp} = \sqrt{1-\rho^2}u_y(t,Y_t).$$

The optimal strategy $\pi_t^* = \pi^*(t, X_t(\pi^*), Y_t)$ is defined by the linear equation

$$\pi_t^* = \frac{1}{1-q}\left(\frac{b(Y_t) + p_\mu \cdot \hat{\nu}_t^*(Z_t)}{p_\sigma^2 \cdot \hat{\nu}_t^*(Z_t)} + \frac{p_\sigma \cdot \hat{\nu}_t^*(Z_t)}{p_\sigma^2 \cdot \hat{\nu}_t^*(Z_t)}Z_t\right)X_t(\pi^*),$$

following from (3.55). As follows from (3.52), the pair $(p_\mu \cdot \hat{\nu}_t^*(z), p_\sigma \cdot \hat{\nu}_t^*(z))$ is defined by

$$(p_\mu \cdot \hat{\nu}_t^*(z), p_\sigma \cdot \hat{\nu}_t^*(z)) = \begin{cases} \left(\mu_+, \dfrac{\mu_+}{z} + \dfrac{\sigma_-\sigma_+}{\sigma_M}\right) & \text{if } z \in \left(-\infty, \dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}\right], \\ (\mu_+, \sigma_-) & \text{if } z \in \left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}, -\dfrac{\mu_+}{\sigma_-}\right], \\ (-z\varphi(z), \varphi(z)) & \text{if } z \in \left(-\dfrac{\mu_+}{\sigma_-}, -\dfrac{\mu_-}{\sigma_+}\right], \\ (\mu_-, \sigma_+) & \text{if } z \in \left(-\dfrac{\mu_-}{\sigma_+}, \dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right], \\ (\mu_-, \dfrac{\mu_+}{z} + \dfrac{\sigma_-\sigma_+}{\sigma_M}) & \text{if } z \in \left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}, \infty\right). \end{cases}$$

Suppose now $U(x, y) = -e^{-\gamma(x-H(y))}$, $\gamma > 0$ and $r = 0$. This case corresponds to the exponential hedging problem of the contingent claim $H(y)$, depending only on the non-tradable asset. We assume that $H \in C_b(\mathbb{R})$. Hence we get the problem

$$\min_{\pi \in \Pi_x} \max_{(\mu,\sigma) \in \mathcal{U}_K} E e^{\gamma(H(Y_T) - X_T^{\mu,\sigma}(\pi))} \tag{3.57}$$

subject to (3.47). It is easy to verify that a solution of (2.35),(2.36) is of the form $v(t, x, y) = -e^{\gamma u(t,y) - \gamma x}$, where $u(t, y)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} u(t, y) + \frac{1}{2} u_{yy}(t, y) + \beta(y) u_y(t, y) + \frac{1}{2}\gamma u_y^2(t, y) & \\ + \frac{1}{2\gamma} \min_{(\mu,\sigma) \in K} \frac{(b(y) + \mu + \rho\gamma\sigma u_y(t, y))^2}{2\sigma_M\sigma - \sigma_-\sigma_+} &= 0, \end{aligned} \tag{3.58}$$

$$u(T, y) = H(y). \tag{3.59}$$

The existence of a classical bounded solution of (3.58),(3.59) with bounded u_y for the case

$$H' \in C_0(\mathbb{R}) \tag{3.60}$$

follows from Proposition B.1. Thus $v(t, x, y)$, $\frac{v_x(t,x,y)}{v_{xx}(t,x,y)} = -\frac{1}{\gamma}$, $\frac{v_{xy}(t,x,y)}{v_{xx}(t,x,y)} = -u_y(t, y)$ are bounded. The robust optimal portfolio is

$$\pi^*(t,y) = -\frac{1}{\gamma}\frac{(b(y)+p_\mu\cdot\nu^*(t,y)) - \gamma\rho(p_\sigma\cdot\nu^*(t,y))u_y(t,y)}{2\sigma_M p_\sigma\cdot\nu^*(t,y) - \sigma_-\sigma_+},$$

where $\nu^*(t,y)$ is defined by (2.39). Therefore all the conditions of Theorem 1 are satisfied. Thus we have proved

