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Effects of Bilateral Tariffs on Currency Exchange Rate and Characterization of Efficient Portfolios

წარდგენილია ბიზნესის ადმინისტრირების დოქტორის აკადემიური ხარისხის მოსაპოვებლად

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As the author of the submitted work, I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published, accepted for publication or written by another person, or substantial proportions of material that have been accepted for the award of any other degree or diploma, except where due acknowledgement is made in the dissertation,

Tsotne Kutalia

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რეზიუმე

თავისუფალი ვაჭრობის პირობებში ვალუტის კურსის მნიშვნელობა დამოკიდებულია ბევრ სხვადასხვა ცვლადზე, რომელთაგან ბევრს სახელმწიფო ვერ აკონტროლებს. ვალუტის გაცვლითი კურსის პროგნოზი მნიშვნელოვანია ფინანსური და ეკონომიკური დაგეგმვისთვის. სავალუტო პორტფელებით ვაჭრობის შედეგი დამოკიდებულია ვალუტის კურსზე. სავალუტო კურსის წონასწორობის წერტილის განსასაზღვრად არსებობს სხვადასხვა მეთოდი. ჩვენთვის საინტერესოა მოდელები, სადაც არსებობს საკონტროლო ცვლადები, რომლებზეც გავლენა აქვთ ვაჭრობაში ჩაბმულ ქვეყნებს და შესაბამისად, განაპირობებენ გაცვლითი კურსის მნიშვნელობას. ამასთან, ამ ცვლადების მნიშვნელობა ქვეყნებმა უნდა შეარჩიონ ვაჭრობის შედეგად მიღებული სარგებლის მაქსიმიზაციით.

ნაშრომი იყოფა ორ ძირითად ნაწილად. პირველი ნაწილი ეფუძნება თამაშის თეორიას და განხილულია მოგების მაქსიმიზაციის ამოცანა ორი ქვეყნისთვის, რომლებიც ჩაბმული არიან საერთაშორისო ვაჭრობაში და ერთმანეთთან არ თანამშრომლობენ. განხილულია პროდუქტების გაცვლის ალბათური მოდელი ორივე ქვეყნის ფასების პირობებში. ვუშვებთ, რომ როგორც გაცვლილი პროდუქტების რაოდენობების, ასევე ორივე ბაზარზე მათი ფასების ალბათური განაწილებები ცნობილია და ვიღებთ ორივე ქვეყნისთვის ვალუტის მოთხოვნის ფუნქციებს, რომელებიც საბოლოოდ განსაზღვრავენ ერთი ქვეყნის უპირატესობას მეორეზე.

გაცვლილი პროდუქტების მოცულობა განსაზღვრავს ერთი ქვეყნის მოთხოვნას მეორე ქვეყნის ვალუტაზე. თუმცა, ქვეყნებს შეუძლიათ მოთხოვნილი ვალუტის რაოდენობაზე გავლენის მოხდენა იმპორტირებულ პროდუქტებზე ტარიფის დაწესებით. ვინაიდან ვაჭრობის შედეგად მოგება განისაზღვრება როგორც ბალანსი იმპორტირებულსა და ექსპორტირებულ

პროდუქტებს შორის, მოგების მაქსიმიზაციის ამოცანა იგივეა რაც ორი კონკურენტის თამაში, სადაც ნეშის წონასწორული ტარიფები დამოკიდებულია ვალუტის მოთხოვნის ფუნქციებზე. საბოლოოდ წონასწორულ ტარიფებზე დაყრდნობით ვიღებთ ვალუტის ოპტიმალურ გაცვლით კურსს.

ორმხრივ ვაჭრობაში ჩაბმული ქვეყნები განსხვავდებიან უცხო ქვეყნის ვალუტის მოთხოვნის ფუნქციების მიხედვით. კერძო შემთხვევა, როდესაც ორივე ქვეყანას ერთნაირი მოთხოვნის ფუნქციები აქვს, ანუ ეკონომიკურად სიმეტრიული ქვეყნები განხილული იყო [15]-ში. ჩვენი მიდგომის სიახლე მდგომარეობს იმაში, რომ განვავრცოთ მოდელი ზოგად, ასიმეტრიულ შემთხვევაზე, როცა ორ ქვეყანას ვალუტის სხვადასხვა მოთხოვნის ფუნქცია აქვს.

ამასთან, განვიხილავთ მოდელს [12]-დან, სადაც ვაჭრობიდან მიღებული მოგების ფუნქცია დამოკიდებულია ზემოთ აღწერილი მოდელისგან განსხვავებულ კომპონენტებზე და ვიღებთ ნეშის წონასწორულ ტარიფების წყვილს. მოდელი გაუმჯობესებულია ვალუტის გაცვლითი კურსის შემოღებით, რაც აჩვენებს კურსის ეფექტს წონასწორულ წერტილზე. ასევე, დახასიათებულია ნეშის წონასწორობის წერტილი ვალუტის კურსის გათვალისწინებით.

ნაშრომში ძირითადად განხილულია ნეშის წონასწორობის მიღების თეორიული გზები ვალუტის მოთხოვნის სხვადასხვა ფუნქციებისთვის. მიღებული შედეგები წარმოდგენილია აბსტრაქტული ფუნქციებით, რომლებიც ექვემდებარებიან ლოგიკურ ეკონომიკურ კანონზომიერებებს. პრაქტიკული თვალსაზრისით, ვალუტის მოთხოვნის ფუნქციები მიიღება გაცვლილი პროდუქტების ფასების და მათი რაოდენობების მიხედვით. ვინაიდან ჩვენი მიზანია მოცემული ფუნქციებისთვის წონასწორობის წერტილების პოვნა, ჩვენი ამოცანა იწყება უკვე განსაზღვრული ფუნქციებიდან.

საბოლოოდ გამოგვყავს ნეშის წონასწორობის საკმარისი პირობა. ეს პირობა შემოწმებულია ვალუტის მოთხოვნის ფუნქციების კერძო შემთხვევაზე ეკონომიკურად ასიმეტრიული ქვეყნებისთვის. ასიმეტრიული შემთხვევისთვის ასევე განვიხილავთ ვალუტის მოთხოვნის ფუნქციების სხვადასხვა ვარიანტს, რაც აჩვენებს მათი ფორმების მნიშვნელობას და გავლენას წონასწორობის წერტილზე.ვალუტის გაცვლითი კურსის განხილვის შემდეგ, ნაშრომის მეორე ნაწილი ეთმობა სავალუტო პორტფელის ოპტიმიზაციის ამოცანას სადაც სტანდარტული კვადრატული საშუალოს მოდელი შეცვლილია საშუალო-მინიმალური ამონაგების დონის მოდელით და დამატებულია კიდევ ერთი მნიშვნელოვანი სიდიდე, მინიმალური ამონაგების დონემდე შემოსაზღვრული პირველი მიღწევის მომენტის მათემატიკური ლოდინი. რისკის განსაზღვრისას ახალი მოდელი ითვალისწინებს როგორც შესაძლო მაქსიმალურ დანაკარგს მოცემული დასაჯერებლობის დონით, ასევე დროს ამ ხდომილების დადგომამდე. წარმოდგენილი მიდგომის სიახლე მდგომარეობს იმაში, რომ შემოღებულია მინიმალური ამონაგების დონემდე შემოსაზღვრული პირველი მიღწევის მომენტი, როგორც ახალი ფაქტორი და მისი მათემატიკური ლოდინი განხილულია როგორც დამატებითი მნიშვნელოვანი ფაქტორი პორტფელის ოპტიმიზაციისთვის. ვუშვებთ რა, რომ აქტივების ფასები მოძრაობს მრავალ-განზომილებიანი ბროუნის ძრაობით, ვიღებთ პორტფელის სიმდიდრის პროცესს მრავალი აქტივისთვის და განვსაზღვრავთ პორტფელის ღირებულების უმცირეს დონეს, რომელსაც შეიძლება მან მიაღწიოს დიდი დასაჯერებლობის დონით. საბოლოოდ პორტფელის ოპტიმიზაციის ამოცანა დაიყვანება რისკიანი პორტფელების ეფექტური ზედაპირის აღწერამდე, სადაც განლაგებულია პორტფელები სხვადასხვა რისკის, მოსალოდნელი ამონაგების და შემოსაზღვრული პირველი მიღწევის მომენტით (მინიმალური ამონაგების დონემდე). ეფექტურ ზედაპირზე განლაგებული ყველა პორტფელი რისკიანობა - ამონაგების თვალსაზრისით ეფექტურია.

ნაშრომის მეორე ნაწილი ძირითადად ეთმობა სავალუტო პორტფელს, თუმცა მოდელი ზოგადი ხასიათისაა და გამოიყენება ნებისმიერი აქტივებისგან შედგენილი პორტფელებისთვის. განხილულია აქტივების ფასების დინამიკა და აირჩევა პორტფელში შემავალი აქტივების წონები საინვესტიციო პერიოდის გათვალისწინებით. ნაშრომში შემუშავებული მიდგომა წარმოადგენს მარკოვიცის [18] მოდელის ერთგვარ განზოგადებას. საბოლოოდ შეგვიძლია დავასკვნათ, რომ მოდელი იძლევა რისკის უფრო გონივრულ საზომს ვიდრე სტანდარტულ მოდელებში წარმოდგენილი ვარიაცია.

შედეგად ნაშრომში წარმოდგენილია ორი ამოცანა - ტარიფების გავლენა ვალუტის გაცვლით კურსზე და პორტფელის ოპტიმიზაცია. ორივე ამოცანის განხილვა შესაძლებელია ინდივიდუალურად და ისინი არ არიან ურთიერთდამოკიდებული. გასაკუთრებით საგულისხმოა პორტფელის ოპტიმიზაციის მოდელი, რომელსაც ნაშრომში განვიხილავთ როგორც ვალუტის პორტფელს. თუმცა მოდელის განზოგადება შესაძლებელია ნებისმიერი სავაჭრო ინსტრუმენტით, რომელთა ფასებიც დაკვირვებადია დინამიკაში. ბოლოს წარმოდგენილია პორტფელების ოპტიმიზაციის მაგალითები.

Effects of Bilateral Tariffs on Currency Exchange Rate and Characterization of Efficient Portfolios

Executive Summary

Currency exchange rate is one of the important factors determining a nation's overall economic health. In a free trade environment, its value depends on multiple variables, many of which are beyond the control of a government. Having a precise prediction of an equilibrium exchange rate helps governments and firms arrange their economic plans safely. Furthermore, economic agents managing portfolios consisting of currency pairs are constantly in need of predicting the exchange rate value. There have been numerous attempts to model the equilibrium based on various factors. We put our attention on a model where the governments of nations involved in a free trade have a control variable to influence the value of the exchange rate. However, the values of these control variables have to be chosen based on maximization of benefit from trade. In addition, once the equilibrium currency exchange rate is determined, for one managing a portfolio with the currency pairs in it, having the values of the rates determined is of great importance.

In order to address the above mentioned problems, this thesis is divided in two main parts. The first part deals with game theory and exploration of gain maximization problem of two nations engaging in non-cooperative bilateral trade. We examine the probabilistic model of an exchange of commodities under different price systems. Assuming the probability distributions of volumes of commodities exchanged and their selling prices in both markets are known, we arrive at currency demand functions for each nation, whose shapes ultimately determine the edge one nation has over another in terms of trading.

Volume of commodities exchanged determines the demand each nation has over the counterparty's currency. However, this quantity can be manipulated by imposing a tariff on imported commodities. As long as the gain from trade is

determined by the balance between imported and exported commodities, such a scenario results in a two party game where Nash equilibrium tariffs are determined for various foreign currency demand functions and ultimately, the exchange rate based on optimal tariffs is obtained.

Nations involved in exchange of commodities differ according to the currency demand functions. A special case when both nations have identical currency demand functions was considered in [15]. Such nations are referred to as economically symmetric ones. The main novelty of our approach is to extend the model into a more general, asymmetric case where two nations involved in non-cooperative bilateral trade have different currency demand functions.

In addition, we consider a model from [12], where the gain from trade is based on completely different components and obtain the Nash equilibrium pair of tariffs. In order to illustrate the effect of currency exchange rate on the equilibrium point, we improve the model by introducing the currency exchange rate and determine its optimal value based on the optimal tariffs.

The thesis lays down the theoretical groundwork for arriving at optimal solutions regardless of the shapes of demand functions. Findings and solutions provided throughout the text are primarily given based on abstract functions following predefined economic patterns. From the practical point of view, these functions would have been derived from probability distributions of the prices and quantities of commodities. Since our task is to analyze the equilibrium point given the currency demand functions, we take some examples of functions as granted.

Lastly, the sufficient conditions are defined for a solution to be the Nash equilibrium point. These conditions are then applied to the special cases of currency demand functions related to economically symmetric and asymmetric nations. For the asymmetric case we also consider different versions of the currency demand functions to emphasize the importance of their shapes and illustrate the dynamic effects of shifting the functions on the equilibrium point.

As long as the exchange rate has been examined, the second part of the thesis explores the selection of optimal portfolio of foreign currency pairs by replacing the standard Mean-Variance model by Mean-Minimum Return Level (MRL) framework and adding one important dimension – expectation of bounded First Passage Time (FPT) towards the MRL. To measure how much a given portfolio is exposed to risk, the new model can capture both, the amount of the largest possible loss at a certain confidence level and time to such an event occurring. The novelty of this approach is the introduction of bounded first passage time towards MRL and taking its expectation into consideration as an additional factor in portfolio selection decision making. Assuming that the asset price dynamics follow multi-dimensional Geometric Brownian Motion with drift, we obtain a portfolio wealth process for multiple assets and we evaluate the lowest possible value to which it can drop by a high confidence level. Then we extend our examination of the optimal portfolio selection by ultimately obtaining the efficient surface of risky portfolios. As a result, we show that the third dimension can make a significant difference while choosing the asset weights compared to classical models ignoring the portfolio return paths as long as they achieve a desired combination of risk and return.

We focus on the portfolio of foreign currency pairs for our purposes, however the model is more general in nature and can easily be applied to portfolios of any assets. We observe the asset price movements in dynamics and decide on the investment plan based on the predefined investment horizon. The model can be thought of as a generalization of a Markowitz [18] model where optimal portfolios are selected based on average return and corresponding variance. From the practical point of view, the three dimensional model we offer gives a more reasonable safety measure and a risk-return combination.

As a result, the thesis offers a comprehensive review of investment plan stretching from economic problem of trade gain maximization to portfolio optimization. Both models are applicable on their own. Especially the three dimensional portfolio model which does not necessarily consist of currency pairs, rather it can include a mix of any trading instruments whose spot prices are dynamically observable. The thesis ends with some examples of portfolios consisting of stocks and ETF (Exchange Traded Funds).

Introduction

Due to different circumstances of production, two nations can produce similar goods and services at different prices. They can both benefit by getting involved in international trade to import commodities, which under their own price system is of relatively low price than domestically produced commodities, which under the same price system is of relatively high price. The volume of commodities imported determines one nation's demand for another nation's currency. Balance of demands of two nations for foreign currency determines an exchange rate.

Gain functions for both nations are made up of the foreign currency demand functions. The foreign currency demand itself for a given nation depends on the volume of commodities imported. A government can affect this quantity by imposing a tariff on imports, thus making it less desirable to buy commodity from another nation's market. For the domestic and foreign nations, annual demand and corresponding prices measured in national currency are d_1, I, d_N, p_1, I, p_N and $d_1^*, I, d_N^*, p_1^*, I, p_N^*$ respectively. If we take x as an exchange rate of a unit of foreign currency in terms of domestic currency units, then the domestic and foreign nations' demand for foreign currency are given by

$$D(x) := \frac{1}{C_N}\sum_{k=1}^{N} \bar{E}(p_k^* d_k, \frac{p_k}{p_k^*} > x)$$

and

$$D^*(x) := \frac{1}{C_N^*}\sum_{k=1}^{N} \bar{E}(p_k d_k^*, \frac{p_k}{p_k^*} < x)$$

respectively, where $C_N = \sum_{k=1}^{N} \bar{E}(p_k^* d_k)$, $C_N^* = \sum_{k=1}^{N} \bar{E}(p_k d_k^*)$ and $\bar{E}$ is the mathematical expectation under $\bar{P}$ on a probability space $(\bar{\Omega}, \bar{F}, \bar{P})$. If we introduce the extended probability space (Ω, F, P), where

$$\Omega = \bar{\Omega} \times \{1, \ldots, N\}, P(A, k) = \frac{1}{N}\, \bar{P}(A), A \in \bar{F}$$

and define random variables p, p^*, d, d^* by

$$p(\omega, k) = p_k(\omega), p^*(\omega, k) = p_k^*(\omega),$$

$$d(\omega, k) = d_k(\omega), d^*(\omega, k) = d_k^*(\omega),$$

then the demand functions above can be rewritten as probability distribution functions

$$D(x) = E\left(p^* d, \frac{p}{p^*} > x\right), \quad D^*(x) = E\left(p d^*, \frac{p}{p^*} < x\right)$$

which indicate that the domestic nation will import the commodity if $\frac{p}{p^*} > x$ and the foreign nation will import if $\frac{p}{p^*} < x$. Since x is the value of an unit of foreign currency in terms of the domestic currency units, increasing the exchange rate makes foreign commodities more expensive for the domestic nation and the domestic commodities less expensive for the foreign nation. Therefore, D is a decreasing function of x and D^* is an increasing function of x. These functions have the following properties

$$D(0) = 1, \qquad D(\infty) = 0, \qquad D^*(0) = 0, \qquad D^*(\infty) = 1.$$

For an exchange rate x, solving the equation

$$xD(x) = D^*(x)$$

for x, yields the equilibrium rate $x = e$.

This equation determines the equilibrium exchange rate when both nations practice an unrestricted free trade policy. Left side of the equation is the foreign currency demand of a domestic nation and the right side is the foreign currency demand of a foreign nation, both measured in domestic currency units.

Now suppose the domestic and foreign governments impose the following tariffs on imported commodities: $1 - \theta$ and $1 - \theta^*$. Then the domestic nation will import the commodity if $\frac{p\theta}{p^*} > x$, and the foreign nation will import if $\frac{p^*\theta^*}{p} > \frac{1}{x}$. Taking tariffs into account, the demand functions defined above now become

$$D\left(\frac{x}{\theta}\right) = \frac{E(p^* d 1_{\{\theta p > x p^*\}})}{E(p^* d)}, \quad D^*(x\theta^*) = \frac{E(p d^* 1_{\{\theta^* p^* x > p\}})}{E(p d^*)},$$

where

$$E\left(p^* d 1_{\{\theta p > x p^*\}}\right) = \frac{1}{N}\sum_{k=1}^{N} \bar{E}\left(p_k^* d_k 1_{\{\theta p_k > x p_k^*\}}\right),$$

$$E\left(p d^* 1_{\{\theta^* p^* x > p\}}\right) = \frac{1}{N}\sum_{k=1}^{N} \bar{E}\left(p_k d_k^* 1_{\{\theta^* p_k^* x > p_k\}}\right),$$

$$E(p^* d) = \frac{C_N}{N}, \quad E(p d^*) = \frac{C_N^*}{N}$$

and so the equilibrium exchange rate adjusted for the tariffs is now determined from

$$x D\left(\frac{x}{\theta}\right) = D^*(\theta^* x),$$

from which it is clear that the equilibrium exchange rate $x = e$ now depends on θ and θ^*. At the same time, according to the Schwartz's [15] model which we develop, gain from competitive trade consists of two components – imported and exported commodities measured in national currency. The gain from trade functions for each nation are defined as follows

$$\begin{aligned} G(e,\theta,\theta^*) &= E\left(pd, \frac{p^*}{p} < \frac{\theta}{e}\right) - E\left(pd^*, \frac{p^*}{p} > \frac{1}{e\theta^*}\right) \\ &= E\left(\frac{p}{p^*} 1_{\left(\frac{p}{p^*} > \frac{e}{\theta}\right)} p^* d\right) - E\left(pd^* 1_{\left(\frac{p}{p^*} > \frac{e}{\theta}\right)}\right) \\ &= -\int_{e/\theta}^{\infty} y D'(y) dy - D^*(\theta^* e), \end{aligned}$$

and

$$\begin{aligned} G^*(e,\theta,\theta^*) &= E\left(p^* d^*, \frac{p}{p^*} < \theta^* e\right) - E\left(p^* d, \frac{p}{p^*} > \frac{e}{\theta}\right) \\ &= \int_{\frac{1}{\theta^* e}}^{\infty} \frac{1}{y} D^{*\prime}\left(\frac{1}{y}\right) dy - D\left(\frac{e}{\theta}\right), \end{aligned}$$

The idea behind imports regarded as beneficial for a nation is that it has an incentive to buy commodities from the foreign market only if it costs them less than it would by buying in the domestic market. And the second component, export is regarded as gain deductible quantity since it benefits the competitor by the same argument.

It is important to realize that within this model, the components of trade gain function involve the commodities exchanged only. So the idea of gain is narrowed down to the advantages competitors take from trade and is not extended throughout the whole economies of two nations. For example, we do not claim that import is necessarily beneficial and export is harming. What we claim is, that import is an advantage taken from the competitor and the export is an advantage taken by the competitor. So this import-export phenomena is considered autonomously only within the trade context without further impact analysis over various parts of economies.

The nations involved in non-cooperative trade are facing the dilemma of how much tariff to impose on imports. Greater the tariff, lower the commodities imported, it hurts the nation's gain, and benefits the competitor's one. From the gain functions, it is obvious that the greatest mutual benefit is achieved when nations cooperate and pursue a free trade policy. We confine ourselves to non-cooperative case and find the pair of optimal tariffs (equilibrium point) which maximizes the gain for both nations.