Theorem 3.2. *Under conditions A1)-A3) and* (3.60) *the Cauchy problem* (3.58), (3.59) *admits a classical solution with bounded* $u_y(t,y)$ *and a saddle point* $(\nu^*(t,y),\pi^*(t,y))$ *of the problem* (2.13), (2.14) *is defined by the equation*

$$\begin{aligned}
&(p_\mu\cdot\nu^*(t,y), p_\sigma\cdot\nu^*(t,y)) \\
&= \begin{cases}
\left(\mu_+, \dfrac{\mu_+}{\rho u_y(t,y)} + \dfrac{\sigma_-\sigma_+}{\sigma_M}\right) & \text{if } \rho\gamma u_y(t,y) \in \left(-\infty, \dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}\right], \\
(\mu_+, \sigma_-) & \text{if } \rho\gamma u_y(t,y) \in \left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}, -\dfrac{\mu_+}{\sigma_-}\right], \\
(-\rho\gamma u_y(t,y)\varphi(\rho\gamma u_y(t,y)), \varphi(\rho\gamma u_y(t,y))) & \text{if } \rho\gamma u_y(t,y) \in \left(-\dfrac{\mu_+}{\sigma_-}, -\dfrac{\mu_-}{\sigma_+}\right], \\
(\mu_-, \sigma_+) & \text{if } \rho\gamma u_y(t,y) \in \left(-\dfrac{\mu_-}{\sigma_+}, \dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right], \\
(\mu_-, \dfrac{\mu_+}{\rho\gamma u_y(t,y)} + \dfrac{\sigma_-\sigma_+}{\sigma_M}) & \text{if } \rho\gamma u_y(t,y) \in \left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}, \infty\right).
\end{cases}
\end{aligned}$$

and by the formula

$$\pi^*(t,y) = -\frac{1}{\gamma}\frac{(b(y)+p_\mu\cdot\nu^*(t,y)) - \gamma\rho(p_\sigma\cdot\nu^*(t,y))u_y(t,y)}{2\sigma_M p_\sigma\cdot\nu^*(t,y) - \sigma_-\sigma_+}.$$

Moreover $\pi^*(t,y)$ *is the optimal strategy of the robust exponential hedging problem* (3.57), (3.47).

1 Appendix A

Each measure ν may be realized as a distribution of a pair of random variables (ξ, η) with the value in K. Simplifying the notation we denote $b(y) + \mu$ by μ again. Our aim is to characterize the dependence of the minimizer of the problem

$$\min_{\nu \in \mathcal{P}(K)} \left[\frac{(p_\mu \cdot \nu + \kappa p_\sigma \cdot \nu)^2}{p_\sigma^2 \cdot \nu} \right] = \min_{(\xi,\eta) \in K} \left[\frac{(E\xi + \kappa E\eta)^2}{E\eta^2} \right]$$

on a parameter $\kappa \in \mathbb{R}$.

Proposition 1.1. *Let*

$$(\xi^*, \eta^*) = \arg \min_{(\xi,\eta) \in K} \left[\frac{(E\xi + \kappa E\eta)^2}{E\eta^2} \right].$$

Then ξ^* is a number, η^* is the Bernoulli random variable with value in the set $\{\sigma_-, \sigma_+\}$ and the expectation of the pair (ξ^*, η^*) is given by the formula

$$(\xi^*, E\eta^*) = \begin{cases} \left(\mu_+, \dfrac{\mu_+}{\kappa} + \dfrac{\sigma_-\sigma_+}{\sigma_M}\right) & \text{if } \kappa \in \left(-\infty, \dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}\right], \\ (\mu_+, \sigma_-) & \text{if } \ \kappa \in \left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}, -\dfrac{\mu_+}{\sigma_-}\right], \\ (\kappa, -1)\text{constant} & \text{if } \kappa \in \left(-\dfrac{\mu_+}{\sigma_-}, -\dfrac{\mu_-}{\sigma_+}\right], \\ (\mu_-, \sigma_+) & \text{if } \kappa \in \left(-\dfrac{\mu_-}{\sigma_+}, \dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right], \\ \left(\mu_-, \dfrac{\mu_-}{\kappa} + \dfrac{\sigma_-\sigma_+}{\sigma_M}\right) & \text{if } \kappa \in \left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}, \infty\right). \end{cases}$$