In the first part, in addition to defining the currency demand functions and illustrating the relation between them based on the currency exchange rate, we also provide a general solution to the gain maximization game and find the Nash equilibrium point for both nations. It is the solution to the system of equations

$$\hat{e}D\left(\frac{\hat{e}}{\hat{\theta}}\right) = D^*\left(\hat{\theta}^*\,\hat{e}\right),$$

$$D\left(\frac{\hat{e}}{\hat{\theta}}\right) = \hat{\theta}^*\left(1-\hat{\theta}\right)D^{*\prime}\left(\hat{\theta}^*\hat{e}\right),$$

$$D\left(\frac{\hat{e}}{\hat{\theta}}\right) = \frac{\hat{e}}{\hat{\theta}}\left(\hat{\theta}^* - 1\right)D'\left(\frac{\hat{e}}{\hat{\theta}}\right).$$

We also provide the sufficient conditions for the solution to be the Nash equilibrium point

$$\hat{\theta}^{*2}\left(1 - \hat{\theta}\right)e_{\hat{\theta}}D^{*\prime\prime}\left(\hat{\theta}^*\hat{e}\right) - \hat{\theta}^*D^{*\prime}\left(\hat{\theta}^*\hat{e}\right) - \frac{e_{\hat{\theta}}\,\hat{\theta} - \hat{e}}{\hat{\theta}^2}D'\left(\frac{\hat{e}}{\hat{\theta}}\right) < 0,$$

$$\hat{\theta}\left(\hat{\theta}^*\hat{e}_{\hat{\theta}^*} + \hat{e}\right)D'\left(\frac{\hat{e}}{\hat{\theta}}\right) - \left(1 - \hat{\theta}^*\right)e_{\hat{\theta}^*}\hat{e}D''\left(\frac{\hat{e}}{\hat{\theta}}\right) > 0.$$

Next we provide definitions for economically symmetric and asymmetric nations based on [15]. Within this part we provide some special cases of functions and find the equilibrium points for those functions. Namely, we solve the trade gain maximization problem for the following symmetric cases: $D(x) = D^*\left(\frac{1}{x}\right) = (1 + x)^{-2}$ and $D(x) = D^*\left(\frac{1}{x}\right) = (1 - \alpha x)^+, \alpha < 1$. More importantly, we consider an asymmetric case based on $D(x) = \exp(-\delta x)\,, D^*(x) = \left(\alpha x \boldsymbol{exp}(\beta x)\right) \wedge 1$. Here we also illustrate the effect of changing the foreign currency demand functions on the equilibrium point.

Later we review an additional model from [12]. Within this model, we solve a similar problem of trade gain maximization where it depends on completely different parameters compared to the first model. Namely, the gain function consists of the consumer surplus, profits made by the local firms from selling commodities in domestic and foreign markets and the tariff revenue collected by the governments from imports

$$W(t, t^*, h, f, h^*, f^*) = \frac{1}{2}Q^2 + \pi(t, t^*, h, f, h^* f^*) + tf^*,$$

$$W^*(t, t^*, h, f, h^*, f^*) = \frac{1}{2}Q^{*2} + \pi^*(t, t^*, h, f, h^*, f^*) + t^* f$$

Where $Q = h + f$ and $Q^* = h^* + f^*$ are the quantities of commodities produced computed as sums of production for the domestic and foreign markets. t and t^* denote the tariffs. And the profit made by the firms are defined as

$$\pi(t,t^*,h,f,h^*,f^*) = (a-h-f^*)h + (a-h^*-f)f - t^*f,$$
$$\pi^*(t,t^*,h,f,h^*,f^*) = (a-h^*-f)h + (a-h-f^*)f - tf^*$$

Here, the firms produce commodities for domestic and foreign markets. Within this part, we also provide a generalized model by introducing the currency exchange rate as $e = \frac{pf^*}{p^*f}$, which redefines the firm profit and gain functions of the foreign nation as

$$\pi^*(t,t^*,h,f,h^*,f^*) = e(a-h^*-f)h^* + e(a-h-f^*)f^* - etf^*$$

and

$$W^*(t,t^*,h,f,h^*,f^*) = \frac{1}{2}eQ^{*2} + \pi^*(t,t^*,h,f,h^*,f^*) + etf^*$$

and find the Nash equilibrium values for production to be

$$\hat{h} = \frac{a-f^*}{2},$$
$$\hat{h}^* = \frac{a+t^*}{3},$$
$$\hat{f} = \frac{a-2t^*}{3},$$
$$\hat{f}^* = \frac{a-h-t+\sqrt{(a-h-t)^2+3(a-h^*-f)h^*}}{3}.$$

Unlike the first model, here the currency exchange rate is not taken as a function of tariffs. Lastly, the proofs of some solutions are provided in appendixes accompanying the work.

Game theory has long been used and is still a widely used tool for problems of competition. Approaches vary depending on the nature of the problem. The outcome of the optimization is the best possible solution for both nations taking into consideration the potential response from another nation. Actions these nations can take are setting tariffs on imported commodities in response to each other. This situation is sometimes referred to as "trade war". The timing of the game in both models is as follows. Both nations trade the quantities based on the necessities they have. Since they do not cooperate, one nation unilaterally sets a tariff on imports and affects the imported quantities of commodities. So this either drops or raises the gain from trade depending on the tariff rate. In response, the second nation chooses a tariff rate that does the same for itself, however the Nash equilibrium pair of tariffs if respected, ensures that they get the most benefit out of the trade.

There are certain assumptions behind the models. Both models assume that the availability of imports does not change regardless of the imposed tariffs. Furthermore, we assume that the prices are not affected by the changed demand. In both cases, the economic intuition would suggest the opposite. Besides that, there are numerous specific mathematical assumptions provided throughout the text.

The novelty of the approach examined in this thesis is to make the commodities and their prices random and solve the gain maximization problem under the Nash's sense. Greatest mutual benefit is achieved when nations cooperate and pursue a free trade policy. However, here we assume the non-cooperative game, so they determine the optimal tariffs which results in greatest benefit for both parties.

The second part of the paper explores the portfolio optimization in three dimensional framework. Portfolio selection theories have gone through various improvements since the introduction of its most prominent theory by Harry Markowitz [18] in 1952. He was first to introduce the risk-return principle with the well-known Mean-Variance framework. The basic idea is to arrive at an efficient frontier curve of risky assets by minimizing volatility for given expected returns. It is shown that taking more than one risky position can eliminate some portion of risk as an investor realizes the effect of diversification. Volatility as a risk measure is ideal when portfolio returns are normally distributed. However, when dealing with asymmetric distributions, it simply leads to misinterpretation of risk. Furthermore, in most of the cases, especially during abnormal economic states, history shows that markets do not follow the logic of normal distribution. Measuring risk by volatility penalizes losses equally to profits of the same magnitude. However, investors are more concerned with a downside risk rather than simple volatility so that they are aware of the worst-case scenario that can be realized with a high degree of confidence. In addition, while aiming to select less correlated assets is a rational approach, there are some downsides we focus on. Slight change in correlation can cause significant change in MRL. At the same time, one may allocate funds into assets in proportions, which while being optimal in Mean-Variance sense, can cause hitting the MRL level faster by having overlooked one important factor – expected time of the portfolio return process towards the minimum level. This may be a source of severe problems for investors who are exposed to margin calls or need to raise funds in a short period of time if such an event is realized. To account for the problem of measuring downside risk rather than simple dispersion, Value at Risk (VaR) is used. VaR is a worst loss that can occur with a high confidence level. While this approach is a step to the right direction when it comes to assessing worst possible risk that can be realized, it still lacks one important factor – expected time when the returns hit the lowest possible value at some confidence level. This is critically important for portfolios exposed to mark to marketing or margin calls. Adding this third dimension makes most of its sense when the portfolio volatility is large enough to cause the expected hitting time move before the investment horizon. In this scenario, one can differentiate portfolios by taking into account the expectation of hitting time bounded by the investment length. In case when portfolio variance is sufficiently low, the expectation of bounded hitting time coincides with the investment horizon and becomes an ignorable factor and an investor can stay within the two-dimensional Mean-MRL framework. Lack of historical data or the complexity of parameter estimation sometimes forces investors to apply non-parametric methods.

Our aim is to construct a model which delivers the best performance in the sense that safety is taken as a priority. In order to concentrate on the contribution of the paper, we use Minimum Return Level as a risk measure instead of VaR or ETL (Expected Tail Loss). Once having MRLs and portfolio expected returns computed for different sets of asset weights, we extend the framework by introducing expected first passage time bounded by investment horizon as a third dimension used for decision making. This is done by computing the expectation of the minimum between the investment horizon and the first passage time of portfolio return process towards the minimum level. Once all three quantities for a given set of portfolio weights are in place, we define the best combination of them by maximizing MRL and the expected bounded first passage time for a given expected return of a portfolio. The ultimate result is the efficient surface of risky portfolios. This can be regarded as the three-dimensional analogue of the efficient frontier in classical Mean-Variance model. As a comparison to the Mean-Variance model, while this model might suggest holding a certain weights in assets allocated within a given portfolio, the Mean-MRL-FPT model may reject it altogether and find a set of weights which outperforms in 3-dimensional sense. In addition, it is quite possible that the optimal portfolio weights found by the Mean-Variance framework produces a negative drift which is avoided by the Mean-MRL-FTP model. In a highly volatile environment, portfolio of assets selected by the Mean-Variance framework will hit the lowest possible return level earlier than the portfolio selected by Mean-MRL-FTP framework and at the same time, the latter includes the risk measured by variance as it is reflected in computation of MRL. So, there is a double benefit from applying MRL and FTP when the available assets are volatile enough.

This chapter is structured in four main sections. The first section examines the differential equations which represent the multi-dimensional Ito's processes and constructs the portfolio wealth process. In particular, we take $n-$dimensional Ito's process which is a vector of asset prices $S^* = (S^1, \dots, S^n)^T$ driven by $n-$dimensional Brownian Motion $B = (B^1, \dots, B^n)^T$, where $B^i = (B_t^i, t \geq 0)$ be the real valued Brownian Motion which starts from 0 on(Ω, F, P):

$$dS^i = S_t^i(\mu^i dt + \sigma^{i1} dB_t^1 + \dots + \sigma^{in} dB_t^n)$$

where μ^i is the drift coefficient and $\sigma^{ij}, j = 1, \ldots, n$ is an element from the row vector $(\sigma^{i1}, \ldots, \sigma^{in})$. Then we define the portfolio wealth process V_t corresponding to self-financing portfolio to follow the differential equation:

$$dV_t = \theta_t^1 dS_t^1 + \cdots + \theta_t^n dS_t^n$$

where θ_t^j denotes the weight of j^{th} asset within the portfolio. Ultimately, we obtain the portfolio wealth process to be driven by the Brownian motion as follows

$$V_t = V(0)\, \boldsymbol{exp}\big(\tilde{\mu}t + \tilde{\sigma}\, \tilde{B}_t\big).$$

Within this section, it is shown that in order for the portfolio wealth to drop to its minimum level, the Geometric Brownian Motion that determines the portfolio wealth must reach the level which we call the Minimum Return Level. This brings us to the next, second section where the MRL is formally defined as a quantile from the probability distribution function of portfolio rates of returns as follows

$$m = F^{-1}(\alpha)$$

with F a probability distribution function of portfolio rates of returns and $1 - \alpha$ as a confidence level. The third section overviews the third dimension of the model – expected bounded First Passage Time towards MRL. This value consists of two parts – the term involving the probability density function and the cumulative probability function of the First Passage Time

$$E[\tau_m \wedge T] = \int_0^T t f_{\tau_m}(t) dt + T[1 - P(\tau_m \leq T)]$$

where τ_m is a first passage time of V_t to the level m, T is the investment horizon and f is the normal probability density function. The final part, section four deals with the model construction. It combines all the three dimensions and obtains the efficient surface of risky portfolios.

Finally, the last sections provide examples of optimal portfolios consisting of two and multiple assets respectively. In particular we review a portfolio consisting of two assets whose prices are observed in dynamics and construct the set of portfolios based on the three dimensional model described above. Later we add a multi-asset portfolio (consisting of four assets) and solve the similar problem inn three dimensional framework and as a comparison we include portfolios from mean-variance framework. In this model the portfolios have expected bounded first passage times all set within the investment horizon. Later we examine an example of a portfolio consisting of three assets whose expected bounded first passage times are equal to the investment horizon. So the portfolios are almost never expected to reach the minimum return level with a high confidence level. In this case, the third dimension can be dropped altogether and switch to the Mean-Variance or Mean-MRL framework. This effect is illustrated on the efficient frontier from the Mean-Variance framework. Numerical techniques are used in computation of the third dimension for portfolios. As a final note, all the examples examined at the end of the thesis are included in the accompanying spreadsheets. Computations are examined in details throughout the text.

Literature Review

Game theory is a discipline studying an interaction of decision makers assuming their rationality. The theory was introduced as a completely new interdisciplinary research by Neumann and Morgenstern [1], [2]. This paper set the philosophical foundation and the mathematical model in combination. Particularly, the paper was motivated by the idea of two-person zero-sum game. The game is called zero-sum if the utility gain of one player is completely eliminated by the utility loss of another. So the participants' wins and losses cancel out each other. The theory was initially applicable in only a few disciplines. Later on, the theory went through the development phase by the original authors as well as many others. Currently there are countless applications of the theory in many disciplines. Namely, the theory is widely applicable in economics, biology, logic and many areas of social and computer sciences.

The decisive development of the theory was a very short paper by Nash [3] in 1950. This paper defined an approach to arrive at an optimal point for competitors at the strategic decision making process. Nash found out that the maximum mutual benefit in a game is attained when the participants cooperate. Furthermore, he concluded that acting based on the selfish interest, leads to an optimal solution commonly referred to as the Nash equilibrium.

Nash [4] developed the model he originally initiated by introducing the bargaining problem in a game. This model still considers two players and deals with sharing the surplus they jointly generate. Nash introduced uncertainty about the utilities attained by each player based on their decisions. They still do not cooperate and make rational decisions. However they might be unaware of the utilities which are feasible following their actions. He claimed that even in this situation, the decisions made by the players again lead to a unique solution known as the bargaining equilibrium. In addition, the model is clearly formulated and axiomatized in the sense that, the conditions are set which have to be satisfied by the solution of the game.

In 1951, Nash [5] developed an idea of two person zero sum game which can be regarded as a generalization of a model introduced by Neumann and Morgenstern described above. This model deals with a scenario where two persons do not cooperate and make rational decisions according to the anticipated utilities they have.

Nash [6] explores the bargaining problem in a more generalized form. In particular, this paper considers a model which is an extension of [4]. However this model assumes that the players do not have opposing plans or aims, neither they have coinciding ones. Furthermore, according to the model, they can agree upon a strategy to mutually attain a solution they both benefit from. Thus the paper is entitled as cooperative game.

Merrill Flood and Melvin Dresher formulated a Prisoner's Dilemma in 1950. This is a hypothetical experiment considering two participants of the game acting purely based on selfish interests which do not lead to mutually optimal outcome. The idea behind this scenario is that cooperation leads to a greater utility received by both participants than the one obtained by pursuing purely self-interests ignoring the interests of the competitor.

Our interests rely on the prisoner's dilemma type of game, where two players do not cooperate and make decisions unilaterally without knowing what decision is made by the competitor. In this type of game, it is obvious that the best mutual benefit would be achieved in case of cooperation, however since the players are not able to communicate, they make decisions based on their individual expected benefits. This type of game is of our primary interest since we only consider a game with competitors not cooperating and not pursuing a free trade policy.

In 1973, Smith and Price [7] made an important application of [8] and defined the mathematical concept for the evolutionary game theory (EGT). The theory is based on the biological context and can be thought of as an application of classical game theory to evolving populations in biology. It had a very limited application initially.

The main idea behind the EGT is the strategies of participants constantly evolving based on the interaction in dynamics. The model uses the Markov chains for switching strategies from states to states. The main difference between EGT and the classical game theory is that, the EGT is more concerned about the dynamics of strategy change which itself is largely affected by the frequency of competing strategies in the population. Within the EGT, the participants do not necessarily have rational strategies. They are only required to have at least some strategy. The goodness of the strategy is ultimately checked based on the alternative strategies making the original one either vulnerable or capable to survive, reproduce and evolve.

The seminal work of Smith [9] was published in 1982 followed by Axelrod's book [10] in 1984. Plenty of material from these two books were later on reflected in many works related to the game theory applications in economics and social sciences. In the modern world, the classical and evolutionary game theories are parts of behavioral economics and other fields where the phenomena of rational decision based non cooperative interactions are involved.

McMillan [11] focuses on business and economics related applications of game theory. Here one can find strategies for rational decision maker managers.

Scientists have studied the trade gain maximization problem from different perspectives. R. Gibbons [12] considered a game model in which total welfare of a country consists of an economic surplus enjoyed by consumers, profit earned by firms within a given country and the tariff revenue collected from the imports. Maximization of the total welfare from trade leads to optimal tariff countries involved in trade should impose.

In [13], a closed economy model is considered in which the country consists of a fixed number of households having preferences as a function of consumption and leisure. Within this model, consumption goods consist of intermediate goods that can be produced by units of labor. Under the closed economy model, quantities of each intermediate good and the tariff a given country imposes on imports are optimized.

The model is then extended to a two country model in which there are large and small countries. Large country consists of many consumers while the small country has only one household. Since technologies of production differ across countries, each has different production capacity based on which they obtain relative price levels for the intermediate goods. The difference in relative prices implies the gain from trade.

Ricardian model [14] of two countries under free trade assumes that the large and the small countries get involved in trade. The large country can meet the demands for a specific intermediate good at a relative price of another intermediate good. Therefore, the small country specializes in the production of such a good. The small country can benefit from trade while the large country has nothing to gain.

The Ricardian model is extended by assuming that the large country imposes a tariff on imported goods. Now it can also benefit from trade. So the small country exports the goods and pays the specified tariff, after which it purchases the intermediate good from the large country. The large country has to optimize the tariff.

J. T. Schwartz [15] considered a model of trade gain maximization where the commodities produced and the prices for those commodities are static. In addition, gain from trade is determined to be the difference between the values of imported and exported commodities measured in national currency. Since importing those commodities which cost less under the national price system is regarded as a benefit for both nations, gain from competitive trade for a given nation is considered to be the difference between the advantage it took over the competitor and the advantage the competitor took over it, thus the difference between imports and exports measured at national currency. The Schwartz's model solves the tariff optimization problem for two nations which are said to be economically symmetric, meaning they have equal demands for each other's currency under a given exchange rate. Additionally, Schwartz considers a special scenario where the currency exchange rate is fixed by one nation's central bank in order to fulfill some economic purposes.

In [17], tariff optimization problem is examined based on maximization of gain function. Two nations involved in non-cooperative trade game is examined and the Nash equilibrium triple of values are obtained. Namely tariffs imposed by each nation on imports of a competitor and currency exchange rate. System of equations solving the trade gain maximization problem is clearly formulated based on abstract foreign currency demand functions for each nation and the ultimate results are illustrated with special examples of these functions.

Within the paper, gain functions for each nation involved in non-cooperative trade are defined. For these functions, the maximization problem is solved under Nash's sense. Next, the exchange rate is defined as a solution to an equation matching the foreign currency demand functions. This leads to the system of equations involving a pair of foreign currency demand functions which play a key role in determining the strength and economic power of a given nation relative to its competitor. Additionally, the paper defines a notion of symmetry of economies and offers special cases of the pairs of foreign currency demand functions by which the trade gain maximization problem is solved. The functions correspond to symmetric and asymmetric cases separately.

Obviously, there are many more models and ideas regarding the definitions of gain from trade and what components should it include. Some of the components are qualitative in nature, like leisure or units of labor measured in qualitative means. Basically, all trade scenario can be transformed as a game as long as there is a strictly defined trade gain function and some control variables like a tariff imposed on imported commodities, by manipulating which, benefits from gain is changing for each player.

Portfolio theory began with the development of ideas which were mostly qualitative in nature. In the beginning of 20th century investors used to mostly put their attention to some qualitative measures of performance. While this approach overlooked may important factors widely known today, it was one of rare options available in the science field by those times. Some of the qualitative measures are still commonly used today.

One of the first attempts to quantify the factors influencing the portfolio performance was introduced by Williams [16] in 1938. In that time, information flow about stocks was too slow and investors simply used to bet on the prices which they thought were at their best. Williams captured time as an important variable to introduce in portfolio construction process. He focused on dividend discount model.

It was until 1952 years that this model was one of the rare quantitative options investors had. In 1952, Harry Markowitz [18] proposed an important idea of portfolio selection, later named as modern portfolio theory. Within this model, risk-return combination is clearly illustrated by the use of quantitative variables only – expected return and variance. This model illuminates the idea of diversification and explains the benefits of holding multiple assets having the same expected return but low correlation. So holding multiple risky assets can eliminate the portion of risk which would be impossible by holding a single asset with the equivalent return. The concept of efficient frontier of risky assets played a significant role in understanding the combination of risk and return. This paper gave rise to the completely new field now known as quantitative finance.

In [19] an efficient diversification scheme is examined. Importance of negatively correlated asset returns is outlined in a portfolio consisting of large number of securities. The effect of correlations on overall portfolio risk measured by variance is illustrated and the benefits of dividing up an investment are shown.

Later developments of the model were seen as the Capital Asset Pricing Model developed by several economists independently. Most prominent of those is Sharpe's paper [20]. Here he proposes a model of market equilibrium taking the risk measured by volatility into account. Introduction of beta was a valuable tool for capturing the relation of an individual asset's risk to the market. Later on, beta found its applications in many other academic researches, most notable of which are the theories related to risk measured as Value at Risk which we later apply. The model can be thought of as an important extension of the Markowitz theory. Although beta is a measure estimating the linear relation of an individual asset's sensitivity to the market, there arose a need for downside risk measure interpreting a risk as a threat to lose. Put another way, risk should have been regarded as a possibility that the returns drop to a certain threshold level making the portfolio (or an individual asset) lose its value.

Portfolio optimization approach with Mean-Minimum Return Level (MRL) – Expected Bounded First Passage Time Framework (FPT) is introduced in [21]. The paper begins with motivation under introducing the FPT as a third dimension for optimal portfolios. The three dimensional model examined within this paper takes the investment horizon into account and computes FPT accordingly. Ultimately, the efficient surface of risky portfolios is obtained. The aim is to construct a model which delivers the best performance in the sense that safety is taken as a priority. In order to concentrate on the contribution of the paper, MRL is taken as a downside risk measure which replaces standard dispersion measures. Once having MRLs and portfolio expected returns computed for different sets of asset weights, the framework is extended by introducing expected First Passage Time bounded by investment horizon as a third dimension used for decision making. This is done by computing the expectation of the minimum between the investment horizon and the First Passage Time of portfolio return process towards the minimum level. Once all three quantities for a given set of portfolio weights are in place, the best combination of them is defined by maximizing MRL and the expected bounded first passage time for a given expected return of a portfolio. The ultimate result is an efficient surface of risky portfolios. This can be regarded as the three-dimensional counterpart to efficient frontier in classical Mean-Variance model.

The paper is structured in four main parts. The second section examines the differential equations which represent the multi-dimensional Ito's processes and constructs the portfolio process. Within this section, it is shown that in order for the portfolio wealth to drop to its minimum level, the Geometric Brownian Motion that determines the portfolio wealth must reach the level which we call the Minimum Return Level. This brings us to the next, third section. In this section the MRL is formally defined according to its probability function. The fourth section overviews the third dimension of the model—expected bounded First Passage Time towards MRL. This value consists of two parts—the probability density function and the cumulative probability function of the First Passage Time. The final part, section five deals with the model construction. It combines all three dimensions and obtains an efficient surface of risky portfolios.

On a final note, as far as applicability of the model is concerned, it is obviously impossible to continuously rebalance the portfolio in order to maintain the constant weights. However, one can adopt some discretization methodology to find the optimal interval for making trades and taking transaction costs into account at the same time.

Value at Risk (VaR) was introduced by JP Morgan in early 1990s. Since then, it has become a major benchmark instrument in the hands of financial institutions and regulators for measuring risk. Some theories appeared in the late 90s which promoted application of VaR and MaxVaR in portfolio management. Bookstaber [22] published a paper with some critical values about classical risk management. In 2004, Boudoukh et al. [23] did research about computing long horizon VaR for portfolios exposed to mark to marketing. In this paper it is shown that VaR is a very useful measure of risk in a mark to market environment and the way to compute it is explained. Basically, VaR is a statistical measure. Specifically, a quantile of losses at some confidence level indicating the highest possible loss that can be incurred in the worst-case scenario. There have been numerous methodologies for computing VaR in different circumstances. Expected Tail Loss (ETL, aka Expected Shortfall), defined as the average loss beyond VaR is a coherent risk measure according to Artzner et al. [24] and is widely used in risk calculation and portfolio optimization problems. This paper provides a list of axioms a risk measure must satisfy in order to be coherent. Classical Markowitz optimization technique was translated into Mean-VaR (or Mean-ETL) framework and the usefulness of ETL was examined by Rockafellar et al. [25], where volatility is replaced by VaR (or ETL) and optimization is done based on minimization of VaR (or ETL) and maximization of expected returns of portfolio.

As a result VaR played an important role in the development of probability based risk models. Today, there are many works related to VaR computation methodologies. VaR is computed based on parametric models as well as empirical models. Popular methods for computing the portfolio VaR is a copula based approach. Copula is a joint probability distribution function of uniformly distributed random variables on a unit square. It has become a powerful tool for simulation techniques. However it still exposes its weakness when it comes to estimate the copula function for high dimensions. Parametric family of copula functions called Archimedean copulas offers a wide range of functions with different dependence structures. Theory of copulas is new and still an emerging field. A good reference to a classic book is [26].

In [27], optimal portfolios with two assets are examined. Copula functions are applied to model the dependence structure between returns of assets as random variables. Portfolio with two assets are taken and an Archimedean copula is chosen to fit the data. Copulas are used to jointly simulate the returns of the assets. Ultimately different portfolios are obtained based on the Conditional Value at Risk and expected return as a combination.

Value at Risk for regulators provides a tool for forcing financial institutions to maintain a certain threshold capital idle in order to ensure safety. [28] is a good reference exploring the application of VaR as a tool for measuring safe level of Capital Requirement. This paper examines the portfolio policies adopted by expected utility maximizing agents under Value at Risk Capital Requirement regulation compared with exogenously imported VaR Limit and Limit Expected Loss regulations. There is a trade-off between the threshold capital required to maintain the solvency and health of a financial institution and the effective management of capital. The results obtained makes the Basel regulations more optimal and rational from the standpoint of regulators on the one hand, and institutions on the other.

Despite the fact that Value at Risk is computationally simple and numerous models developed so far give satisfactory results, Engle and Manganelli [29] proposed a new estimation of a quantile level of future portfolio values conditioned on current information named Conditional Value at Risk by Quantile Regression. This model does not require assumption that portfolio returns are from some particular distribution or they are independently and identically distributed. Within the paper, portfolio risks are defined based on this new measure and the advantage of the approach is supported by examples constructed by evolutionary generic algorithms producing empirical evidence of this methodology being able to adapt to new risk environment.

In [30], Engle and Manganelli extend [29] and show that the historical simulation method which they provide is just a special case of CAViar framework. In addition, they introduce the extreme value theory for CAViar and compute the expected loss as a quantile level conditioned by the VaR level. So expected loss within the CAViar framework is defined similarly as for ordinary VaR. The performance is then checked by Monte Carlo simulation.

Statistical theory of extremes is used in [31] to justify it being more natural and robust approach in risk management computations. Specifically, this paper deals with extreme tails and uses probability first as a measure of extremeness of events and then to determine the proper threshold level for capital.

In [32], extreme stock price movements are presented. The author investigates the economic booms and crashes followed by abnormal stock price movements. The approach defined is supported by the example consuming data from the most traded stocks on the New York Stock Exchange from 1885-1990. Finally, it is shown empirically that returns of stocks in during extreme cases follow a Frechet distribution.

Chen et al. [33] proposes the portfolio optimization problem based on semi variance of uncertain variables. Within this model, the returns of assets are estimated based on experts' subjective views. Models like uncertain semi-variance have parameters which are hard to quantify, but in uncertain situations subjective views are useful or at least the only solution. Closely related idea to the uncertain semi- variance model is the semi-absolute deviation model proposed by Qin et al [34]. Within this paper, authors examine the portfolio selection by several mean-semi absolute deviation adjusting models to measure tradeoff between risk and return. Views about the asset returns are obtained from expert opinions like in semi-variance model.

The concept of Brownian motion arose from an experiment by a British botanist, Robert Brown in 1828 where he observed irregular movement of suspended pollen grain in water. Water molecules cause motion of the pollen grain which is described by the famous Brownian motion model, otherwise known as the Wiener Process [35]. Wiener constructed the first mathematically rigorous description of Brownian motion.

Levy [36], [37] is credited with the discoveries of important properties of Brownian motion. Within these works, some quite non-intuitive properties are presented.

More of famous works dedicated to the topics described above are listed in the bibliography for a convenient reference.

Chapter 1 Equations for Nash-equilibrium tariffs and exchange rate

1.1 Foreign Currency Demand Functions

This chapter deals with generalization of a model proposed by Schwartz [15]. Let us assume two nations exchange N different commodities for which the demand and prices are known. For the domestic and foreign nations, annual demand and corresponding prices measured in national currency are $d_1, \dots, d_N, p_1, \dots, p_N$ and $d_1^*, \dots, d_N^*, p_1^*, \dots, p_N^*$ respectively. If we take x as an exchange rate of a unit of foreign currency in terms of domestic currency units, then the domestic and foreign nations' demand for foreign currency are given by

$$D(x) := \frac{1}{C_N} \sum_{k=1}^{N} \bar{E}(p_k^* d_k, \frac{p_k}{p_k^*} > x) \tag{1.1}$$

and

$$D^*(x) := \frac{1}{C_N^*} \sum_{k=1}^{N} \bar{E}(p_k d_k^*, \frac{p_k}{p_k^*} < x) \tag{1.2}$$

respectively, where $C_N = \sum_{k=1}^{N} \bar{E}(p_k^* d_k)$, $C_N^* = \sum_{k=1}^{N} \bar{E}(p_k d_k^*)$ and $\bar{E}$ is the mathematical expectation under $\bar{P}$ on a probability space $(\bar{\Omega}, \bar{F}, \bar{P})$. If we introduce the extended probability space (Ω, F, P), where

$$\Omega = \bar{\Omega} \times \{1, \dots, N\}, P(A, k) = \frac{1}{N}\, \bar{P}(A), A \in \bar{F}$$

and define random variables p, p^*, d, d^* by

$$p(\omega, k) = p_k(\omega), p^*(\omega, k) = p_k^*(\omega),$$

$$d(\omega, k) = d_k(\omega), d^*(\omega, k) = d^*(\omega),$$

then (1.1), (1.2) demand functions above can be rewritten as probability distribution functions

$$D(x) = E\left(p^* d, \frac{p}{p^*} > x\right), \quad D^*(x) = E\left(p d^*, \frac{p}{p^*} < x\right) \tag{1.3}$$

which indicate that the domestic nation will import the commodity if $\frac{p}{p^*} > x$ and the foreign nation will import if $\frac{p}{p^*} < x$. Since x is the value of an unit of foreign currency in terms of the domestic currency units, increasing the exchange rate makes foreign commodities more expensive for the domestic nation and the domestic commodities less expensive for the foreign nation. Therefore, D is a decreasing function of x and D^* is an increasing function of x. These functions have the following properties

$$D(0) = 1, \qquad D(\infty) = 0, \qquad D^*(0) = 0, \qquad D^*(\infty) = 1.$$

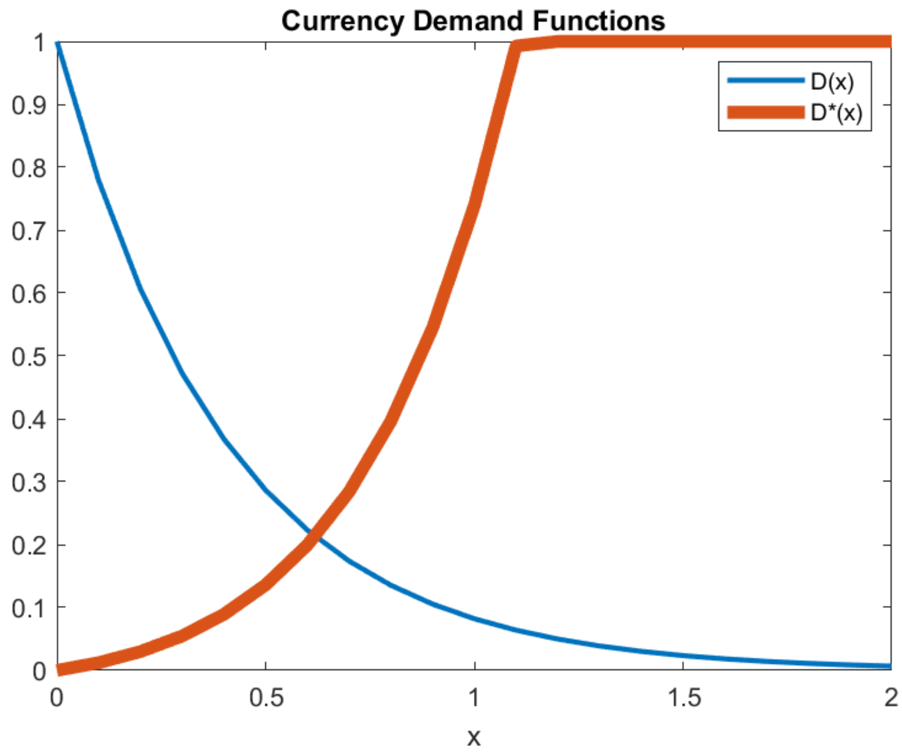

Figure 1.1: $D(x)$ and $D^*(x)$

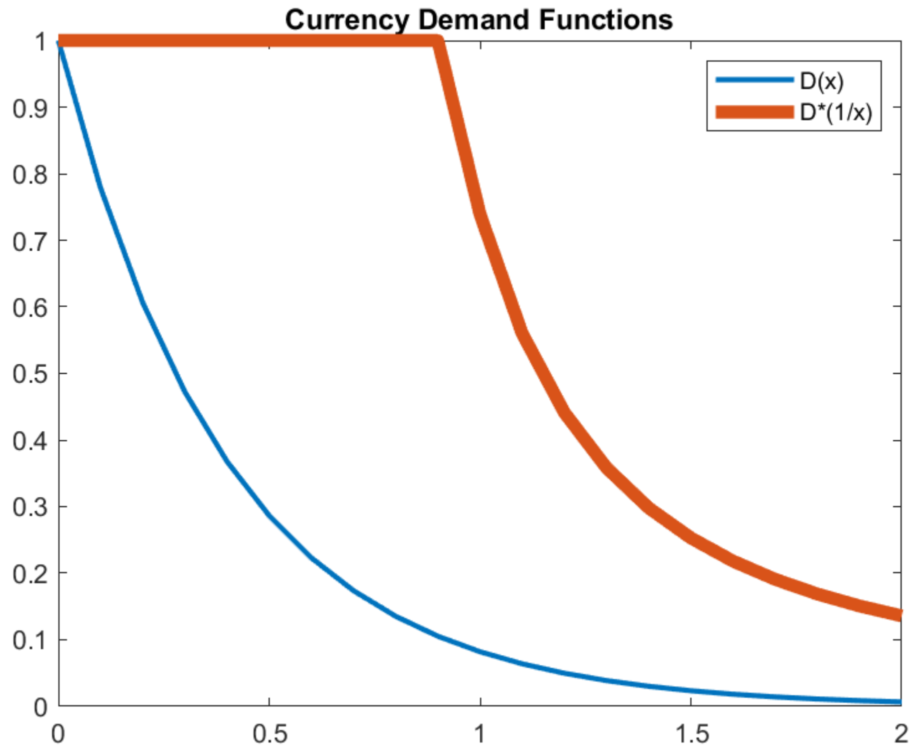

Figure 1.2: $D(x)$ and $D^*(1/x)$

From (1.1) and (1.2) it is clear that the foreign currency demand functions are derived from the known probability distributions of prices and quantities of the exchanged commodities. Our main interest lies in the shape of these functions, not how they are obtained. So we mostly focus on the ready-made functions themselves and introduce some special cases of them carrying intuitive economic interpretations. From the practical perspective, one should obtain the functions from the equations above. While it is true that the economic strength and dominance of one nation over another is determined by these functions, as we will see later, from these functions alone, it is impossible to anticipate which nation will be able to impose a greater tariff over the competitor.

1.2 Introducing Currency Exchange Rate

For an exchange rate x, solving the equation

$$xD(x) = D^*(x) \tag{1.4}$$

for x yields the equilibrium rate $x = e$.

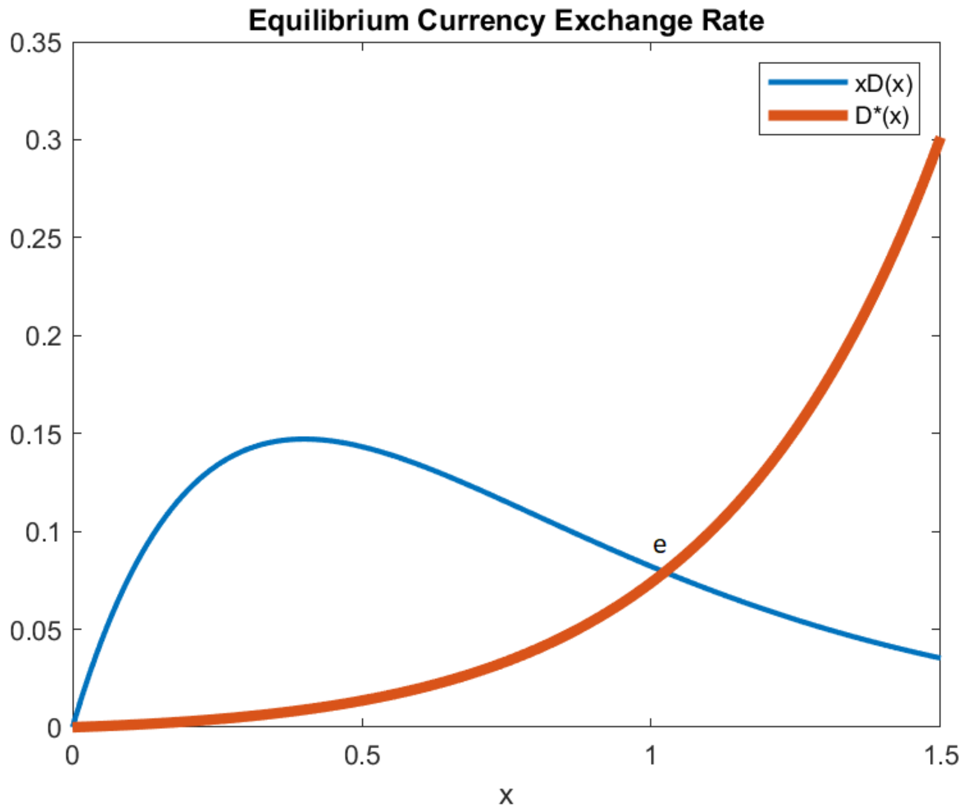

Figure 1.3: Solution to $xD(x) = D^*(x), D(x)$ and $D^*(x)$ are taken arbitrarily

This equation determines the equilibrium exchange rate when both nations practice an unrestricted free trade policy. Left side of the equation is the foreign currency demand of a domestic nation and the right side is the foreign currency demand of a foreign nation, both measured in domestic currency units.

Now suppose the domestic and foreign governments impose the following tariffs on imported commodities: $1-\theta$ and $1-\theta^*$. Then the domestic nation will import the commodity if $\frac{p\theta}{p^*} > x$, and the foreign nation will import if $\frac{p^*\theta^*}{p} > \frac{1}{x}$. Taking tariffs into account, the demand functions (1.3) now become

$$D\left(\frac{x}{\theta}\right) = \frac{E(p^* d 1_{\{\theta p > x p^*\}})}{E(p^* d)}, \quad D^*(x\theta^*) = \frac{E(p d^* 1_{\{\theta^* p^* x > p\}})}{E(p d^*)},$$

where

$$E\left(p^* d 1_{\{\theta p > x p^*\}}\right) = \frac{1}{N} \sum_{k=1}^{N} \bar{E}\left(p_k^* d_k 1_{\{\theta p_k > x p_k^*\}}\right),$$

$$E\left(pd^*1_{\{\theta^* p^* x > p\}}\right) = \frac{1}{N}\sum_{k=1}^{N} \bar{E}\left(p_k d_k^* 1_{\{\theta^* p_k^* x > p_k\}}\right),$$

$$E(p^*d) = \frac{C_N}{N}, \quad E(pd^*) = \frac{C_N^*}{N}$$

and the relation (1.4) is rewritten as

$$xD\left(\frac{x}{\theta}\right) = D^*(\theta^* x) \tag{1.5}$$

from which it is clear that the equilibrium exchange rate $x = e$ now depends on θ and θ^*. Equation (1.5) always has the solution $e = 0$, $\frac{1}{e} = 0$, or $\theta = \theta^* = 0$, which do not carry any useful economic sense. Such conditions would restrict the involvement of both nations in trade. To rule out these possibilities, we claim $\frac{1}{M} \leq e \leq M$, for some large number M and $\frac{1}{M} \leq \theta \leq 1$, $\frac{1}{M} \leq \theta^* \leq 1$.

One important note here is that, although we defined the upper and lower bounds for the tariffs that can be imposed by both nations, and the mathematics is fine with any solution within that range, normally the numbers θ and θ^* are close to 1. In realistic scenario, it is hard to find a number significantly less than one. Regardless of that, we do not restrict ourselves to obtain numbers which are very realistic. Our goal is to obtain numbers those fit well within the mathematical restrictions and give a good meaning to the ultimate results as a whole.

1.3 Gain Functions

Since the ultimate goal of both nations is to set the tariffs unilaterally which will maximize their gain from trade, we have to find the Nash equilibrium point, the pair $(\hat{\theta}, \hat{\theta}^*)$. The gain functions of each nation are given by

$$\begin{aligned} G(e,\theta,\theta^*) &= E\left(pd, \frac{p^*}{p} < \frac{\theta}{e}\right) - E\left(pd^*, \frac{p^*}{p} > \frac{1}{e\theta^*}\right) \\ &= E\left(\frac{p}{p^*} 1_{\left(\frac{p}{p^*} > \frac{e}{\theta}\right)} p^* d\right) - E\left(pd^* 1_{\left(\frac{p}{p^*} > \frac{e}{\theta}\right)}\right) \\ &= -\int_{e/\theta}^{\infty} yD'(y)dy - D^*(\theta^* e), \end{aligned} \tag{1.6}$$

and

$$G^*(e,\theta,\theta^*) = E\left(p^*d^*, \frac{p}{p^*} < \theta^* e\right) - E\left(p^*d, \frac{p}{p^*} > \frac{e}{\theta}\right) \tag{1.7}$$

$$= \int_{\frac{1}{\theta^* e}}^{\infty} \frac{1}{y} D^{*\prime}\left(\frac{1}{y}\right) dy - D\left(\frac{e}{\theta}\right),$$

respectively. Since the equilibrium exchange rate is the function of tariffs, we have $e = e(\theta, \theta^*)$. Our goal is to find the Nash equilibrium for the nations, i.e. such pair $(\hat{\theta}, \hat{\theta}^*)$. Our goal is to find the Nash equilibrium for the nations, i.e. such pair $(\hat{\theta}, \hat{\theta}^*)$ that relations

$$\max_{\theta} G\big(e(\theta, \hat{\theta}^*), \theta, \hat{\theta}^*\big) = G\big(e(\hat{\theta}, \hat{\theta}^*), \hat{\theta}, \hat{\theta}^*\big),$$

$$\max_{\theta^*} G^*\big(e(\hat{\theta}, \theta^*), \hat{\theta}, \theta^*\big) = G^*\big(e(\hat{\theta}, \hat{\theta}^*), \hat{\theta}, \hat{\theta}^*\big)$$

hold. The Nash pair is found from the system of equations

$$\frac{\partial}{\partial \theta} G(e,\theta,\theta^*) = 0, \tag{1.8}$$

$$\frac{\partial}{\partial \theta^*} G^*(e,\theta,\theta^*) = 0, \tag{1.9}$$

Given the currency demand functions $D(x)$ and $D^*(x)$, solution to the system of equations (1.8),(1.9) leads to yet another system of equations (see Appendix A)

$$D\left(\frac{e}{\theta}\right) = \theta^*(1-\theta) D^{*\prime}(\theta^* e) \tag{1.10}$$

$$D\left(\frac{e}{\theta}\right) = \frac{e}{\theta}(\theta^* - 1) D'\left(\frac{e}{\theta}\right) \tag{1.11}$$

Remark: According to (1.5), $D\left(\frac{e}{\theta}\right) = \frac{D^*(\theta^* e)}{e}$. Then (1.10) can be rewritten as

$$D^*(\theta^* e) = e\theta^*(1-\theta) D^{*\prime}(\theta^* e) \tag{1.12}$$

Denoting $\bar{e} = \frac{1}{e}$, $\bar{D}(x) = D^*\left(\frac{1}{x}\right)$, (1.12) now becomes

$$\bar{D}\left(\frac{\bar{e}}{\theta^*}\right) = \frac{\bar{e}}{\theta^*}(\theta - 1)\, \bar{D}'\left(\frac{\bar{e}}{\theta^*}\right),$$

which is similar to (1.11).

Note that θ plays a role in the first component of (1.6) and the second component of (1.7). At the first glance it seems that increasing the value of θ (meaning absorbing less portion from the value of imported commodities) causes the gain of the first nation to decrease and vice versa. Similarly, θ^* participates in the second component of (1.6) and the first component of (1.7). However, since the currency exchange rate e is the function of θ and θ^*, by only observing these equations, it is unclear what values of tariffs will be beneficial for each nation.