Moreover,

$$\frac{(\xi^* + \kappa E\eta^*)^2}{E\eta^{*2}} = \begin{cases} \dfrac{\kappa(2\mu_+\sigma_M + \kappa\sigma_-\sigma_+)}{\sigma_M^2} & \text{if } \kappa \in \left(-\infty, \dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}\right], \\ \dfrac{(\mu_+ + \kappa\sigma_-)^2}{\sigma_-^2} & \text{if } \kappa \in \left(\dfrac{\mu_+\sigma_M}{\sigma_M\sigma_- - \sigma_+\sigma_-}, -\dfrac{\mu_+}{\sigma_-}\right], \\ 0 & \text{if } \kappa \in \left(-\dfrac{\mu_+}{\sigma_-}, -\dfrac{\mu_-}{\sigma_+}\right], \\ \dfrac{(\mu_- + \kappa\sigma_+)^2}{\sigma_+^2} & \text{if } \kappa \in \left(-\dfrac{\mu_-}{\sigma_+}, \dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}\right], \\ \dfrac{\kappa(2\mu_-\sigma_M + \kappa\sigma_-\sigma_+)}{\sigma_M^2} & \text{if } \kappa \in \left(\dfrac{\mu_-\sigma_M}{\sigma_M\sigma_+ - \sigma_+\sigma_-}, \infty\right). \end{cases}$$

Proof. Let $(\mu_+ + \kappa\sigma_-)(\mu_- + \kappa\sigma_+) \leq 0$. Then by the continuity of a function $\mu + \kappa\sigma, (\mu, \sigma) \in K$, there exists $(\hat{\mu}, \hat{\sigma})$ such that $\hat{\mu} + \kappa\hat{\sigma} = 0$. Thus $(\hat{\mu}, \hat{\sigma})$ is proportional to $(\kappa, -1)$ and $\left[\frac{(E\xi^* + \kappa E\eta^*)^2}{E\eta^{*2}}\right] = 0$. If $(\mu_+ + \kappa\sigma_-)(\mu_- + \kappa\sigma_+) > 0$, then either $\kappa > \frac{\mu_-}{\sigma_+}$ and $\xi^* = \mu_-$ or $\kappa < -\frac{\mu_+}{\sigma_-}$ and $\xi^* = \mu_+$. Thus it is sufficient to study the minimization problem

$$\min_{\eta \in [\sigma_-, \sigma_+]} \left[\frac{(\mu_a + \kappa E\eta)^2}{E\eta^2}\right] \quad \text{for} \quad a = +, -.$$

Now show that η^* is of the form $\eta^* = \sigma_- \chi_B + \sigma_+ \chi_{B^c}$ for some event B. Indeed, if $E\eta^* = y$ then $E\eta^{*2} = 2\sigma_M y - \sigma_-\sigma_+$ and η^* is the maximizer of the problem

$$\max_{\eta, E\eta = y} E\eta^2,$$

since for any η, with $E\eta = y$ we have

$$\begin{aligned} E\eta^2 &= E(\eta - \sigma_M)^2 + 2\sigma_M y - \sigma_M^2 \\ &\leq \left(\frac{\sigma_+ - \sigma_-}{2}\right)^2 + 2\sigma_M y - \sigma_M^2 \\ &= 2\sigma_M y - \sigma_-\sigma_+ = E\eta^{*2}. \end{aligned}$$

Hence

$$\min_{\eta \in [\sigma_-, \sigma_+]} \left[\frac{(\mu_a + \kappa E\eta)^2}{E\eta^2}\right] = \min_{\sigma_- \leq y \leq \sigma_+} \psi_a(y),$$

where $\psi_a(y) = \frac{(\mu_a + \kappa y)^2}{2\sigma_M y - \sigma_-\sigma_+}$. Since

$$\psi_a'(y) = \frac{\kappa^2}{2\sigma_M} - \frac{\kappa^2}{2\sigma_M} \frac{(2\sigma_M \frac{\mu_a}{\kappa} + \sigma_-\sigma_+)^2}{(2\sigma_M y - \sigma_-\sigma_+)^2}$$

the equation $\psi_a'(y) = 0$ has two roots:

$$y_1^a = -\frac{\mu_a}{\kappa}, \; y_2^a = \frac{\mu_a}{\kappa} + \frac{\sigma_-\sigma_+}{\sigma_M}.$$