1.4 Solution to the Gain Maximization Problem

At this point, if the demand functions for foreign currency of each nation are known, from (1.10) and (1.11) the Nash equilibrium pair $(\hat{\theta}, \hat{\theta}^*)$ can be found. Ultimately putting these values in (1.5) and solving for x will result in the equilibrium triple $\left(\hat{e}, \hat{\theta}, \hat{\theta}^*\right) = \left(e\left(\hat{\theta}, \hat{\theta}^*\right), \hat{\theta}, \hat{\theta}^*\right)$. Hence the triple satisfy

$$\hat{e}D\left(\frac{\hat{e}}{\hat{\theta}}\right) = D^*(\hat{\theta}^*\hat{e}) \tag{1.13}$$

$$D\left(\frac{\hat{e}}{\hat{\theta}}\right) = \hat{\theta}^*\left(1 - \hat{\theta}\right)D^{*\prime}(\hat{\theta}^*\hat{e}) \tag{1.14}$$

$$D\left(\frac{\hat{e}}{\hat{\theta}}\right) = \frac{\hat{e}}{\hat{\theta}}\left(\hat{\theta}^* - 1\right)D'\left(\frac{\hat{e}}{\hat{\theta}}\right) \tag{1.15}$$

Obviously, one should check whether the extremum points given by the solution to (1.10) and (1.11) are really maximums. Differentiating the derivatives of the gain functions once again and checking the signs for the equilibrium points serve this purpose. So the following inequalities must hold

$$\frac{\partial^2}{\partial\theta^2}G\left(\hat{e}, \hat{\theta}, \hat{\theta}^*\right) < 0,$$
$$\frac{\partial^2}{\partial{\theta^*}^2}G^*\left(\hat{e}, \hat{\theta}, \hat{\theta}^*\right) < 0$$

which means (see Appendix C)

$$\hat{\theta}^{*2}\left(1 - \hat{\theta}\right)e_{\hat{\theta}}D^{*\prime\prime}\left(\hat{\theta}^*\hat{e}\right) - \hat{\theta}^*D^{*\prime}\left(\hat{\theta}^*\hat{e}\right) - \frac{e_{\hat{\theta}}\,\hat{\theta} - \hat{e}}{\hat{\theta}^2}D'\left(\frac{\hat{e}}{\hat{\theta}}\right) < 0 \tag{1.16}$$

$$\hat{\theta}\left(\hat{\theta}^*e_{\hat{\theta}^*} + \hat{e}\right)D'\left(\frac{\hat{e}}{\hat{\theta}}\right) - \left(1 - \hat{\theta}^*\right)e_{\hat{\theta}^*}\hat{e}D''\left(\frac{\hat{e}}{\hat{\theta}}\right) > 0 \tag{1.17}$$

Hence we can formulate our main result: If pair $\left(\hat{\theta}, \hat{\theta}^*\right) \epsilon \left(\frac{1}{M}, 1\right)^2$ is a unique solution (1.13), (1.14), (1.15), (1.16), (1.17), then it is the Nash equilibrium of the game.

As an important note, there can be two kinds of intersections for (1.4). According to Schwartz, the intersection occurring over the decreasing interval of $xD(x)$ gives some useful economic and mathematical insights (see [15]). This is called a normal configuration and was illustrated in Figure 1.3. Another case is the intersection point occurring at an increasing interval of the function. The following figure illustrates this case.

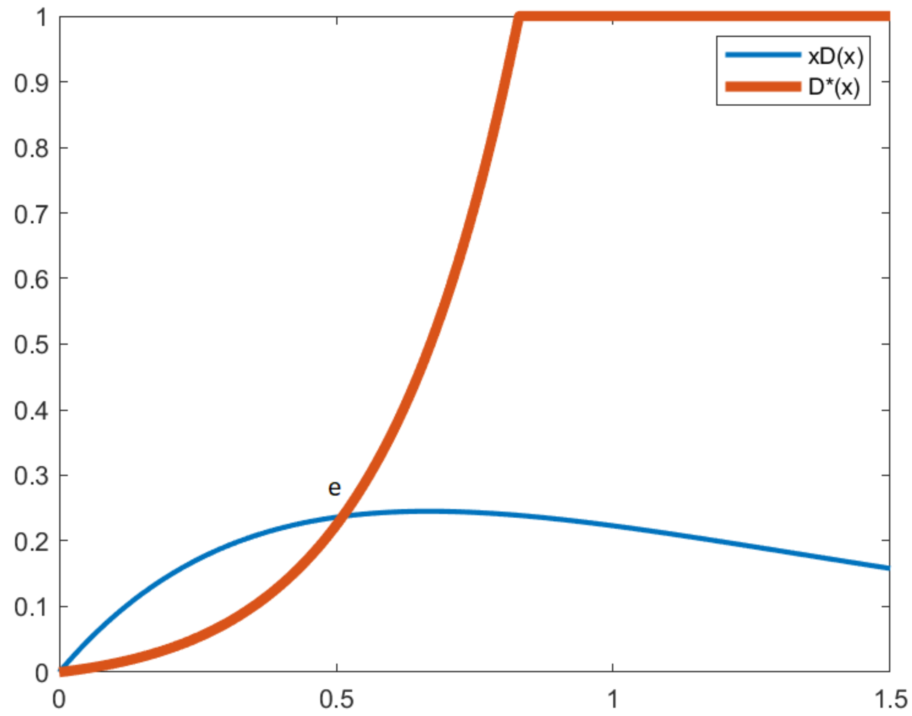

Figure 1.4: Solution to $xD(x) = D^*(x)$, Abnormal configuration

Economically interpreted, this scenario leads to misconception since if both functions are increasing, their derivatives with respect to the exchange rate is positive which does not make sense. The functions must be responding to changing exchange rate in an opposite way.

1.5 Symmetry and Asymmetry of Economies

Demand functions differ from nation to nation. Specifically, two nations are said to be economically symmetric if

$$D(x) = D^*\left(\frac{1}{x}\right), \tag{1.18}$$

which implies that their demand for each other's currency are equal under any given exchange rate. Economically, this means that on average they produce and exchange commodities of equal value. In case of symmetric nations, from (1.5) we can simply conclude that $e(\theta^*, \theta) = \frac{1}{e(\theta,\theta^*)}$. Thus, $e(\theta, \theta^*) = 1$, which makes perfect sense. Since two nations have equal demand for each other's currency, neither is able to employ dominant economic power over the counter party, so the Nash equilibrium will occur at equal tariffs and a unit exchange rate. More rigorously, since $G^*\left(\frac{1}{e(\theta,\theta^*)}, \theta^*, \theta\right) = G(e(\theta, \theta^*), \theta, \theta^*)$, from the result of Game Theory ([38],p.134) follows that $\hat{\theta} = \hat{\theta}^*$ for Nash point $(\hat{\theta}, \hat{\theta}^*)$. This fact simplifies the computations above. Specifically, taking $\theta = \theta^*$ and $e = 1$, (1.11) becomes

$$\theta D\left(\frac{1}{\theta}\right) = (\theta - 1)D'(\frac{1}{\theta}) \tag{1.19}$$

Given the function $D(x)$, the equilibrium pair $(\hat{\theta}, \hat{\theta}^*)$ is found.

However, more realistic case is economically asymmetric nations having different demands for each other's currency. In this case, the equality (1.18) no longer holds. So the nations will have different tariffs imposed on imported commodities.

In [15], the following demand functions for symmetric nations were considered: $D(x) = D^*\left(\frac{1}{x}\right) = (1 + x)^{-2}$. Since two nations are economically symmetric, we have $\hat{\theta} = \hat{\theta}^*, \hat{e} = 1$. We use (1.19) to solve the equation for θ. Given

$$D(x) = D^*\left(\frac{1}{x}\right) = (1 + x)^{-2}$$

the derivative of the function is

$$D'(x) = -\frac{2}{(1 + x)^3},$$

putting it into (1.19) gives

$$\frac{\theta}{\left(1 + \frac{1}{\theta}\right)^2} = (\theta - 1)(-\frac{2}{\left(1 + \frac{1}{\theta}\right)^3}),$$

solving for θ yields

$$-\frac{2\theta}{1+\theta}(\theta-1)=1$$

from which we trivially get

$$\theta=\theta^*=\frac{1}{3}$$

The solution $(\hat{e},\hat{\theta},\hat{\theta}^*)=(1,\frac{1}{3},\frac{1}{3})$ agrees with Schwartz's results. So in case of symmetric nations with the foreign currency demand functions given by $D(x)$ and $D^*(\frac{1}{x})$, we obtained the Nash equilibrium point at equal tariffs to be imposed that maximize the gain for both nations from trade. However, neither is able to tax the competitor by a greater amount than itself being taxed by. In addition, we consider two more examples.

Symmetric Case:

Here we consider one more symmetric case. Suppose $D(x)=D^*\left(\frac{1}{x}\right)=(1-\alpha x)^+, \alpha<1$. Similarly applying (1.19) leads to the following solution. Given

$$D(x)=D^*\left(\frac{1}{x}\right)=(1-\alpha x)^+,$$

the derivative is

$$D'(x)=-\alpha,$$

putting it in (1.19) gives

$$\theta\left(1-\frac{\alpha}{\theta}\right)=(\theta-1)(-\alpha),$$

from which

$$\theta-\alpha=\alpha-\alpha\theta,$$

solving for θ yields

$$\theta=\theta^*=\frac{2\alpha}{1+\alpha}.$$

So the Nash equilibrium is $(\hat{e},\hat{\theta},\hat{\theta}^*)=\left(1,\frac{2\alpha}{1+\alpha},\frac{2\alpha}{1+\alpha}\right)$. Similarly, given any value α, which defines the shapes of the demand functions, the equilibrium point will occur at the same tariffs for both nations.

Asymmetric case:

Now we generalize the problem to a more common asymmetric case. Suppose $D(x) = \mathbf{exp}(-\delta x)\,, \delta > 0, D^*(x) = \big(\alpha x \boldsymbol{exp}(\beta x)\big) \wedge 1. \alpha, \beta > 0$. Then solving (1.5) yields the equilibrium exchange rate

$$e = \frac{-\theta \ln{(\alpha\theta^*)}}{\theta\theta^*\beta + \delta}.$$

The Nash equilibrium condition (1.10), (1.11) gives

$$\theta = \frac{\delta(\theta^* - 1)\ln(\alpha\theta^*) - \delta}{\theta^*\beta}$$

(see Appendix B) and

$$\beta\theta^*(\theta^* - 1) = (\theta^*\beta - \delta(\theta^* - 1)\ln(\alpha\theta^*) + \delta)(\theta^* - (\theta^* - 1)\ln(\alpha\theta^*)).$$

Specifically, if $\alpha = 0.01, \beta = 2, \delta = 2.5$, the Nash equilibrium pint is $\big(\hat{e}, \hat{\theta}, \hat{\theta}^*\big) = (0.81, 0.54, 0.73)$. The equilibrium exchange rate which is the solution of (1.5) is illustrated in Figure (1.5).

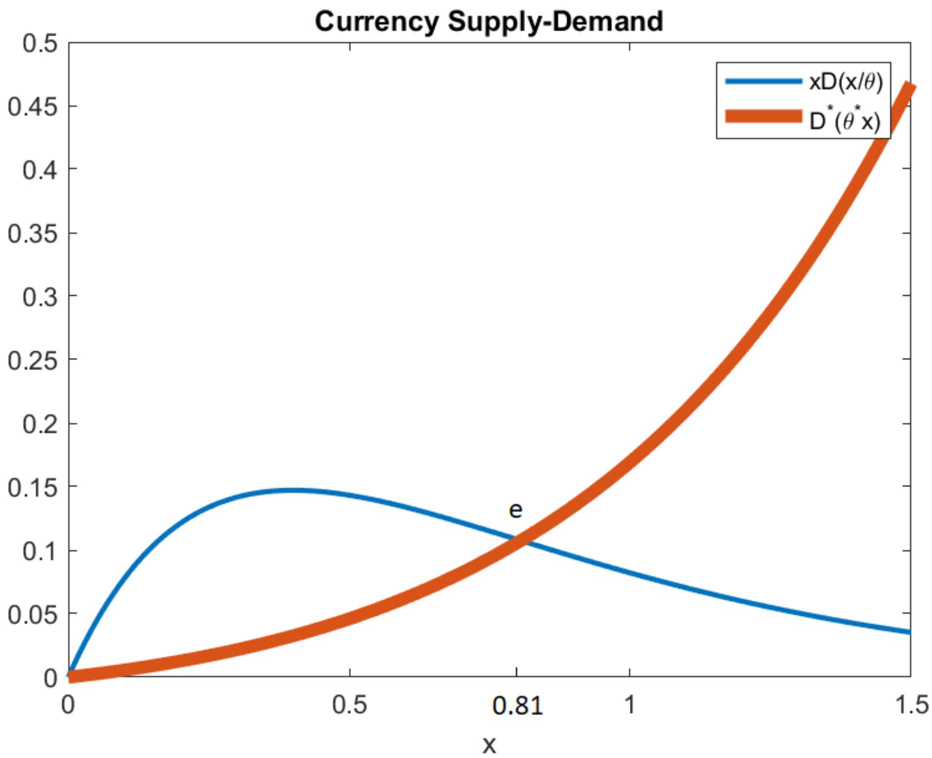

Figure 1.5: Solution to $xD(x) = D^*(x)$ for $D(x) = \boldsymbol{exp}(-2.5\delta)\,, D^*(x) = 0.01x\boldsymbol{exp}(2x) \wedge 1$

Here the Nash equilibrium point was obtained for the case where $D(x) \neq D^*(\frac{1}{x})$. Solution to the system of equations (1.13), (1.14), (1.15) leads to a domestic nation imposing greater tariff than the foreign nation. Initially, based only on the shapes of the functions $D(x)$ and $D^*(x)$, it is impossible to identify which nation is "economically stronger" and therefore will have a greater optimal tariff.

From the asymmetric case here, we intentionally took the functions $D(x)$ and $D^*(x)$ such that solving the system of equations (1.13), (1.14), (1.15) explicitly for the currency exchange rate and at least one of the thetas (in this case θ) were possible. Obviously, unlike the situation above, depending on the functions $D(x)$ and $D^*(x)$, is might be impossible to provide an explicit solution.

1.6 Effects of Currency Demand Functions on Equilibrium Point

Clearly, the economic strength of a nation relative to its competitor is determined by the foreign currency demand function. Economists may give a precise answer to whether greater demand for the competitor's currency is advantageous or not for a given nation over another, however in our model we rely completely on mathematical outcome and we define economic dominant power of a nation over the competitor as the ability to impose a greater tariff according to the system of equations (1.13), (1.14), (1.15). Put another way, a nation is said to be economically dominant over the competitor if after solving the system of equations above, it has the Nash equilibrium tariff greater than that of a competitor. However, this fact is not directly observable from the currency demand functions. It would be desirable to be able to identify patterns defining which nation is stronger economically in that sense by comparing the currency needs they have relative to each other. Ultimately it all depends on the outcome of the system. We illustrate some examples of changing demand functions and their effects on the equilibrium.

Case 1: Stronger demand function for the second nation

Originally we had $D^*(x) = \alpha x \boldsymbol{exp}(\beta x)$ where $\alpha = 0.01$ and $\beta = 2$. Now strengthen the demand of the second nation by making $\beta = 3$. The graphs of $xD(x) = D^*(x)$ would now be slightly different

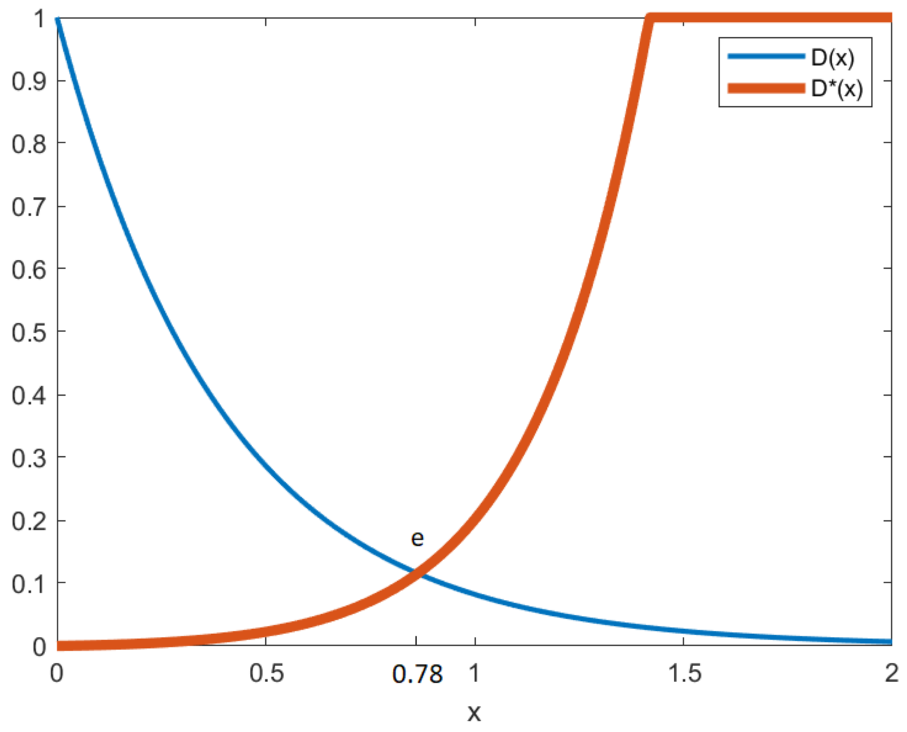

Figure 1.6: $D(x) = \boldsymbol{exp}(-2.5x)\,, D^*(x) = 0.01x\boldsymbol{exp}(3x)$

The Nash Equilibrium point we obtain based on this modified function now is: $(\hat{\theta}, \hat{\theta}^*) = (0.63, 0.69)$. Since the tariffs are defined as $(1 - \hat{\theta}, 1 - \hat{\theta}^*) = (0.37, 0.31)$, we have the first nation still being able to impose a greater tariff than the second nation. Recall that the initial results were $(\hat{\theta}, \hat{\theta}^*) = (0.54, 0.73)$, so $(1 - \hat{\theta}, 1 - \hat{\theta}^*)$ = (0.46, 0.27).

One might have concluded before, that greater foreign currency demand function causes a nation to be trapped by the necessity of the imports from the competing nation and therefore, this gives the competing nation some economic dominant power resulting in greater tariff to be imposed. However this logic does not directly translate into the outcome of the system of equations above.

Case 2: Stronger demand function for the second nation

Now let us take $\beta = 4$ to make $D^*(x)$ even stronger. So, now $D^*(x) = 0.01x\boldsymbol{exp}(4x)$.

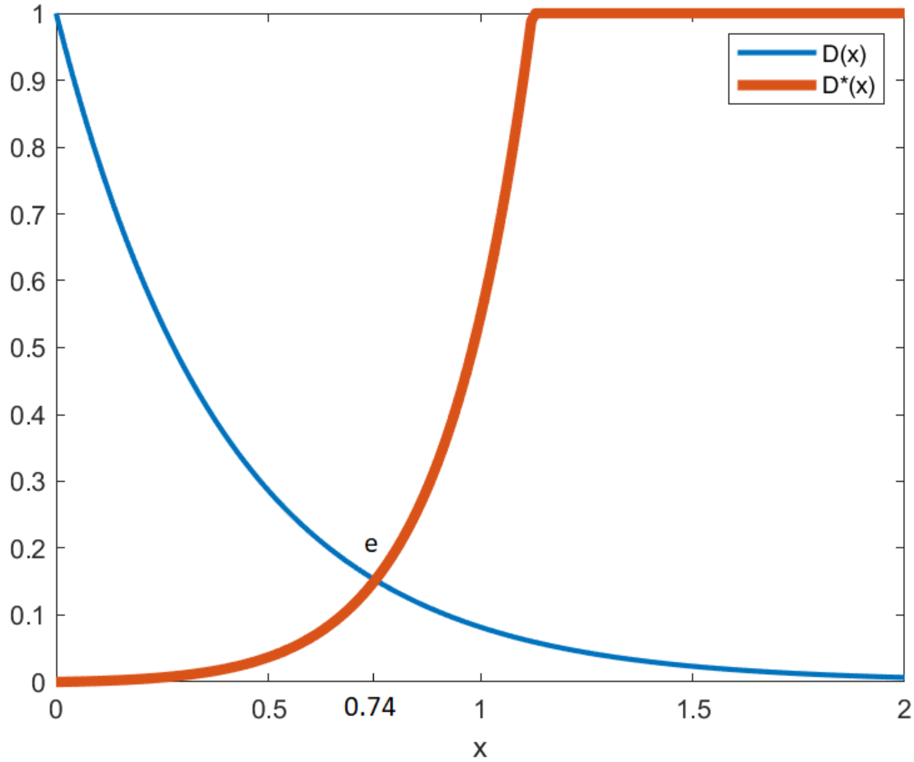

Figure 1.7: $D(x) = \boldsymbol{exp}(-2.5x)\,, D^*(x) = 0.01x\boldsymbol{exp}(4x)$

The Nash equilibrium point now is $(\hat{\theta}, \hat{\theta}^*) = (0.68, 0.66)$. So the tariffs are $(1 - \hat{\theta}, 1 - \hat{\theta}^*) = (0.32, 0.34)$. Here the second nation is already able to tax the competitor by a greater amount. We can conclude that changing the constant parameters of the function, and therefore its shape does not give a predictable answer to what it might result in.

The value of β which would make the tariffs equal for the first and the second nation under these functions is 1.48. In this case we would have $(\hat{\theta}, \hat{\theta}^*) = (0.73, 0.73)$ and $(1 - \hat{\theta}, 1 - \hat{\theta}^*) = (0.27, 0.27)$. The functions are illustrated in the following figure.