If $y_1^a = -\frac{\mu_a}{\kappa} \in [\sigma_-, \sigma_+]$ then $y_2^a = \frac{\mu_a}{\kappa} + \frac{\sigma_-\sigma_+}{\sigma_M} \in [-\sigma_+ + \frac{\sigma_-\sigma_+}{\sigma_M}, -\sigma_- + \frac{\sigma_-\sigma_+}{\sigma_M}]$ and vice versa. Moreover, $[\sigma_-, \sigma_+] \cap [-\sigma_+ + \frac{\sigma_-\sigma_+}{\sigma_M}, -\sigma_- + \frac{\sigma_-\sigma_+}{\sigma_M}] = \emptyset$. Since $\lim_{y\to\pm\infty} \psi_a(y) = \pm\infty$, the smallest root is the maximizer and the biggest one is the minimizer. The case of $y_1^a \in [\sigma_-, \sigma_+]$ is equivalent to the case

$$\kappa \in [-\frac{\sigma_+}{\mu_a}, -\frac{\sigma_-}{\mu_a}],$$

which yields $\min \psi_a(y) = \psi_a(y_1^a) = 0$. From the relation $y_2^a \in [\sigma_-, \sigma_+]$ follows that $-\sigma_+ + \frac{\sigma_-\sigma_+}{\sigma_M} \leq -\frac{\mu_a}{\kappa} \leq -\sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}$, which is equivalent to the relation

$$\kappa \in (-\infty, \frac{\mu_a}{\sigma_- - \frac{\sigma_-\sigma_+}{\sigma_M}}] \cup [\frac{\mu_a}{\sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}}, \infty).$$

In that case, $\min_{\sigma_-\leq y\leq\sigma_+} \psi_a(y) = \psi_a(y_2^a) = \kappa\frac{2\mu_a+\kappa\sigma_-\sigma_+}{\sigma_M^2}$.

Now we will consider step by step all the possibilities of displacement of κ in the intervals formulated in the proposition.

1) $\kappa \in (-\infty, \frac{\mu_a}{\sigma_- - \frac{\sigma_-\sigma_+}{\sigma_M}}]$. Since $\frac{\mu_a}{\sigma_- - \frac{\sigma_-\sigma_+}{\sigma_M}} \leq -\frac{\mu_+}{\sigma_-}$, we have $\kappa \in (-\infty, -\frac{\mu_+}{\sigma_-}]$ and $\xi^* = \mu_+$. Moreover, $\min \psi_+(y) = \psi_+(y_2^+) = \kappa\frac{2\mu_+ + \kappa\sigma_-\sigma_+}{\sigma_M^2}$.

2) $\kappa \in (\frac{\mu_+}{\sigma_- - \frac{\sigma_-\sigma_+}{\sigma_M}}, -\frac{\mu_+}{\sigma_-}]$. From $\kappa \leq -\frac{\mu_+}{\sigma_-}$ it follows that $y_1^+ = -\frac{\mu_+}{\kappa} < \sigma_-$ and from $\kappa > \frac{\mu_+}{\sigma_- - \frac{\sigma_-\sigma_+}{\sigma_M}}$ it follows that $y_2^+ = \frac{\mu_+}{\kappa} + \frac{\sigma_-\sigma_+}{\sigma_M} < \sigma_-$. Hence $\psi_+(y)$ is increasing on $[\sigma_-, \sigma_+]$ and $\arg\min_{\sigma_-\leq y\leq\sigma_+} \psi_+(y) = \sigma_-$.

3) $\kappa \in (-\frac{\mu_+}{\sigma_-}, -\frac{\mu_-}{\sigma_+}]$. Then $y_1^+ = -\frac{\mu_+}{\kappa} \in [\sigma_-, \sigma_+]$ and $\min \psi_+(y) = 0$.

4) $\kappa \in (-\frac{\mu_-}{\sigma_+}, \frac{\mu_-}{\sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}}]$. Then $\frac{\mu_-}{\kappa} > \sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}$ and $y_1^- = -\frac{\mu_-}{\kappa} < -\sigma_+ + \frac{\sigma_-\sigma_+}{\sigma_M} < \sigma_-$, $y_2^- = \frac{\mu_-}{\kappa} + \frac{\sigma_-\sigma_+}{\sigma_M} > \sigma_+$. Hence $\psi_-(y)$ is decreasing on $[\sigma_-, \sigma_+]$ and $\arg\min \psi_+(y) = \sigma_+$.

5) $\kappa \in (\frac{\mu_-}{\sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}}, \infty]$. Then $\kappa > \frac{\mu_-}{\sigma_+}$ and $\xi^* = \mu_-$. On the other hand, from $\frac{\mu_-}{\kappa} < \sigma_+ - \frac{\sigma_-\sigma_+}{\sigma_M}$ it follows that $y_2^- \in [\sigma_-, \sigma_+]$. Hence $\min_{\sigma_- \le y \le \sigma_+} \psi_-(y) = \psi_-(y_2^-)$.

Lemma 1.1. *Let $u(y)$, $f_1(z), f_2(z), , , , f_N(z)$ be Lipschitz functions and $-\infty = a_0 < a_1 < ... < a_N = \infty$ are such points that $f_k(a_k) = f_{k+1}(a_k)$, $k = 1, ..., N-1$. Then the function*

$$\nu(y) = f_k(u(y)), \text{ if } a_{k-1} < u(y) \le a_k, \ k = 2, ..., N$$

is also a Lipschitz function.

Proof. For the sake of simplicity we consider the case $N = 2$. It is clear that $f_k(u(y))$, $k = 1, 2, 3$, are Lipschitz functions, i.e. there exists a constant $C > 0$ such that $|f_k(u(y_1) - f_k(u(y_2))| \le C|y_1 - y_2|$. Suppose $A_1\{y : u(y) \le a_1\}$, $A_2 = \{y : u(y) > a_1\}$ and set $y_1 \in A_1$ $y_2 \in A_2$. Since $u(y_1) \le a_1 \le u(y_2)$, by the continuity of u there exists $\bar{y}$ such that $u(\bar{y}) = a_1$. Hence we have

$$|\nu(y_1) - \nu(y_2)| = |f_1(u(y_1)) - f_2(u(y_2))| = |f_1(u(y_1)) - f_1(a_1) + f_2(a_2) - f_2(u(y_2))|$$

$$\le |f_1(u(y_1)) - f_1(u(\bar{y}))| + |f_2(u(\bar{y})) - f_2(u(y_2))| \le C|y_1 - \bar{y}| + C|y_2 - \bar{y}| = C(y_2 - y_1).$$

2 Appendix B

Let $\beta, a, b, H \in C_b(\mathbb{R})$ and γ, c, g be some constants. We consider the Cauchy problem

$$\frac{\partial}{\partial t}u(t,y) + \frac{1}{2}u_{yy}(t,y) + \beta(y)u_y(t,y) + \gamma u_y^2(t,y) + a(y) + c\min_{(\mu,\sigma)\in K}\frac{(b(y)+\mu+g\sigma u_y(t,y))^2}{2\sigma_M\sigma - \sigma_-\sigma_+} = 0, \tag{2.61}$$

$$u(T,y) = H(y). \tag{2.62}$$

Proposition 2.1. *Let β, a, b, H be such that $a', b', H' \in C_0(\mathbb{R})$. Then the Cauchy problem* (2.61), (2.62) *admits a classical solution with bounded $u_y(t,y)$.*

Proof. By condition of the proposition there exists $N \geq 0$ such that $a'(y), b'(y) = 0,\ H'(y) = 0$, if $|y| > N$. Thus $a(y) = a^+,\ b(y) = b^+,\ H(y) = H^+$, if $y \geq N$ and $a(y) = a^-,\ b(y) = b^-,\ H(y) = H^-$, if $y \leq -N$ for some constants $a^+, a^-, b^+, b^-, H^+, H^-$. The solutions of (3.50) on the intervals $(-\infty, -N]$ and $[N, \infty)$ are $u^-(t) = a^-(T-t) + c\frac{(b^-+\mu_-)^2}{\sigma_+^2}(T-t) + H^-$ and $u^+(t) = a^+(T-t) + c\frac{(b^++\mu_-)^2}{\sigma_+^2}(T-t) + H^+$, respectively. Now let us consider the Cauchy-Dirichlet problem on the bounded domain $(0,T)\times(-N,N)$

$$\frac{\partial}{\partial t}u(t,y) + \frac{1}{2}u_{yy}(t,y) + \beta(y)u_y(t,y) + \gamma u_y^2(t,y) + a(y) + c\min_{(\mu,\sigma)\in K}\frac{(b(y)+\mu+g\sigma u_y(t,y))^2}{2\sigma_M\sigma - \sigma_-\sigma_+} = 0,$$

$$u(T,y) = H(y),\ u(t,\pm N) = u^{\pm}(t).$$

Suppose

$$A_1(t,y,u,p) = \frac{1}{2}p,$$

$$A(t, y, u, p) = \beta(y)p + \frac{1}{2}p + a(y) + c \min_{(\mu,\sigma)\in K} \frac{(b(y) + \mu + g\sigma p)^2}{2\sigma_M\sigma - \sigma_-\sigma_+}.$$