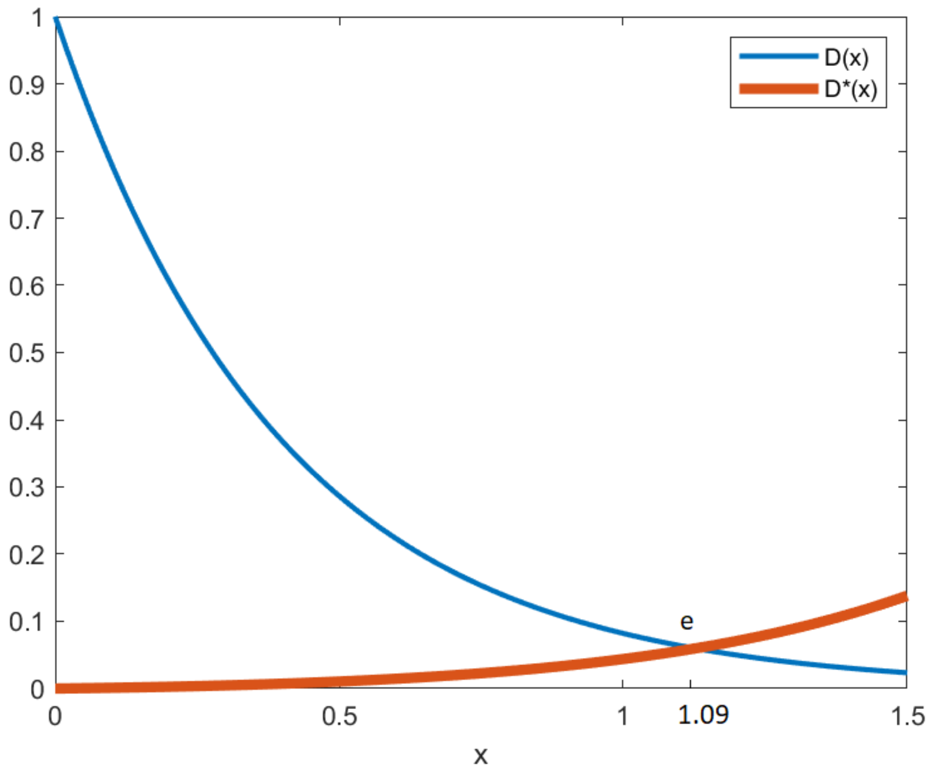

Figure 1.8: $D(x) = \boldsymbol{exp}(-2.5x)\,, D^{*}(x) = 0.01x\boldsymbol{exp}(1.48x)$

1.7 Dominant Foreign Currency Demand Functions

The economy of a given nation, whose equilibrium tariff to be imposed on imported commodities is greater than the one of its competitor, is called economically dominant. We have stated that the foreign currency demand functions ultimately determine which nation will be able to impose a greater tariff. Regardless of inability to anticipate which nation will possess such economic power based only on the demand functions, we can review some examples of functions leading to an economic dominance. We refer to the foreign currency demand function leading to an economic dominance (in that sense) as dominant function.

Basically, the shapes of these functions expose the influence of the exchange rate on the currency demand. So different shapes can be interpreted differently. In particular, more concave or smooth the function is, less sensitive it is to the change of the equilibrium. Likewise, more convex function exposes much sensitivity to the exchange rate.

The solution to the system of equations (1.13), (1.14), (1.15) can expose a total dominance of one nation over another. This is reflected in one tariff to be zero, thus one nation being unable to tax the competitor while the other nation can impose any tariff. This case leads to an economic nonsense. Namely, if the dominant nation imposes a tariff equal to 1, which means that it absorbs 100% of the value of imported commodities. This would lead to another nation abandon trading with such a competitor altogether. At the same time, since there is no commodities flow from one nation to another, the exchange rate also loses its sense. Since the mathematical model we provide assumes that the trading continues as long as the indicator conditions are met in (1.3), we have to logically restrict such possibilities.

In Section 1.2, we claimed that $\frac{1}{M} \leq e \leq M$, for some large number M and $\frac{1}{M} \leq \theta \leq 1, \frac{1}{M} \leq \theta^* \leq 1$. Also the conditions $D(0) = 1, D(\infty) = 0, D^*(0), D^*(\infty) = 1$ must be respected. This restriction averts the possibility of the scenario described above. However, the demand functions which can cause such situations deserve some attention. Such functions are not generalized in this thesis, so we do not provide the properties they must satisfy in order to lead to such case. So instead of abstraction, here is a list of such pairs of $D(x)$ and $D^*(x)$. We are interested within the domain of $0 \leq x \leq 1$.

Example 1:

Foreign currency demand function of the first and the second nations are given by

$$D(x) = (1-x)^+, D^*(x) = x \wedge 1$$

where $(.)^+$ denotes the maximum between the expression inside the parenthesis and zero. $\wedge$ denotes the minimum between the two sides of the symbol. These functions are illustrated in Figure 1.9. Their derivatives are $D'(x) = -1$ and $D^{*\prime}(x) = 1$ respectively. Putting $D(x)$ and $D^*(x)$ in (1.5) yields

$$e\left(1 - \frac{e}{\theta}\right) = \theta^* e$$

from which eliminating e and rearranging the terms gives

$$\theta - e = \theta\theta^*,$$

solving for e results in

$$e = \theta(1 - \theta^*). \tag{1.20}$$

Once having e expressed in terms of θ and θ^*, we can solve the equation (1.10) which looks like

$$1 - \frac{e}{\theta} = \theta^*(1 - \theta), \tag{1.21}$$

putting (1.20) into (1.21) gives

$$1 - \frac{1}{\theta}\theta(1 - \theta^*) = \theta^*(1 - \theta),$$

canceling and rearranging some terms yields

$$\theta^* = \theta^*(1 - \theta)$$

from which $\theta = 0$, while the value of θ^* can be anything within the range of [0,1].

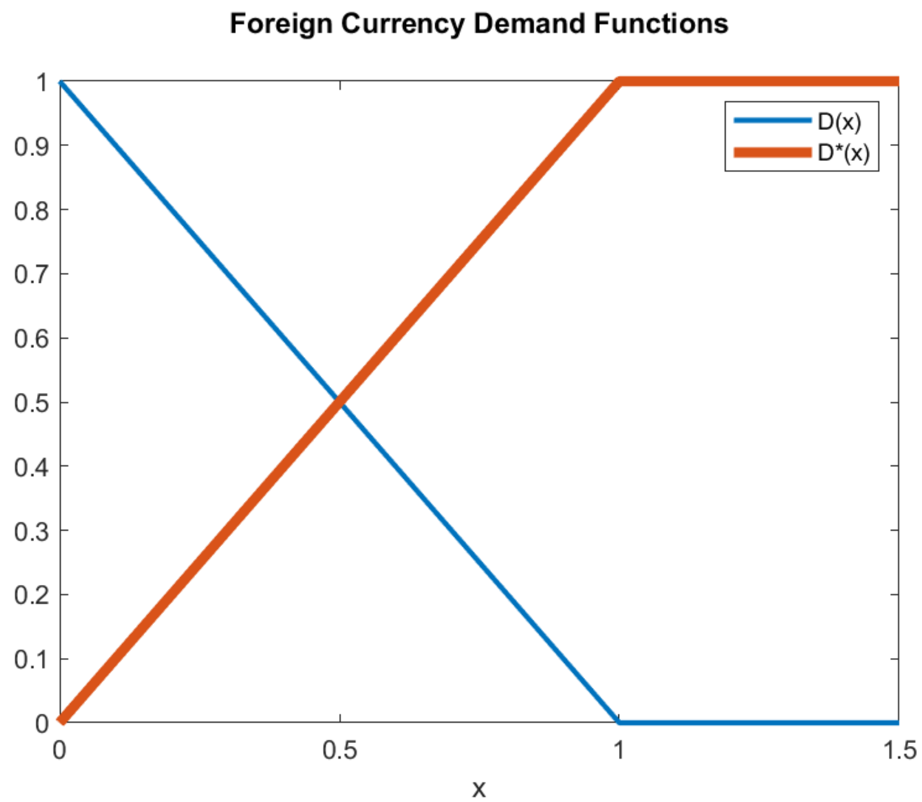

Figure 1.9: $D(x) = (1 - x)^+,\ D^*(x) = x \wedge 1$

Example 2:

Here we provide a square root function for one nation

$$D(x) = 1 - \sqrt{x}, \quad D^*(x) = x \wedge 1$$

where the second function is still the minimum between x and 1.

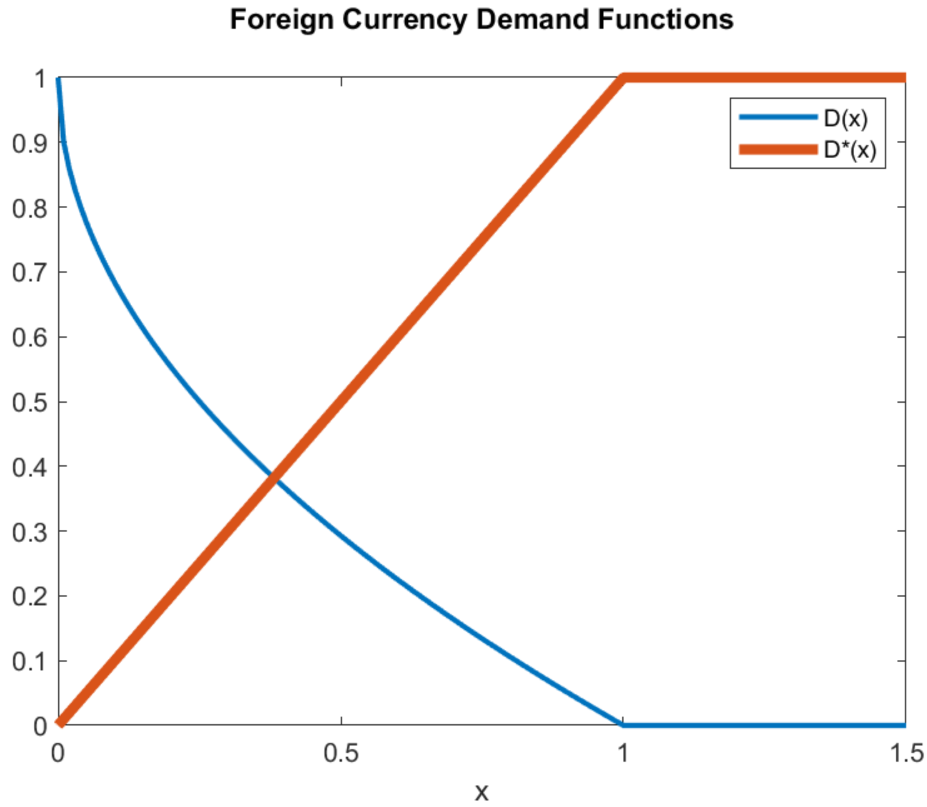

Figure 1.10: $D(x) = 1 - \sqrt{x}, \quad D^*(x) = x \wedge 1$

Their derivatives are given by $D'(x) = -\frac{1}{2\sqrt{x}}$ and $D^{*\prime}(x) = 1$. Applying (1.5) for these functions gives the solution to the exchange rate as follows

$$e\left(1 - \sqrt{\frac{e}{\theta}}\right) = \theta^* e,$$

eliminating e on both sides gives

$$\sqrt{\theta} - \sqrt{e} = \theta^* \sqrt{\theta},$$

from which e is obtained to be the following expression

$$e = \theta(1 - \theta^*)^2. \tag{1.22}$$

Applying $D(x)$ and $D^*(x)$ in (1.10) results in

$$1 - \frac{\sqrt{e}}{\sqrt{\theta}} = \theta^*(1-\theta), \tag{1.23}$$

putting (1.22) into (1.23) yields

$$1 - \frac{\sqrt{e}}{\sqrt{\theta}} = \theta^*(1-\theta),$$

this equation simplifies to

$$1-(1-\theta^*) = \theta^*(1-\theta)$$

and ultimately

$$\theta^* = \theta^*(1-\theta)$$

from which we can conclude that $\theta = 0, \theta^* \epsilon [0,1]$.

Example 3:

The next pair of functions which lead to the similar scenario are

$$D(x) = \mathbf{exp}(-x)\,, D^*(x) = x \wedge 1$$

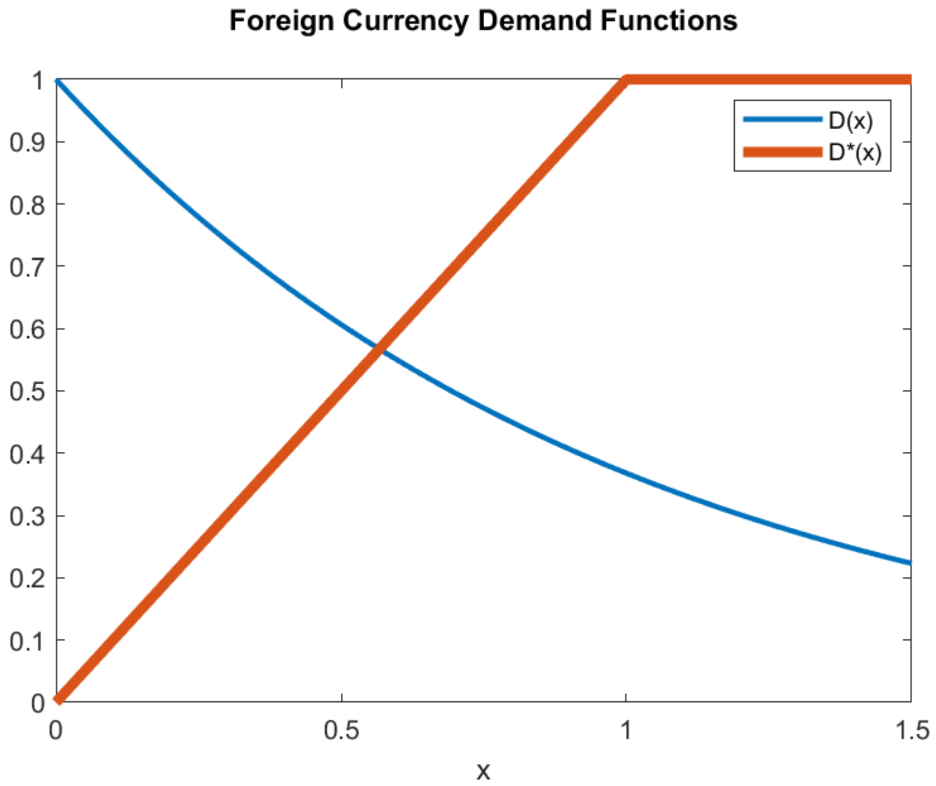

Figure 1.11: $D(x) = \boldsymbol{exp}(-x)\,, D^*(x) = x \wedge 1$

Differentiating of both functions gives $D'(x) = -\boldsymbol{exp}\,(-x)$ and $D^{*\prime}(x) = 1$ from (1.5)

$$e\boldsymbol{exp}\left(-\frac{e}{\theta}\right) = \theta^* e$$

taking natural logarithms from both sides leaves

$$-\frac{e}{\theta} = ln\theta^*,$$

solving for e gives

$$e = -\theta ln\theta^*. \tag{1.24}$$

Applying (1.10) yields

$$\boldsymbol{exp}\left(-\frac{e}{\theta}\right) = \theta^*(1-\theta), \tag{1.25}$$

solving for θ^*

$$\theta^* = \frac{\boldsymbol{exp}\,(-\frac{e}{\theta})}{1-\theta}. \tag{1.26}$$

Putting (1.24) in (1.26) gives

$$\theta^* = \boldsymbol{exp}\left(\frac{\theta ln\theta^*}{\theta}\right)\frac{1}{1-\theta},$$

simplifying the numerator,

$$\theta^* = \theta^*(1-\theta),$$

so the result is $\theta = 0, \theta^* \epsilon [0,1]$.

Example 4:

Here, one of the functions is quadratic

$$D(x) = 1 - x^2, D^*(x) = x \wedge 1$$

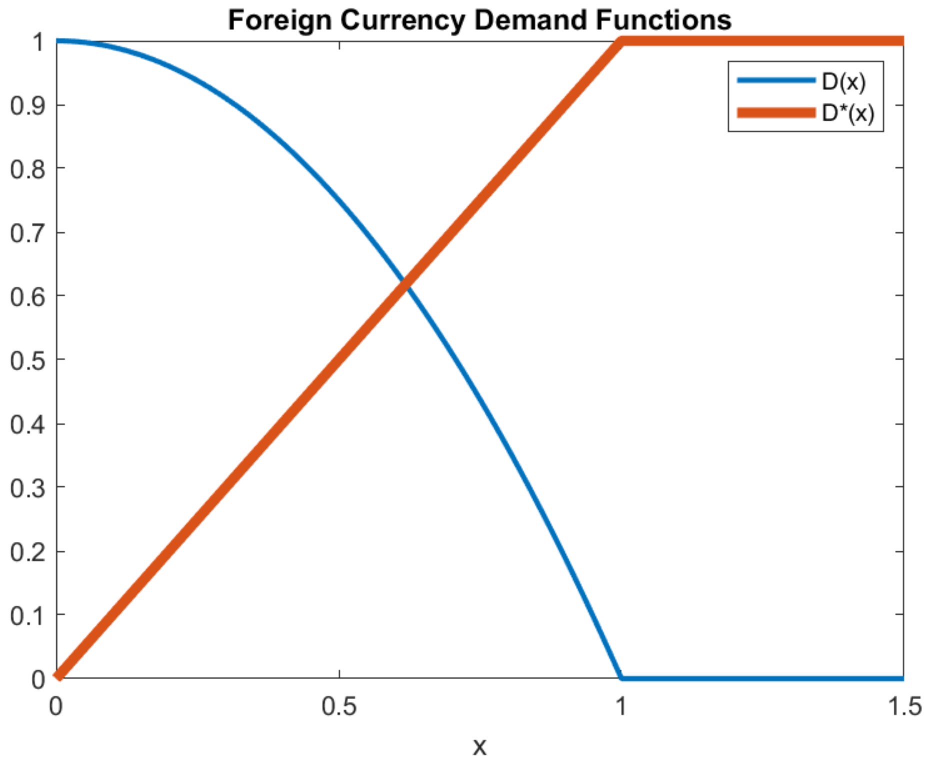

Figure 1.12: $D(x) = 1 - x^2, D^*(x) = x \wedge 1$

Their derivatives are

$$D'(x) = -2x, D^*(x) = 1,$$

(1.5) gives

$$e\left(1 - \frac{e^2}{\theta^2}\right) = \theta^* e,$$

by solving this expression for e, we arrive at

$$e = \sqrt{\theta^2(1 - \theta^*)}, \tag{1.27}$$

by (1.10)

$$1 - \frac{e^2}{\theta^2} = \theta^*(1 - \theta)$$

and correspondingly by (1.27) and (1.10)

$$\theta^* = \theta^*(1 - \theta)$$

from which $\theta = 0, \theta^* \epsilon [0,1]$.

Example 5:

Consider a logarithmic function

$$D(x) = 1 - \ln(x+1)\,, \qquad D^*(x) = x \wedge 1$$

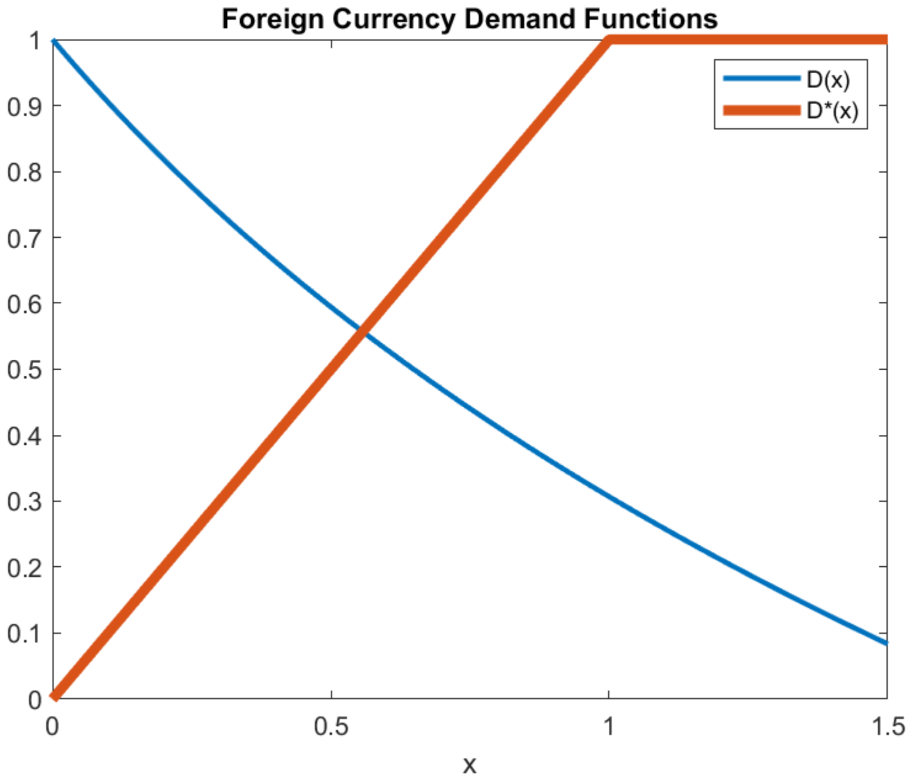

Figure 1.13: $D(x) = 1 - \ln(x+1)\,, D^*(x) = x \wedge 1$

Derivatives of $D(x)$ and $D^*(x)$ are respectively given by

$$D'(x) = -\frac{1}{x+1}, \quad D^{*\prime}(x) = 1,$$

again by (1.5)

$$e\left(1 - \ln\left(\frac{e}{\theta}\right)\right) = \theta^* e,$$

rearranging the terms and removing logarithm gives

$$\boldsymbol{exp}(1 - \theta^*) = \frac{e}{\theta}$$

from which

$$e = \theta\, \boldsymbol{exp}(1 - \theta^*), \tag{1.28}$$

by (1.10)

$$1-\ln\frac{e}{\theta}=\theta^*(1-\theta), \tag{1.29}$$

putting (1.28) in (1.29) results in

$$\theta^* = \theta^*(1-\theta),$$

as a result $\theta = 0, \theta^* \epsilon [0,1]$.

Example 6:

Here we have a combination of logarithmic and cubic functions

$$D(x)=1-\ln(x+1)\,, D^*(x^3 \wedge 1)$$

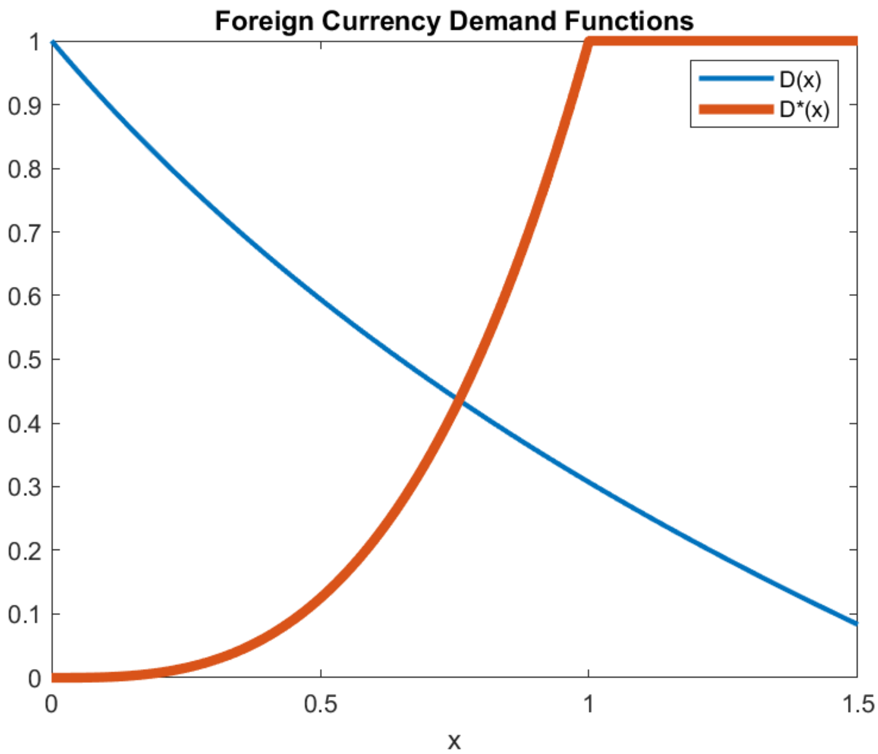

Figure 1.14: $D(x)=1-\ln(x+1)\,, D^*(x)=x^3 \wedge 1$

Derivatives are

$$D'(x)=-\frac{1}{x+1}, \qquad D^*(x)=3x^2.$$

Applying (1.5)

$$e\left(1-\ln\frac{e}{\theta}\right)=(\theta^* e)^3,$$

canceling e leaves the following expression

$$1 - \ln\frac{e}{\theta} = \theta^{*3}e^2,$$

from which

$$\theta = \frac{e}{\boldsymbol{exp}\,(1 - \theta^{*3}e^2)}.$$

Applying (1.10)

$$1 - \ln\frac{e}{\theta} = \theta^*(1-\theta)3(\theta^* e)^2,$$

replacing θ with (1.30) ultimately yields

$$\theta^* = \theta^*(1-\theta)$$

and therefore $\theta = 0, \theta^* \epsilon [0,1]$.

1.8 Empirical Illustration

This section illustrates the application of the model described above. Specifically, we analyze the trade data of Georgia and Turkey. Because of the free trade agreement, we have $\theta = 1$ and $\theta^* = 1$. The data is taken from the National Statistics Office of Georgia[1]. In order to observe the quantities of products along with their prices, we extract HS-4 and HS-6[2] classifications which contain products imported and exported filling a significant portion of total trade. In addition, since the traded products are measured in US dollars, we convert those amounts into national currencies at an exchange rate for 3.01.2020. In particular, we take USD/GEL[3] = 2.8661, USD/TRY[4]=5.9705. USD/TRY exchange rate is taken as an average of that day's bid and ask values. Once these quantities are in place, we can obtain prices of per units of products in national currency.

1. Source of Data: www.ex-trade.geostat.ge
2. HS-4, HS-6 classification of products contain quality measurable products
3. www.nbg.gov.ge
4. www.tcmb.gov.tr

Product Code	p	d^*	pUSD	pd^*	$pd^* 1_{\{\theta^* p^* x > p\}}$
0208	0.5732	22.5	4.5	12.8975	0
0302	1.1575	1225.7	495	1418.72	1418.72
0601	8.4333	13.9	40.9	117.2235	117.22
0602	0.5236	31.2	5.7	16.3368	16.34
0703	3.8460	23.4	31.4	89.9955	90.00
0703	1.6507	709.1	408.4	1170.52	1170.52
0713	3.4269	4.6	5.5	15.7636	15.76
0802	17.4832	22	134.2	384.63	384.63
0802	16.4890	24.3	139.8	400.68	400.68
0810	1.4032	158.3	77.5	222.12	222.12
0713	8.1574	6.5	18.5	53.02	53.02
0901	28.2789	1.5	14.8	42.42	42.42
0902	7.1656	434.2	1085.4	3110.87	0
1106	0.5720	94.2	18.8	53.88	53.88
1209	0.4379	7.2	1.1	3.15	3.15
1211	6.1839	596.4	1286.8	3688.10	3688.10
1401	0.2166	22.5	1.7	4.8724	0
1502	3.5314	16.8	20.7	59.33	0
1504	4.6143	2702.5	4350.9	12470.11	12470.11
1515	2.5814	382.6	344.6	987.66	987.66
1516	9.7447	0.5	1.7	4.87	4.87
1522	1.0058	49.3	17.3	49.59	0
1806	6.4691	59.9	135.2	387.50	387.50
2005	0.0000	0	0	0	0
2007	0.0000	0	0	0	0
2103	0.0000	0	0	0	0
2106	59.4716	8.3	8.3	23.79	0

2201	0.8571	73	73	209.23	209.23
2204	8.4297	20	20	57.32	0
2204	7.7026	30.1	30.1	86.27	0

Table 1.1 Exported products from Georgia to Turkey

The first column of Table 1.1 contains the unique codes of products. The second column p is the GEL value of $pUSD$ column values which are the prices of the exported products measured in USD. The third column d^* is the quantity of exported products. The fifth column pd^* is the GEL value of exported product and the last one contains the same value filtered by the given indicator. Similarly, Table 1.2 indicates the imported data in Georgia from Turkey. Columns d and p^*USD contain the imported amounts of products and their total values measured in USD. Once converted into TRY, we have the second column named p^*. Amounts of products exported from Turkey to Georgia measured in TRY is given by the column p^*d and the same amount given the indicator function is in the column $p^*d1_{\{\theta p > x p^*\}}$.

An important note to take into account is that the Tables 1.1 and 1.2 represent the extract from the original data. The full data is contained in the accompanying spreadsheet file. Here we have several assumptions. Firstly, the random variables d_k, p_k, d_k^*, p_k^* are assumed to be realized in year 2019 data. So we take the values of those random variables as defined on the extended probability space and compute the desired quantities accordingly. Since $\theta = 1$ and $\theta^* = 1$, the only relevant equation from the system (1.13), (1.14), (1.15) is (1.13). However, as shown later, the quantities of $D'\left(\frac{\hat{e}}{\hat{\theta}}\right) = -0.93$ and $D^*\left(\hat{\theta}^*\hat{e}\right) = 0.11$.

In addition, because of the individual product prices and quantities grouped within the classified data, individual product prices and quantities are found by averaging the price for the quantity given. That provides an approximation to individual product prices. Similarly, the quantities contain the total quantities for all products within a given category. So, the quantities cannot be approximated to individual product quantities. However, since we are interested in the currency flow between the two countries, we approximate the amount of currency exchanged by taking the product category as a single product and applying its observed quantity along with the average price.

Product Code	p^*	d	p^*USD	p^*d	$p^*d1_{\{\theta p>xp^*\}}$
4010	25.3656	135.6	576.1	3439.58	3439.58
4011	44.2944	610.6	4530	27046.14	0
4011	21.7291	2573	9364.3	55909.08	0
4012	14.2939	88.3	211.4	1262.15	0
4013	37.8894	2.6	16.5	98.51	0
4014	77.6159	1.6	20.8	124.19	0
4015	0	0	0	0	0
4015	31.5858	83	439.1	2621.63	0
4016	298.5225	0.4	20	119.41	0
4016	9.2271	1.1	1.7	10.15	10.1498
4016	16.5688	1222.1	3391.5	20248.78	20248.78
4017	15.6898	8.6	22.6	134.93	134.9322
4104	0	0	1.1	0	0
4106	126.8721	0.8	17	101.50	0
4107	48.0541	133.6	1075.3	6420.02	6420.025
4112	182.0987	0.2	6.1	36.42	0
4113	76.2891	6.3	80.5	480.62	0
4114	41.7932	0.2	1.4	8.36	8.3586
4115	44.7784	1.8	13.5	80.60	80.6011
4201	32.8385	0.2	1.1	6.57	6.5675
4201	55.7242	0.3	2.8	16.72	16.7173
4202	194.0396	0.2	6.5	38.81	0

Table 1.2: Imported products from Turkey to Georgia

In order to obtain the functions $D(x)$ and $D^*(x)$, we construct the exponential and logarithmic regressions respectively. The following table illustrates the currency demand functions for varying values of the exchange rate.

x	0	0.2	0.4	0.6	0.8	1
$D(x)$	1	0.6442	0.5017	0.3046	0.2416	0.1965
$D^*(x)$	0	0.1593	0.3575	0.5789	0.6097	0.6143

Table 1.3: Currency Demand

Corresponding regression plots are given in Figures 1.15 and 1.16.

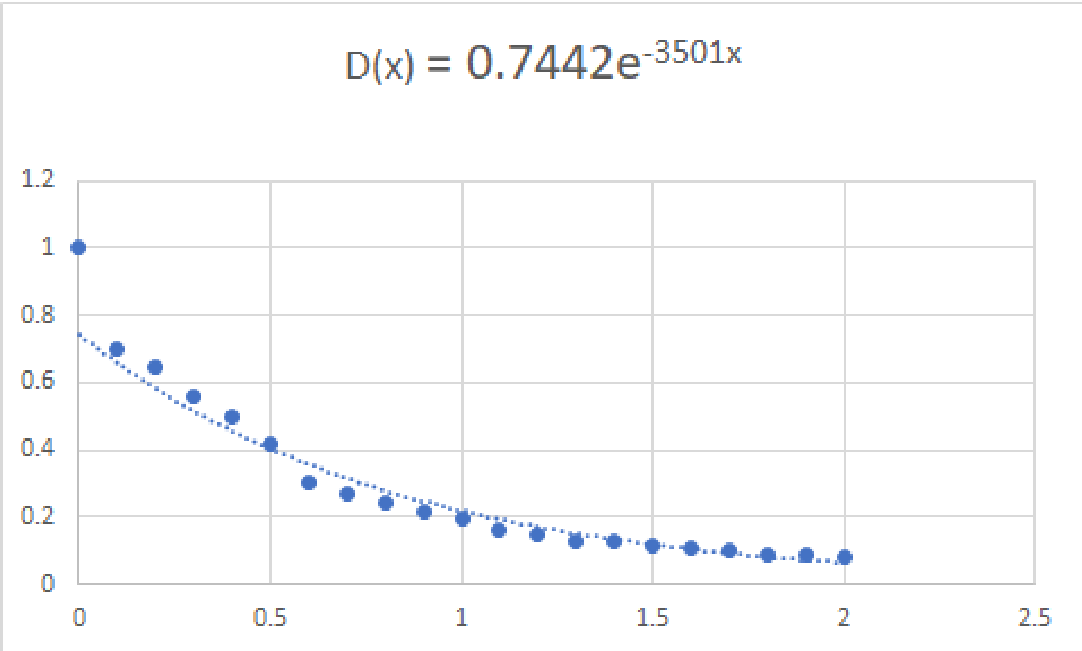

Figure 1.15: Exponential Regression for $D(x)$

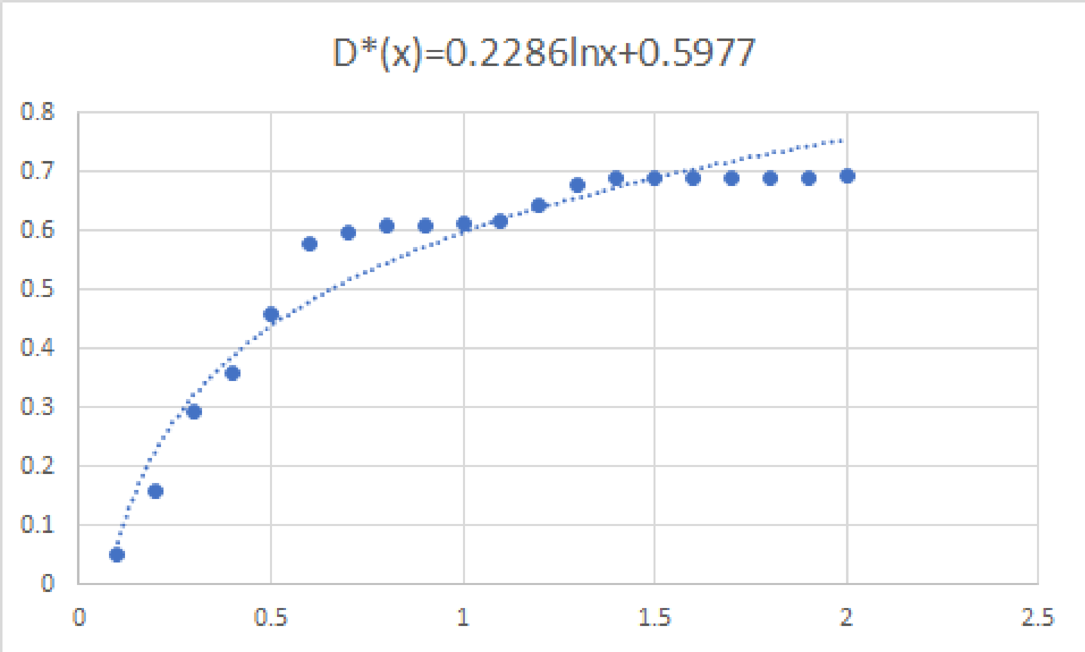

Figure 1.16: Logarithmic Regression for $D^*(x)$

From the definition of (1.1), (1.2), we compute C_N and C_N^* and from (1.3) we have the following table.

C_N	C_N^*	$E(pd^*1_{\{\theta^*p^*x>p\}})$	$E(p^*d1_{\{\theta p>xp^*\}})$	e	$D(\frac{e}{\theta})$	$D^*(e\theta^*)$
3861	1351	935.07	310.95	2.0339	0.37	0.76

Table 1.4: Components of the system (1.13), (1.14), (1.15)

Since we have a free trade agreement implying $\theta = 1$ and $\theta^* = 1$, we only have to solve (1.13) from the system of equations and obtain an optimal exchange rate. The solution to this equation yields $\hat{e} = 2.0339$ while the cross rate computed is $GEL/TRY = 2.0831$.

Chapter 2 Gibbons Model

2.1. Description of the Model

In this chapter we describe one additional model proposed by R. Gibbons [12]. The primary difference between this model and the one developed before is that, this model defines the gain function based on completely different parameters. The approach to find the Nash Equilibrium point is identical.

R. Gibbons considered a model in which the economic welfare of a nation is determined by consumers' surplus enjoyed by the consumers within a given country, profit made by the local firms from selling goods on the domestic and foreign markets and the tariff revenue collected by the government from foreign imports. Two nations enter into an unrestricted bilateral trade. Competitive game between the nations is based on the sequence of decisions made by the firms and governments. Initially, the governments impose tariffs on imported products and the firms in both countries respond by deciding on the profit maximizing quantities of products for the home consumption and exports. The governments' aim to maximize economic welfare results in a prisoner's dilemma type of non-cooperative game.

According to the model, total production within the domestic and foreign markets are given by

$$Q = h + f^{*}, Q^{*} = h^{*} + f$$

respectively, where h and h^{*} are the production for the domestic consumption while f and f^{*} are the exported production for the domestic and foreign firms. Let us assume the market clearing prices for the production are given by

$$P = a - Q, P^{*} = a - Q^{*},$$

where a is a positive constant satisfying $a > Q$ and $a > Q^{*}$. Given t and t^{*} are the tariffs imposed on imported production for the domestic and foreign countries, profits for the firms are defined respectively as

$$\pi(t,t^*,h,f,h^*,f^*) = (a-h-f^*)h + (a-h^*-f)f - t^*f, \qquad (2.1)$$
$$\pi^*(t,t^*,h,f,h^*,f^*) = (a-h^*-f)h + (a-h-f^*)f - tf^*$$

Having defined these profits, the welfare of nations is determined as follows

$$W(t,t^*,h,f,h^*,f^*) = \frac{1}{2}Q^2 + \pi(t,t^*,h,f,h^*,f^*) + tf^* \qquad (2.2)$$
$$W^*(t,t^*,h,f,h^*,f^*) = \frac{1}{2}Q^{*2} + \pi^*(t.t^*,h,f,h^*,f^*) + t^*f$$

So here the nations are facing a dilemma to set an optimal tariff on imported commodities and take into consideration the optimal response from the competitor.

2.2 Game Without Exchange Rate

In this section we aim to solve the tariff optimization problem in order to maximize the welfare functions. Here the nations not only have to optimize the tariffs giving maximum welfare, but also to optimize the quantities of commodities produced for the domestic and foreign markets.

For a given set of tariffs t, t^*, the Nash equilibrium point for the firms is found by solving the system of equations

$$\frac{\partial}{\partial h}\pi(t,t^*,h,f,h^*,f^*) = 0, \qquad (2.3)$$
$$\frac{\partial}{\partial h}\pi^*(t,t^*,h,f,h^*,f^*) = 0,$$
$$\frac{\partial}{\partial f}\pi(t,t^*,h,f,h^*,f^*) = 0,$$
$$\frac{\partial}{\partial f^*}\pi^*(t,t^*,h,f,h^*,f^*) = 0,$$

which yields (see Appendix D)

$$\hat{h} = \frac{a+t}{3}, \hat{f} = \frac{a-2t^*}{3}, \hat{h}^* = \frac{a+t^*}{3}, \hat{f}^* = \frac{a-2t}{3} \qquad (2.4)$$

So, depending on the tariffs imposed by the governments, firms produce goods according to (2.4). The governments themselves select the tariffs to be imposed on imports by maximizing their economic welfare functions. Having defined h, f, h^*, f^* in terms of t and t^*, (2.2) can be redefined as the functions of tariffs

$$\widehat{W}(t,t^*) = W\left(t,t^*,\frac{a+t}{3},\frac{a-2t}{3},\frac{a+t^*}{3},\frac{a-2t}{3}\right), \tag{2.5}$$

$$\widehat{W}^*(t,t^*) = W\left(t,t^*,\frac{a+t}{3},\frac{a-2t}{3},\frac{a+t^*}{3},\frac{a-2t}{3}\right).$$

The Nash equilibrium point for the game is obtained from solving the system

$$\frac{\partial}{\partial t}(t,t^*) = 0, \frac{\partial}{\partial t^*}W^*(t,t^*) = 0 \tag{2.6}$$

which gives $(\hat{t},\hat{t}^*) = \left(\frac{a}{3},\frac{a}{3}\right)$. (See Appendix D) So in case of imposing tariffs, total production for each market are

$$\hat{Q} = \hat{h} + \hat{f}^* = \frac{2a-\hat{t}}{3},$$

$$\hat{Q}^* = \hat{h}^* + \hat{f} = \frac{2a-\hat{t}^*}{3}$$

compared to

$$\hat{Q} = \frac{2a}{3}, \hat{Q}^* = \frac{2a}{3}$$

in case of zero tariffs. Since the non-cooperative solution is not Pareto-optimal, the tariff game is a prisoner's dilemma type of problem. Here we obtain the similar result as we have in Schwartz's model. Free trade policy is beneficial for both parties. It is possible to draw a parallel to the results obtained in the asymmetric case of the previous model. When the parameters in the model are identical, neither nation has the dominant power over the competitor so they will have to impose equal tariffs.

2.3 Game With Exchange Rate

In this section we introduce a currency exchange rate, redefine the welfare and the firms profit functions of one nation taking the exchange rate into account, and find the Nash Equilibrium point for the welfare functions. We define the currency exchange rate as it is done in the original model. Within Gibbon's model, not only the tariffs to be imposed on imported commodities are optimized along with the exchange rate, but the quantities to be produced as well. Here the currency exchange rate is not directly a function of tariffs, but it is a function of exported commodities of domestic and foreign firms which themselves are the functions of tariffs.

Let us introduce a currency exchange rate $e = \frac{pf^*}{p^*f}$ where p and $p^*$are the market clearing prices for products on domestic and foreign markets while f and f^* are the exported production for the domestic and foreign firms. Taking the currency exchange rate into account, profits made by firms in domestic and foreign markets are defined as follows

$$\pi(t,t^*,h,f,h^*,f^*) = (a-h-f^*)h + (a-h^*-f)f - t^*f, \qquad (2.7)$$
$$\pi(t,t^*,h,f,h^*,f^*) = e(a-h^*-f)h^* + e(a-h-f^*)f^* - etf^*$$

The economic welfare of countries are

$$W(t,t^*,h,f,h^*,f^*) = \frac{1}{2}Q^2 + \pi(t,t^*,h,f,h^*,f^*) + tf^*, \qquad (2.8)$$
$$W^*(t,t^*,h,f,h^*,f^*) = \frac{1}{2}eQ^{*2} + \pi^*(t,t^*,h,f,h^*,f^*) + etf^*.$$

Solving the system of equations (2.6) does not yield an explicit solution for t and t^*. They can only be solved by numerical methods. However, given the optimal values of t and t^*, we can solve for $\hat{h},\hat{f},\hat{h}^*,\hat{f}^*$ (See Appendix D)

$$\hat{h} = \frac{a-f^*}{2}, \qquad (2.9)$$
$$\hat{h}^* = \frac{a+t^*}{3},$$
$$\hat{f} = \frac{a-2t^*}{3},$$
$$\hat{f}^* = \frac{a-h-t+\sqrt{(a-h-t)^2+3(a-h^*-f)h^*}}{3}$$

These are the quantities of commodities produced by firms on both sides for domestic and foreign markets. Obviously differentiating the welfare functions with respect to tariffs would not give explicit solutions, so we do not provide the expressions here.

Unlike the previous model, Gibbons model does not explicitly account for foreign currency demand functions. Rather it focuses on firm specific details and expresses the welfare function for a given country as sums of firms' profits, tax revenue made by the country and the consumers' surplus within a given country. As a comparison with the previous model, it can be noted that once the objective functions are defined, maximization problems are identical in both models and both lead to an optimal Nash equilibrium. From the practical point of view, measuring all the quantities making up the welfare functions are quite feasible. Namely, the last two parameters, tax revenue and firms' profits are trivially measured. Consumers' surplus depends on the willingness of consumer's to pay for particular products. So there must be estimated price demand functions compared with actual prices for particular products. However, according to the welfare functions defined above, the first parameter named as consumers' surplus is not really a consumers' surplus classically. The idea behind using quantities produced for the domestic and foreign markets as consumer's surplus is that economic benefit is enjoyed by the nation producing and selling these goods. Accurate estimation of consumers' surplus is a matter of scientific research. Related papers are listed in bibliography.

Once the exchange rate is determined, we continue the thesis with portfolio optimization model which can be applied to any trading instrument with observable price dynamics including the currency exchange pairs.

2.4 Empirical Illustration

Since there is a zero tariffs imposed on imported products, instead of computing the optimal production level for domestic and foreign consumption, we estimate the exchange rate that is determined by the exported and imported products along with their market clearing prices. Here we assume that the market clearing prices are the ones observed by the National Statistics Office of Georgia. In particular, we have $e = \frac{pf^*}{p^*f}$. We can translate this fraction into the terms of the previous model.

In particular, since the numerator and denominator represent the total amounts of exported and imported products, we take $e = \frac{E(p^* d1_{\{\theta p > xp^*\}})}{E(pd^* 1_{\{\theta^* p^* x > p\}})}$.

Since only the observation on whole classification of products is available with total prices measured in USD, here again as in previous example, we divide the total amount of exported and imported products measured in USD by the quantities and convert them into the national currencies of both countries at the given official exchange rate for 3.01.2020.