It is easy to see that A is a Lipschitz function on each ball of its domain, $A(t, y, u, 0)$ is bounded from below and all the conditions of Theorem 6.2 of ([37], Chapt. V) are satisfied. Therefore there exists a classical solution of (3.50),(3.51) with bounded $u_y(t, y)$.

Remark 2.1. The existence of a classical solution follows also from Example 3.6 of [31] if we consider a mixed problem with boundary conditions $u(T, y) = 0,\ u_y(t, \pm N) + u(t, \pm N) = u^{\pm}(t)$).

Bibliography

[1] R. Tevzadze, T. Uzunashvili, Robust Mean-Variance Hedging In The Two Period Model, (2010), reports of the seminar of I. Vekua Institute of Applied Mathematics, v.24.

[2] 2. R. Tevzadze, T. Uzunashvili, Robust Mean-Variance Hedging And Pricing Of Contingent Claims In A One Period Model, (2012), Int. J. Theor. Appl. Finan. 15, 1250024.

[3] R. Tevzadze, T. Toronjadze, T. Uzunashvili, Robust Utility Maximization For A Diffusion Market Model With Misspecified Coefficients, (2013) Finance and Stochastics: Volume 17, Issue 3, Page 535-563.

[4] Avinash K. Dixit and Robert S. Pindyck, Investment under Uncertainty, (1994), Princeton University Press.

[5] Lenos Trigeorgis, Real options: Managerial flexibility and strategy in resource allocation, (1996), MIT Press

[6] Stewart C. Myers, Determinants of Corporate Borrowings, (1977), MIT Press

[7] Michael J. Brennan and Eduardo S. Schwartz, Evaluating Natural Resource Investments, (1985), The Journal of Business, Vol. 58, No. 2, 135-157, University of Chicago Press.

[8] Robert McDonald and Daniel Siegel, The Value of Waiting to Invest, (1986), The Quarterly Journal of Economics, Vol. 101, No. 4, 707-728, MIT Press

[9] Saman Majd and Robert S. Pindyck, Time to Build, Option Value, and Investment Decisions, (1987), MIT Press

[10] Avinash K. Dixit, Entry and Exit Decisions under Uncertainty, (1989), Journal of Political Economy, Vol. 97, No. 3, pp. 620-638, The University of Chicago Press

[11] Alexander J. Triantis and James E. Hodder, Valuing Flexibility as a Complex Option, (1990), Journal of Finance, Vol/ 45, No. 2, 549-565.

[12] S. Titman, Urban Land Prices Under Uncertainty, (1985), American Economic Review, Vol. 75, No. 3, pp. 505-514.

[13] J. Williams, Real State Development as an Option, (1991), Journal of Real Estate Finance and Economics, v.4, n.2, p.191-208.

[14] I. Karatzas and S. Shreve, Methods of Mathematical Finance, (1991), Springer

[15] S. Shreve, Stochastic Calculus for Finance II, Continuous-Time Models,(2000), Springer

[16] J.P. Aubin and I. Ekeland, Applied nonlinear analysis, (1984), Wiley.

[17] M. Avelaneda, A. Levy and A. Paras, Pricing and hedging derivative securities in market with uncertain volatilities, Applied Mathematical Finance, vol.2, (1995), 73-88.

[18] H. Ahn, A. Muni and G. Swindle, Misspecified asset price models and robust hedging strategies, Applied Mathematical Finance, vol.4, (1997), 21-36.

[19] H. Follmer and M. Schweizer, Hedging of Contingent Claims under Incomplete Information, in: M. H. A. Davis and R. J. Elliott (eds.), Applied Stochastic Analysis, Stochastics Monographs, Vol. 5, Gordon and Breach, London/New York (1991), 389-414

[20] H. Follmer and D. Sondermann, Hedging of Non-Redundant Contingent Claims, in: W. Hiloenbrand and A. Mas-colell (eds.), Contributions to Mathematical Economics North- Holland (1986), 205-223.