Since we assume pf^* and $p^* f$ are the quantities defined already on an extended probability space, we take the sum of the last column values in Tables 1.1 and 1.2 which are

$$E\big(p^* d1_{\{\theta p > xp^*\}}\big) = \frac{1}{N}\sum_{k=1}^{N} \bar{E}\left(p_k^* d_k 1_{\{\theta p_k > xp_k^*\}}\right) = 1423820,$$

$$E\big(pd^* 1_{\{\theta^* p^* x > p\}}\big) = \frac{1}{N}\sum_{k=1}^{N} \bar{E}\left(p_k d_k^* 1_{\{\theta^* p_k^* x > p_k\}}\right) = 773305.$$

Both of these quantities are given in thousands of national currency units. As a final note, since in the previous model, we have computed GEL/TRY, here we reverse the exchange rate fraction and get $e = 1.8412$.

In order to define optimal quantities, we have to make several assumptions for reasonable approximation. Since the information about profits made by firms in a given country is not available (domestic and foreign countries), we assume that the profits made from selling products on a domestic market is some portion of exports. In particular, according to the National Statistics Office of Georgia, in 2019 total imports equate 9 120.4 million USD and exports to 3 766.4 million USD[1]. At the same time, Turkish total exports as per 2018 was 168 023 390 million USD and imports reached 223 039 038 million USD[2]. As an approximation, we take Georgian exports made by firms being 2.4 times lower than imports. Similarly, we assume that Turkish firms export represent only ¾ of total imports. So, since we have $f = 334\,870$ and $f^* = 2\,891\,306$, we take $h = 139\,529$ and $h^* =$

2 178 125. Before we have all we need in order to estimate the optimal quantities of commodities for domestic and foreign consumption from (2.9), the only quantity to be estimated remains the constant a. We already have observations on $Q = h + f^*$ and $Q^* = h^* + f$. Applying these quantities and the fact that $P = a - Q$ and $P^* = a - Q^*$, we approximate a by computing it for both price equations and averaging it out such that it satisfies the inequalities $a > Q$ and $a > Q^*$. This way we obtain approximate $a = 3\ 051\ 782$. Using the equations (2.4), we get the final quantities. If Georgia chooses to produce 1 million for the domestic production, i.e. $\hat{h} = 1\ 000\ 000$, then we obtain $\hat{h}^* = 1\ 017\ 261$, $\hat{f} = 1\ 017\ 261$ and $\hat{f}^* = 2\ 953\ 279$. Obviously, the quantities $\hat{h}^*$ and $\hat{f}$ coincide because of the absence of tariffs imposed by any country. At the same time, the result of f^* being almost three times higher than f is intuitive and is directly visible from the original export/import data.

1.Source of Data: www.geostat.ge

2.Source of Data: www.wits.worldbank.org

Chapter 3 Portfolio Optimization

3.1 Portfolio Wealth Process

Consider a portfolio consisting of n risky assets. [39] examines the multi-dimensional Brownian motions for self-financing portfolios. To model the asset price movements, we take n – dimensional Ito's process which is a vector of asset prices $S^* = (S^1, \dots, S^n)^T$ driven by n – dimensional Brownian motion $B = (B^1, \dots, B^n)^T$, where $B^i = (B_t^i, t \geq 0)$ be the real valued Brownian motion which starts from 0 on (Ω, F, P):

$$dS_t^i = S_t^i(\mu^i dt + \sigma^i dB_i) \tag{3.1}$$

where μ^i is the drift and σ^i is the row vector $(\sigma^{i1}, \dots, \sigma^{in})$. For more convenient notation we can convert the differential equation into the following form:

$$dS_t^i = S_t^i(\mu^i dt + \sigma^{i1} dB_i^1 + \dots + \sigma^{in} dB_t^n) \tag{3.2}$$

Define the portfolio wealth process V_t corresponding to self-financing portfolio to follow the differential equation:

$$dV_t = \theta_t^1 dS_t^1 + \dots + \theta_t^n dS_t^n \tag{3.3}$$

Since we only consider long portfolios, here θ_t^j denotes the number of j^{th} asset purchased at time t and it is a finite variance process. To solve this process, we extend the differential equation and introduce some notations. Let $\pi_t^i = \theta_t^i S_t^i$ be the cash position of i^{th} asset and let $q_t^i = \frac{\pi_t^i}{V_t}$ be the weight of i^{th} asset within a portfolio at time t. Having defined these quantities, we can proceed to solve the portfolio wealth process as follows:

$$\begin{aligned} dV_t = \theta_t^1 S_t^1(\mu^1 dt + \sigma^{11} dB_t^1 + \sigma^{12} dB_t^2 + \dots + \sigma^{1n} dB_t^n) \\ + \theta_t^2 S_t^2(\mu^2 dt + \sigma^{21} dB_t^1 + \sigma^{22} dB_t^2 + \dots + \sigma^{2n} dB_t^n) + \cdots \\ + \theta_t^n S_t^n(\mu^n dt + \sigma^{n1} dB_t^1 + \sigma^{n2} dB_t^2 + \dots + \sigma^{nn} dB_t^n). \end{aligned} \tag{3.4}$$

Multiplying the terms, factoring out the like terms and converting the equation into π_t^i terms yields:

$$dV_t = (\pi_t^1\mu^1 + \pi_t^2\mu^2 + \dots + \pi_t^n\mu^n)dt \\ + (\pi_t^1\sigma^{11} + \pi_t^2\sigma^{21} + \dots + \pi_t^n\sigma^{n1})dB_t^1 \\ + (\pi_t^1\sigma^{12} + \pi_t^2\sigma^{22} + \dots + \pi_t^n\sigma^{n2})dB_t^2 + \dots + (\pi_t^1\sigma^{1n} \\ + \pi_t^2\sigma^{2n} + \dots + \pi_t^n\sigma^{nn})dB_t^n \tag{3.5}$$

At this point we have arrived to an equation defined in terms of dollar positions in each asset within a portfolio. However, since our ultimate goal is to optimize the asset weights, we need to convert this equation into the terms of q_t^j. This is achieved by multiplying and diving the right side of the equation by V_t at the same time. So, the result is an equation translated into weight terms:

$$dV_t = V_t[(q_t^1\mu^1 + q_t^2\mu^2 + \dots + q_t^n\mu^n)dt \\ + (q_t^1\sigma^{11} + q_t^2\sigma^{21} + \dots + q_t^n\sigma^{n1})dB_t^1 \\ + (q_t^1\sigma^{12} + q_t^2\sigma^{22} + \dots + q_t^n\sigma^{n2})dB_t^2 + \dots + (q_t^1\sigma^{1n} \\ + q_t^2\sigma^{2n} + \dots + q_t^n\sigma^{nn})dB_t^n] \tag{3.6}$$

Since the optimal weights imply an investor should hold these weights constant during an investment horizon, it means an investor should constantly re-balance the portfolio in order to maintain the once selected weights. So, assuming that weights are held constant at any point in time t, we can correspondingly update the equation (3.6) into the form:

$$dV_t = V_t[(q_1\mu^1 + q_2\mu^2 + \dots + q_n\mu^n)dt \\ + (q_1\sigma^{11} + q_2\sigma^{21} + \dots + q_n\sigma^{n1})dB_t^1 \\ + (q_1\sigma^{12} + q_2\sigma^{22} + \dots + q_n\sigma^{n2})dB_t^2 + \dots + (q_1\sigma^{1n} \\ + q_2\sigma^{2n} + \dots + q_n\sigma^{nn})dB_t^n] \tag{3.7}$$

In this equation, all sums within the parenthesis are constants, so we can shorten the notation by introducing the new notations. Let

$$\bar{\mu} = q_1\mu^1 + q_2\mu^2 + \dots + q_n\mu^n \tag{3.8}$$

and

$$\bar{\sigma}_j = q_1\sigma^{1j} + q_2\sigma^{2j} + \dots + q_n\sigma^{nj} \tag{3.9}$$

for all $j = 1, \dots, n$. Equation (3.7) now becomes

$$dV_t = V_t[\bar{\mu}dt + \bar{\sigma}_1 dB_t^1 + \bar{\sigma}_2 dB_t^2 + \dots + \bar{\sigma}^n dB_t^n]. \tag{3.10}$$

Solution to this differential equation by [39] is:

$$V_t = V_0\boldsymbol{exp}\ (\left[\bar{\mu} - \frac{1}{2}(\bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \ldots + \bar{\sigma}_n^2)\right] t + \bar{\sigma}_1 B_t^1 + \bar{\sigma}_2 B_t^2 + \ldots + \bar{\sigma}_n B_t^n) \tag{3.11}$$

Power can be simplified once more if we let $\tilde{\mu} = \bar{\mu} - \frac{1}{2}(\bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \ldots + \bar{\sigma}_n^2)$ and represent the sum of Brownian motions as a single Brownian motion by adjusting the coefficients accordingly. So

$$\bar{\sigma}_1 B_t^1 + \bar{\sigma}_2 B_t^2 + \ldots + \bar{\sigma}_n B_t^n = \tilde{\sigma}\tilde{B}_t$$

where

$$\tilde{\sigma} = \sqrt{\bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \ldots + \bar{\sigma}_n^2}. \tag{3.12}$$

Finally, the portfolio wealth process is

$$V_t = V_0\boldsymbol{exp}\ (\tilde{\mu}t + \tilde{\sigma}\ \widetilde{B_t}) \tag{3.13}$$

At this point, it is clear that the power

$$R_t = \tilde{\mu}t + \tilde{\mu}\ \widetilde{B_t} \tag{3.14}$$

so called return of the portfolio is a Brownian motion with drift and diffusion coefficients. Since it represents the rate at which the portfolio wealth is changing, $R(0) = 0$.

3.2 Minimum Return Level

Given the portfolio wealth process by (3.13), it is clear that minimum portfolio wealth by high confidence level is reached when (3.14) obtains the lowest value by the same confidence level. In order to measure it, we need to know the probability distribution function of portfolio returns. Once we have estimated the probability distribution function F for portfolio returns, we can extract the quantile $F^{-1}(\alpha)$, where alpha is a significance level, usually taken to be 1% or 5%. The key improvement brought by the First Passage Time is that, if the estimated portfolio return probability density function does not turn out to be symmetric while the volatility is significantly large, then the portfolios' expected bounded FPTs will often differ a lot. Graphically, if we denote MRL as $m = F^{-1}(\alpha)$, on a normal distribution density function, it looks as follows

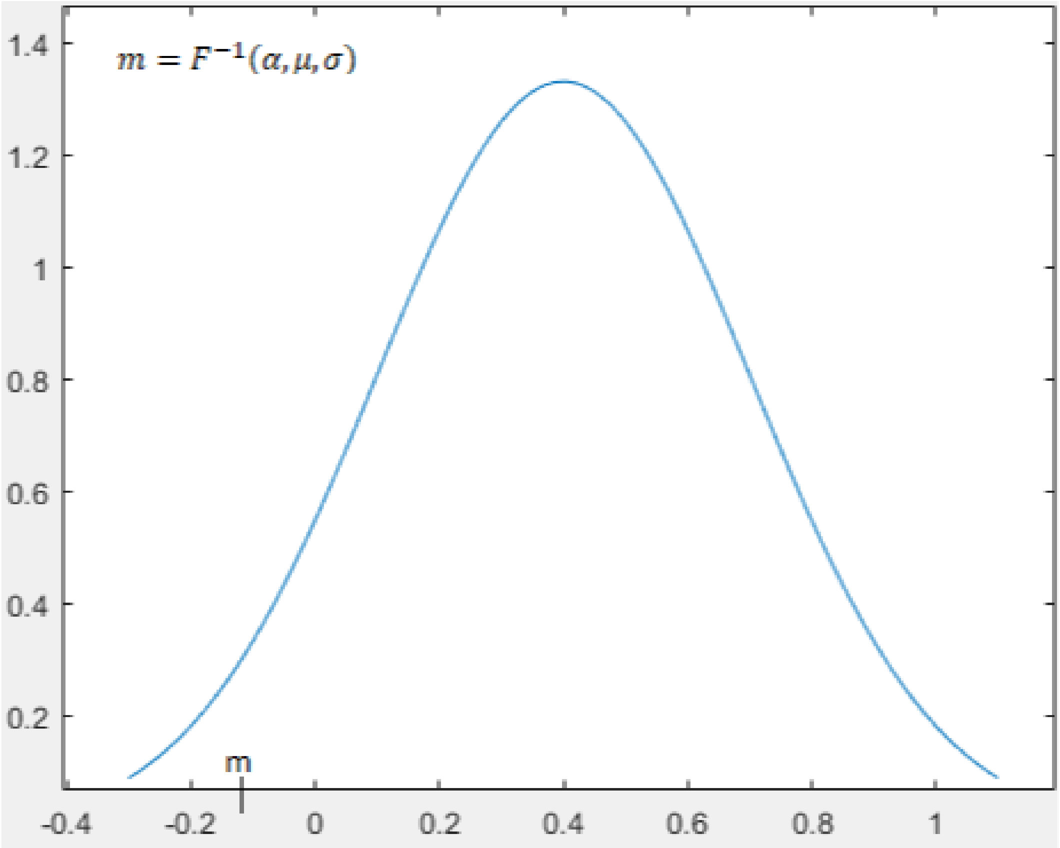

Figure 3.1: m level taken from the Normal distribution density function

In the figure above, $m = F^{-1}(\alpha, \mu, \sigma), -0.1428 = F^{-1}(0.05, 0.4, 0.33)$. From now on we will use m as the lowest level for the returns process (3.14) to reach in order to obtain the lowest portfolio wealth.

Here we compute the MRL assuming that the probability distribution density function is known. However it has to be estimated in real world scenario.

3.3 Expectation of Bounded First Passage Time

Next step is to define the new dimension – expectation of bounded first passage time. For a Brownian motion with drift

$$X_t = \mu t + W_t \tag{3.15}$$

if we denote the minimum value of this process till time t as:

$$M_t^x = \inf_{s \le t} X_s \tag{3.16}$$

and let $\tau_y = \min\{t \geq 0; X_t \leq y\}$ be the first passage time to the level y, then it is shown from [6] that the probability distribution function for τ_y is given by

$$P(M_t^x \leq y) = P(\tau_y \leq t)N\left(\frac{y - \mu t}{\sqrt{t}}\right) + \exp(2\mu y)\, N(\frac{y + \mu t}{\sqrt{t}}) \tag{3.17}$$

where $N(x)$ is the cumulative standard normal probability distribution function.

We are looking for the first passage time for the returns process given by (3.14) towards the level m (which we called MRL). m is usually a negative quantity.

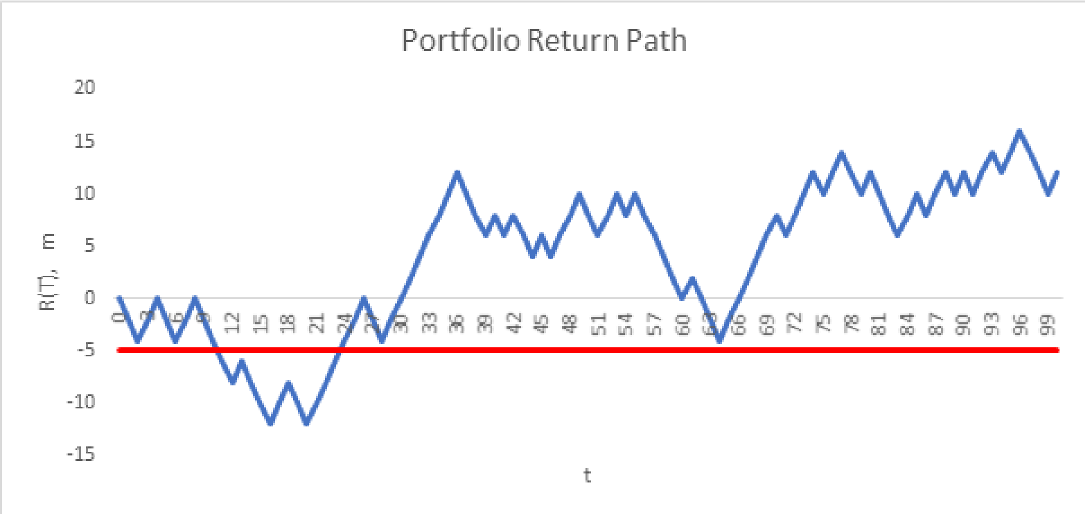

Figure 3.2: Brownian motion path and the m level

In Figure 3.2, MRL: $m = -5\%$, return process: $R_t = 0.35t + 0.25\,\tilde{B}_t$, positive drift: $\mu = 0.35$, diffusion: $\sigma = 0.25$.

We know that $R(0) = 0$. In order for (3.14) to reach the m level, the following equation must be satisfied

$$\frac{m}{\tilde{\sigma}} = \frac{\tilde{\mu}}{\tilde{\sigma}}t + \tilde{B}_t \tag{3.18}$$

So, the first passage time $\tau_m = \min\{t \geq 0; R_t \leq m\}$ has the probability distribution function

$$P(R_t \leq m) = N\left(\frac{m - \tilde{\mu}t}{\tilde{\sigma}\sqrt{t}}\right) + \exp\left(\frac{2\tilde{\mu}m}{\tilde{\sigma}\tilde{\sigma}}\right) N(\frac{m + \mu t}{\tilde{\sigma}\sqrt{t}}) \tag{3.19}$$

If we have a Brownian motion with drift and diffusion given by

$$dX_t = \mu dt + \sigma dW_t \tag{3.20}$$

and $\tau = \min\{t \geq 0; X_t \leq y\}$, then it is shown in [40] that the probability density function of τ_y is given by

$$f_{\tau_y}(t) = \frac{|y - X_0|}{2\pi\sigma^2 t^3} exp\ (\frac{(\mu t - y + X_0)^2}{2\sigma^2 t} \tag{3.21}$$

Correspondingly by [41]

$$E[\tau_y \wedge T] = \int_0^T t f_{\tau_y}(t)\, dt + T[1 - P(\tau_y \leq T)] \tag{3.22}$$

Converting (3.21) into the terms of R yields

$$f_{\tau_m}(t) - \frac{|m - R(0)|}{2\pi\tilde{\sigma}^2 t^3} \exp\ (- \frac{\left(\tilde{\mu} t - m + +R(0)\right)^2}{2\ \tilde{\sigma}^2 t} \tag{3.23}$$

thus

$$E[\tau_m \wedge T] = \int_0^T t f_{\tau_m}(t) dt + T[1 - P(\tau_m \leq T)] \tag{3.24}$$

The reason we switch to the bounded first passage time is that since $R(0) = 0 = m$, from (3.14) it can be shown that for $\tilde{\mu} > 0, E(\tau_m) = \infty$. We always consider portfolio return process which has a positive drift, because we examine only long portfolios in this paper.

3.4 Mean-MRL-FPT Framework

After having defined the portfolio wealth process, and MRL and expectation of the bounded first passage time, we can construct the model of portfolio optimization. The goal is to find the maximum MRL and bounded First Passage Time for a given expected return for the investment end time T: $E[R_T] = \tilde{\mu}T$.

$$\begin{gathered} E[\tau_m \wedge T] \\ E[R_T] \\ m \end{gathered} \tag{3.25}$$

Varying the weights $q_1, q_2, \ldots, q_n$ allocated in the assets gives us the set of different portfolios from which selecting the best combination of the above quantities yields the efficient surface

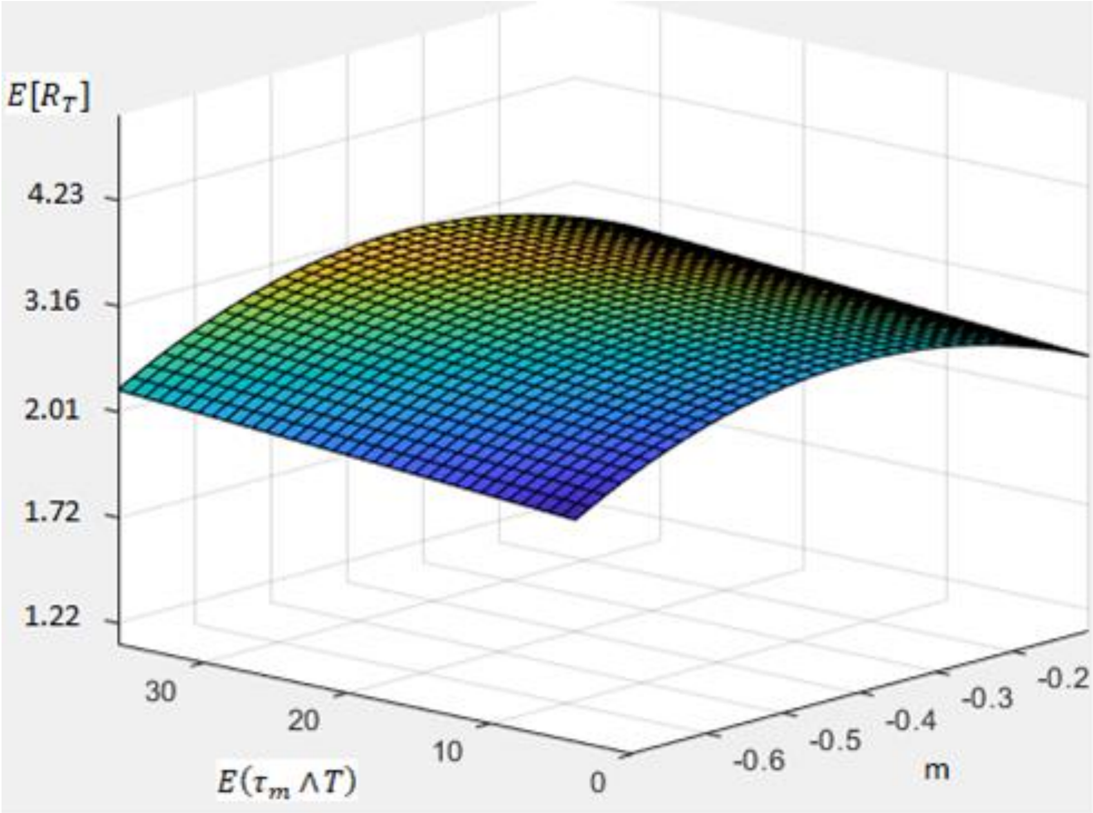

Figure 3.3: Surface of portfolios obtained by different weights allocated in the assets included

On this surface, all risky portfolios are optimal in Mean-MRL-FPT sense since it is impossible to find a better combination of given quantities for each.