[21] H. Ahn, A. Muni and G. Swindle, Optimal hedging strategies for misspecified asset price models, Applied Mathematical Finance, vol.6, (1999), 197-208.

[22] G. Bordigoni, A. Matoussi and M. Schweizer, A Stochastic Control Approach to a Robust Utility Maximization Problem, in: F. E. Benth et al. (eds.), Stochastic Analysis and Applications. Proceedings of the Second Abel Symposium, Oslo, (2007), Springer, 125-151.

[23] M. C. Pinar, On Robust Quadratic Hedging of Contingent Claims in Incomplete Markets under Ambiguous Uncertainty, March 2006, presented at the First Conference on Advanced Mathematical Methods in Finance, April 2006

[24] R. Buff, Uncertain volatility models–Theory and applications, (2002), Springer-Verlag.

[25] J. Cvitanic and I. Karatzas, On dynamic measures of risk, Finance and Stochastics, 3(4), (1999), 451-482.

[26] L. Denis and C. Martini, A Theoretical framework for the pricing of contingent claims in the presence of model uncertainty, The Annals of Applied Probability (2006), vol. 16, No. 2, 827-852

[27] H. Föllmer and A. Gundel, Robust projections in the class of martingale measures. Ill J Math 50(2), (2006), 439-472.

[28] A. Gundel , Robust utility maximization for complete and incomplete market models, Finance and Stochastics 9, (2005), 151-176.

[29] D. Hernandez-Hernandez and A. Schied, Robust utility maximization in a stochastic factor model, Statistics and Decitions, 24, No. 3, (2006), 109-125 .

[30] D. Hernndez-Hernndez and A. Schied, A control approach to robust utility maximization with logarithmic utility and time-consistent penalties, Stoch. Proc. Appl. Vol. 117, 8,(2007), 980-1000.

[31] D. Henry, Geometric theory of semilinear parabolic equations, (1981), Springer-Verlag.

[32] P. J. Huber, Robust Statistics. N.-Y.(1981), Wiley.

[33] D. Kramkov and W. Schachermayer, The asymptotic elasticity of utility functions and optimal investment in incomplete markets. The Annals of Applied Probability, Vol 9, No 9, (1999),904-950.

[34] M. Kobylanski, Backward stochastic differential equations and partial differential equations with quadratic growth, The Annals of Probability, vol. 28, N2,(2000), 558-602.

[35] R. Korn and O. Menkens, Worst-case scenario portfolio optimization: a new stochastic control approach. Math. Methods Oper. Res., 62(1), (2005), 123-140.

[36] R. Korn and P. Wilmott, Optimal portfolios under the threat of a crash. Int. J. Theor. Appl. Finances 5(2), (2002), 171-187.

[37] O.A. Ladizenskaia, V.A. Solonnikov, and N.N. Uraltseva, Linear and quasi-linear equations of parabolic type (Russian), (1967).

[38] N. Lazrieva and T. Toronjadze, Optimal robust mean-variance hedging in incomplete financial markets, Journal of Mathematical Sciences, 153,3, (2008), 262-290.

[39] R.Sh. Liptzer and A.N. Shiryayev, Martingale theory, Nauka, Moscow, 1986.

[40] K.P. Parthasarathy, Introduction to probability and measure, (1980), Hindustan Book Agency.

[41] M-C. Quenez , Optimal portfolio in a multiple-priors model. Seminar on Stochastic Analysis, Random Fields and Applications IV, Progr Probab, 58, (2004), 291-321, Birkhuser, Basel.

[42] A. Schied, Robust optimal control for a consumption-investment problem, Math. Meth. Oper. Res. (2008), 1-20.

[43] D. Talay, Z. Zheng, Worst case model risk management, (2002), Finance and Stochastics, 6(4), 517–537.

[44] R. Tevzadze, Solvability of backward stochastic differential equations with quadratic growth, Stochastic Proc. Appl., 118 (2008), 503–515.

[45] J. Warga, The optimal control of differential and functional equations, (1976), Academic press.

[46] V.F. Demyanov, V.N. Malazemov, Introduction to minimax, Dover Publicatons, Inc., (1990), Russian edition: Izdatelstvo Nauka, Moscow, (1972)