As an important note, the model is particularly useful when the individual assets within a portfolio have large variance causing the portfolio variance to be large as well. This makes the portfolio returns likely to hit the minimum level before the investment horizon. So, in this case $E[\tau_m \wedge T] < T$ and it makes sense to compare such portfolios. Otherwise, if the individual volatilities are sufficiently low, then no matter what weights are allocated in each asset, the expected bounded first passage time almost always coincides to the investment horizon-T. In this case, $E[\tau_m \wedge T] = T$ for any set of weights allocated to different assets and the first passage time can be dropped altogether and the decision is to be made solely on two dimensions – Mean and Minimum Return Level. In such a situation, we would obtain the two-dimensional curve that looks much like the efficient frontier

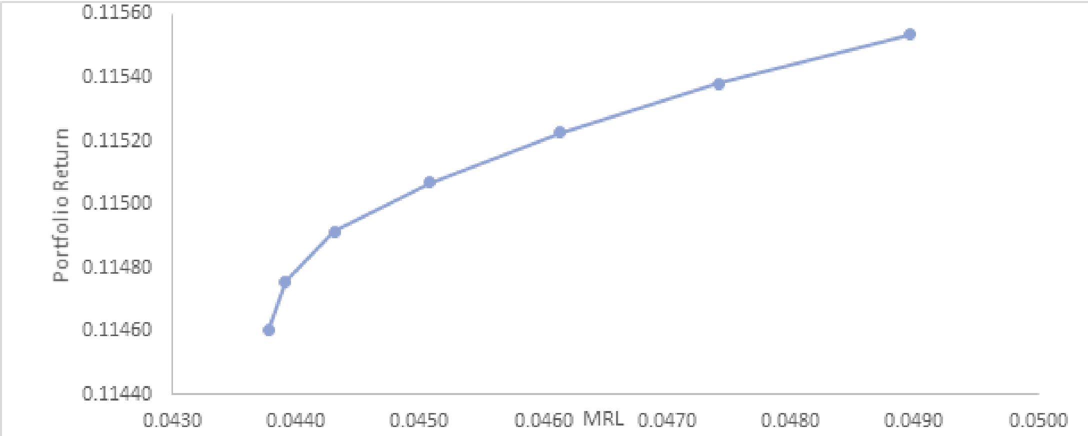

Figure 3.4: Efficient frontier constructed by $E[R_T]$ and $m = F^{-1}(\alpha)$ for different portfolios

m	0.042	0.045	0.0448	0.0453	0.0463	0.0478	0.0489
$E[R_T]$	0.1145	0.1149	0.1152	0.1152	0.1152	0.1154	0.1156

Table 3.1: Efficient frontier

3.5 Example of Two-Asset Portfolio

Here we apply the theory described above for a portfolio consisting of two assets. We take two Exchange Traded Funds (ETFs) from NASDAQ and observe their spot prices on a daily bases for four months. Table E1 shows a historical data for Vanguard Total Stock Market ETF (VTI) and iShares 7-10 Year Treasury Bond ETF (EIF) with the daily returns computed.

Considering the investment horizon of $T = 30$ days, we compute $\bar{\mu}$ and $\bar{\sigma}_j$ for $j = 1,2$ from (3.8) and (3.9) for various combinations of weights. In addition we compute $\tilde{\sigma}$ from (3.12) and $P(\tau_m \leq T)$ from (3.17). This information is given in Table E2.

Since the portfolio wealth process obtains the minimum level by a given confidence level wherever (3.14) drops to its minimum m by the same confidence level, we observe the rate instead of the portfolio value itself. Ultimately the three dimensions are generated in Table E3. This table corresponds to the efficient surface similar to Figure 3.4. See Appendix E for the tables.

3.6 Example of Four-Asset Portfolio

This section covers the portfolio consisting of multiple assets. In particular, we consider a portfolio consisting of four common stocks – Travelzoo (TZOO), AXT Inc. (AXTI), Universal Forest Products (UFPI), Advanced Micro Devices (AMD). The investment horizon is taken to be 10 years. Unlike the previous example, here we take the annual spot prices. Table E4 illustrates the annual spot prices from 2009 to 2019 taken from NASDAQ.

We compute $\bar{\mu}$ and $\bar{\sigma}_j$ for $j = 1,2,3,4$ from (3.8) and (3.9) for various combinations of weights. We also compute $\tilde{\sigma}$ from (3.12). This information is given in Table E5. Ultimately the three dimensions are generated in Table E6. This table corresponds to the efficient surface similar to Figure 3.4. Since there are too many combinations of weights constructing different portfolios, Table E6 illustrates some combinations of weights of four assets. Detailed results are provided in the accompanying spreadsheet.

If we consider a hypothetical scenario where the third dimension – the expected bounded first passage time equals the investment horizon, this dimension would be dropped and the entire model would be replaced by the two dimensional analogue as discussed above. Ultimately the result would again be the efficient frontier of risky assets with the expected returns and the minimum return levels measured for them. This case is illustrated in the following example. See Appendix E for the tables.

3.7 Example of Three-Asset Portfolio

This section covers the portfolio consisting of three assets whose expected bounded first passage times made up from different portfolios turn out to be quite different. So the portfolio returns are expected to hit the low barrier in different times before the investment horizon. In Particular, we consider a portfolio consisting of three common stocks – Apple Inc. (AAPL), JPMorgan Chase Co. (JPM) and Walmart Inc.

(WMT). The investment horizon is taken to be 30 months. Here we take the monthly spot prices again. Table E7 illustrates the monthly spot prices from April the 28th, 2017 to 1st of April, 2019, taken from NASDAQ.

We compute $\bar{\mu}$ and $\bar{\sigma}$ for $j = 1,2,3$ from (3.8) and (3.9) for various combinations of weights. We also compute $\tilde{\sigma}$ from (3.12). This information is given in Table E8. Ultimately the three dimensions are generated in Table E9. This table corresponds to the efficient surface similar to Figure 3.4. If we had done the portfolio optimization consisting of these three assets in mean-variance framework, the results would be as shown in Table E10 with the corresponding scatter plot in Figure 3.5. All tables for this example are illustrated in Appendix E.

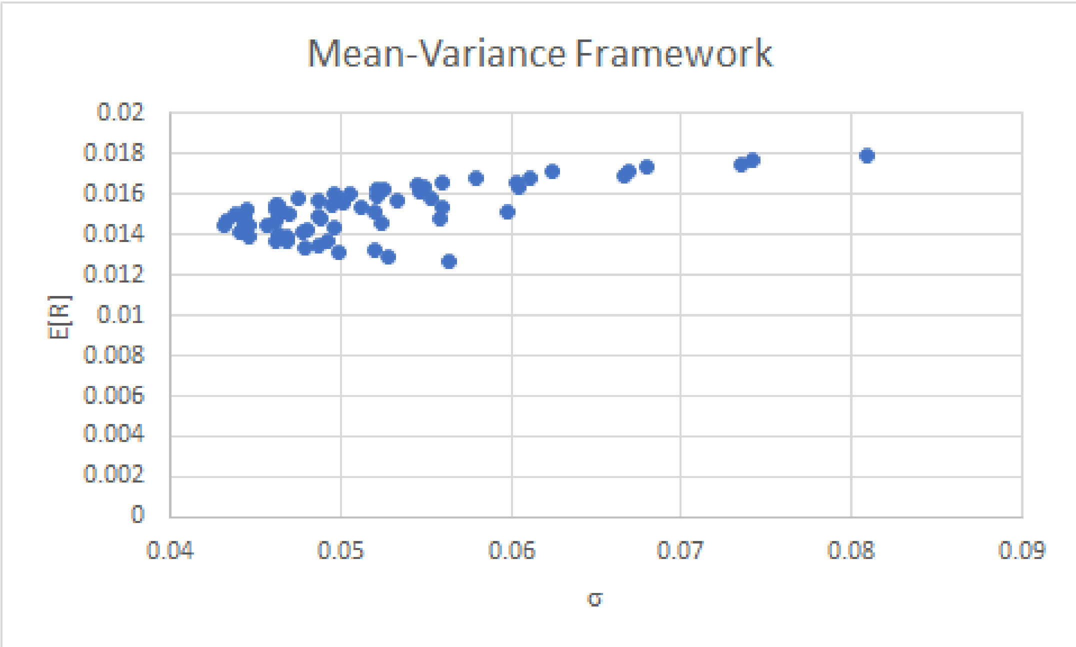

Figure 3.5: All Portfolios, Mean-Variane Framework

3.8 Example of Multi-Asset Portfolio with Large FPTs

This section covers the portfolio consisting of three assets whose expected bounded first passage times made up from different portfolios turn out to be very close to the investment horizon. So the portfolio returns are not expected to hit the low barrier throughout the investment period. In Particular, we consider a portfolio consisting of three common stocks – YUMA Energy Inc (YUMA), Immunic Inc.

(IMUX), Savara Inc (SVRA). The investment horizon is taken to be 30 days. Unlike the previous example, here we take the daily spot prices. Table E11 illustrates the daily spot prices from March the 25th, 2019 to 23rd of April, 2019 taken from NASDAQ.

We compute $\bar{\mu}$ and $\bar{\sigma}_j$ for $j = 1{,}2$ from (3.8) and (3.9) for various combinations of weights. We also compute $\tilde{\sigma}$ from (3.12). This information is summarized in Table E12. Ultimately the three dimensions are generated in Table E13. This table corresponds to the efficient surface similar to Figure 3.4.

Since expected bounded first passage time coincides with the investment horizon for any portfolio made up of these assets. This means that we can ignore this dimension altogether. So instead of the three dimensional surface, here we draw the efficient frontier in two dimensions. In particular, in the last column of the Table E13, all values would be 30. In this case we obtain an efficient frontier illustrated in Figure 3.6. Figure 3.4 where MRL is taken as a risk measure instead of volatility is a preferable option though. Detailed results are provided in the accompanying spreadsheet. See Appendix E for the tables.

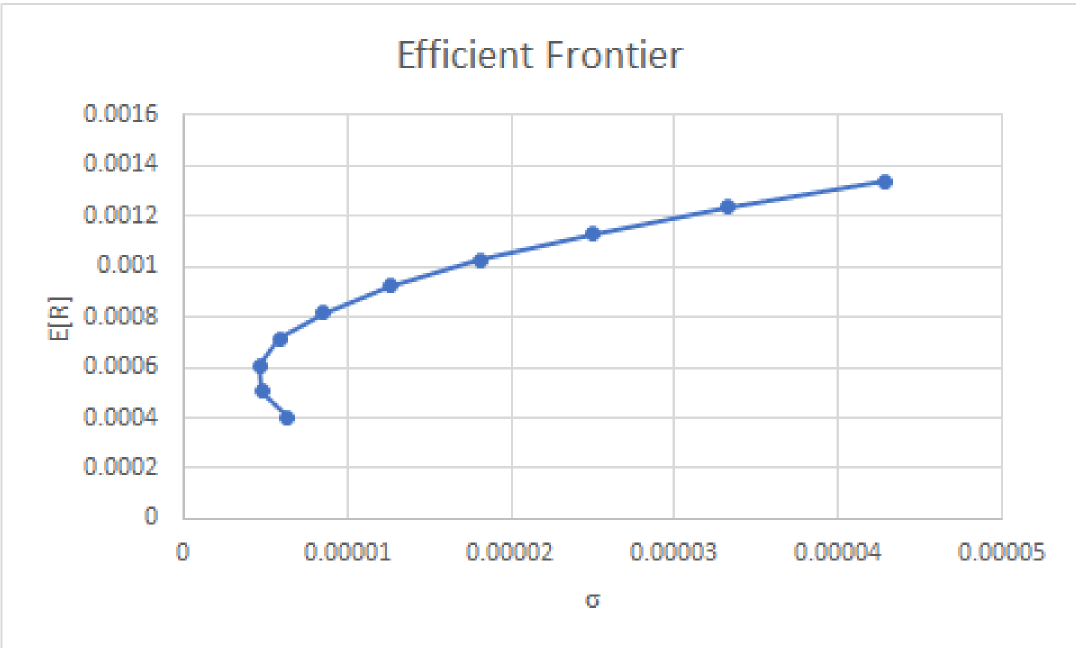

Figure 3.6: Efficient Frontier, Mean-Variance Framework

Conclusion

It can be concluded that the non-cooperative trade game results in a problem of optimizing tariffs on imported commodities. We derived system of equations and sufficient conditions for the Nash equilibrium point which gives the optimal triple – tariffs imposed by nations on imported commodities and the currency exchange rate. We illustrated that shapes of the demand functions determine the economic power one nation has over another. However, from the demand functions alone, it is impossible to predict which nation will have a greater optimal tariff to be imposed. It is assumed that the distributions of commodities exchanged and the prices for those commodities are known. The examples provided are intended to illustrate the typical cases of economically symmetric and asymmetric nations involved in non-cooperative bilateral trade game. Obviously, in real world scenario, the demand functions are not predetermined. Rather, they are derived from (1.1) and (1.2). Next we provided several pairs of the foreign currency demand functions where one nation dominates the other.

Within the same context, the second chapter dealt with the Gibbon's model where a similar problem is solved. Namely, economic welfare functions for the nations differ from those in the first model. Here we solved the economic welfare maximization problem with and without the concept of a currency exchange rate and obtained the optimal production volumes for each nation aimed for the domestic and foreign markets.

The final part of the thesis explored the portfolio selection process by introducing the framework involving three dimensions. The basic idea was to extend the two-dimensional framework by an additional one – the expected bounded first passage time. The usefulness of the approach is evident once the individual assets within a portfolio have large volatilities causing the returns to hit the minimum level before the investment horizon. The paper only concerned itself with optimizing risky portfolios. There can be numerous continuations to the problem. If an investor decides to allocate part of the investment amount into some risk-free assets, then the optimal weights must be modified according to some criteria. In two-dimensional Mean-Variance model, maximization of Sharpe ratio and building a Capital Allocation Line (CAL) is one possible development. Similarly, one may think of capital allocation plane as an analogue to the CAL in 3D. However, this model is restricted to risky portfolio optimization. Within this chapter, we provide an example where a portfolio consists of stocks and construct the efficient surface for that particular example. This example can easily be extended to portfolios of any assets including the currency exchange pairs.

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Appendix A

The following system of equations determine the tariffs each nation has to set in order to obtain the maximum gains from trade

$$\frac{\partial}{\partial\theta}G(e,\theta,\theta^*) = G_\theta(e,\theta,\theta^*) + G_e(e,\theta,\theta^*)e_\theta = 0, \tag{26}$$

$$\frac{\partial}{\partial\theta}G^*(e,\theta,\theta^*) = G^*_{\theta^*}(e,\theta,\theta^*) + G^*_e(e,\theta,\theta^*)e_\theta = 0 \tag{27}$$

where the index denotes the partial derivative of a given function with respect to a given variable.

If we define $g(e,\theta) = \frac{e}{\theta}$ and $g(e,\theta) = \frac{1}{\theta^* e}$, the individual components of (26) and (27) become

$$G_\theta(e,\theta,\theta^*) = \frac{\partial}{\partial g}[-\int_g^\infty yD'(y)dy]\frac{\partial g}{\partial\theta} - \frac{\partial}{\partial\theta}D^*(\theta^* e) = gD'(g)g'(\theta) \tag{28}$$

$$= \frac{e}{\theta}D'\left(\frac{e}{\theta}\right)\left(-\frac{e}{\theta^2}\right) = -\frac{e^2}{\theta^3}D'(\frac{e}{\theta}),$$

$$G_e(e,\theta,\theta^*) = \frac{\partial}{\partial g}[-\int_g^\infty yD'(y)dy]\frac{\partial g}{\partial e} - \frac{\partial}{\partial e}D^*(\theta^* e) \tag{29}$$

$$= gD'(g)g'(e) - \theta^* D^{*\prime}(\theta^* e) = \frac{e}{\theta^2}D'\left(\frac{e}{\theta}\right) - \theta^* D^{*\prime}(\theta^* e),$$

$$G^*_{\theta^*}(e,\theta,\theta^*) = \frac{\partial}{\partial g}[\int_g^\infty \frac{1}{y}D^{*\prime}\left(\frac{1}{y}\right)dy]\frac{\partial g}{\partial\theta^*} - \frac{\partial}{\partial\theta^*}D\left(\frac{e}{\theta}\right) \tag{30}$$

$$= -\theta^* e D^{*\prime}(\theta^* e)\left(-\frac{1}{\theta^{*2}e}\right) = \frac{1}{\theta^*}D^{*\prime}(\theta^* e),$$

$$G^*_e(e,\theta,\theta^*) = \frac{\partial}{\partial g}[\int_g^\infty \frac{1}{y}D^{*\prime}\left(\frac{1}{y}\right)dy]\frac{\partial g}{\partial e} - \frac{\partial}{\partial e}D\left(\frac{e}{\theta}\right) \tag{31}$$

$$= -\theta^* e D^{*\prime}(\theta^* e)\left(-\frac{1}{\theta^* e^2}\right) - \frac{1}{\theta}D'\left(\frac{e}{\theta}\right)$$

$$= \frac{1}{e}D^{*\prime}(\theta^* e) - \frac{1}{\theta}D'\left(\frac{e}{\theta}\right),$$

e_θ and e_{θ^*} can be found from (1.5) as described next.

Let us define

$$F(e,\theta,\theta^*) = eD\left(\frac{e}{\theta}\right) - D^*(\theta^* e). \tag{32}$$

Differentiating (32) with respect to θ and θ^* separately and equating them to zero yields the system of equations

$$F_\theta(e,\theta,\theta^*) + F_e(e,\theta,\theta^*)e_\theta = 0,$$

$$F_{\theta^*}(e,\theta,\theta^*) + F_e(e,\theta,\theta^*)e_{\theta^*} = 0,$$

from which solving for e_θ and e_{θ^*} gives

$$e_\theta = -\frac{F_\theta(e,\theta,\theta^*)}{F_e(e,\theta,\theta^*)} = \frac{\frac{e^2}{\theta^2}D'\left(\frac{e}{\theta}\right)}{D\left(\frac{e}{\theta}\right) + \frac{e}{\theta}D'\left(\frac{e}{\theta}\right) - \theta^* {D^*}'(\theta^* e)}, \tag{33}$$

$$e_{\theta^*} = -\frac{F_{\theta^*}(e,\theta,\theta^*)}{F_e(e,\theta,\theta^*)} = \frac{e{D^*}'(\theta^* e)}{D\left(\frac{e}{\theta}\right) + \frac{e}{\theta}D'\left(\frac{e}{\theta}\right) - \theta^* {D^*}'(\theta^* e)}, \tag{34}$$

Putting these solutions into the system of equations (26), (27) yields the following results. From (26) we have

$$\begin{aligned}
&\frac{\partial}{\partial\theta}G(e(\theta,\theta^*),\theta,\theta^*) \\
&\quad = -\frac{e^2}{\theta^3}D'\left(\frac{e}{\theta}\right) \\
&\quad + \left[\frac{e}{\theta^2}D'\left(\frac{e}{\theta}\right)\right. \\
&\quad \left. - \theta^* {D^*}'(\theta^* e)\right]\left[\frac{\frac{e^2}{\theta^2}D'\left(\frac{e}{\theta}\right)}{D\left(\frac{e}{\theta}\right) + \frac{e}{\theta}D'\left(\frac{e}{\theta}\right) - \theta^* {D^*}'(\theta^* e)}\right] \\
&\quad = \frac{\frac{e^2}{\theta^2}D'\left(\frac{e}{\theta}\right)}{D\left(\frac{e}{\theta}\right) + \frac{e}{\theta}D'\left(\frac{e}{\theta}\right) - \theta^* {D^*}'(\theta^* e)}\left[\frac{e}{\theta^2}D'\left(\frac{e}{\theta}\right)\right. \\
&\quad \left. - \theta^* {D^*}'(\theta^* e) - \frac{1}{\theta}\left[D\left(\frac{e}{\theta}\right) + \frac{e}{\theta}D'\left(\frac{e}{\theta}\right) - \theta^* {D^*}'(\theta^* e)\right]\right] \\
&\quad = \frac{e_\theta}{\theta^2}\left[eD'\left(\frac{e}{\theta}\right) - \theta^2\theta^* {D^*}'(\theta^* e) - \theta D\left(\frac{e}{\theta}\right) - eD'\left(\frac{e}{\theta}\right)\right. \\
&\quad \left. + \theta\theta^* {D^*}'(\theta^* e)\right] = \frac{e_\theta}{\theta}\left[-D\left(\frac{e}{\theta}\right) + \theta^*(1-\theta){D^*}'(\theta^* e)\right] = 0
\end{aligned} \tag{35}$$

and from (1.9) we have

$$\frac{\partial}{\partial\theta^*}G^*(e(\theta,\theta^*),\theta,\theta^*) \tag{36}$$

$$=\frac{1}{\theta^*}D^{*\prime}(\theta^* e)$$

$$+\left[\frac{1}{e}D^{*\prime}(\theta^* e)\right.$$

$$\left.-\frac{1}{\theta}D'\left(\frac{e}{\theta}\right)\right]\left[\frac{eD^{*\prime}(\theta^* e)}{D\left(\frac{e}{\theta}\right)+\frac{e}{\theta}D'\left(\frac{e}{\theta}\right)-\theta^* D^{*\prime}(\theta^* e)}\right]$$

$$=\frac{eD^{*\prime}(\theta^* e)}{D\left(\frac{e}{\theta}\right)+\frac{e}{\theta}D'\left(\frac{e}{\theta}\right)-\theta^* D^{*\prime}(\theta^* e)}\left[\frac{1}{e\theta^*}\left[D\left(\frac{e}{\theta}\right)+\frac{e}{\theta}D'\left(\frac{e}{\theta}\right)\right.\right.$$

$$\left.\left.-\theta^* D^{*\prime}(\theta^* e)\right]+\left[\frac{1}{e}D^{*\prime}(\theta^* e)-\frac{1}{\theta}D'\left(\frac{e}{\theta}\right)\right]\right]$$

$$=\frac{e_{\theta^*}}{e}\left[\frac{1}{\theta^*}D\left(\frac{e}{\theta}\right)+\frac{e}{\theta^*\theta}D'\left(\frac{e}{\theta}\right)-D^{*\prime}(\theta^* e)+D^{*\prime}(\theta^* e)\right.$$

$$\left.-\frac{e}{\theta}D'\left(\frac{e}{\theta}\right)\right]=\frac{e_{\theta^*}}{\theta^* e}\left[D\left(\frac{e}{\theta}\right)-\frac{e}{\theta}(\theta^*-1)D'\left(\frac{e}{\theta}\right)\right]=0.$$

Hence

$$D\left(\frac{e}{\theta}\right)=\theta^*(1-\theta)D^{*\prime}(\theta^* e),$$

$$D\left(\frac{e}{\theta}\right)=\frac{e}{\theta}(\theta^*-1)D'\left(\frac{e}{\theta}\right).$$

For the second derivatives, we have

$$\frac{\partial^2}{\partial\theta^2}G(e(\theta,\theta^*),\theta,\theta^*) \tag{37}$$

$$=\frac{\theta e_{\theta\theta}-e_\theta}{\theta^2}\left[\theta^*(1-\theta)D^{*\prime}(\theta^* e)-D\left(\frac{e}{\theta}\right)\right]$$

$$+\frac{e_\theta}{\theta}\left[\theta^{*2}(1-\theta)e_\theta D^{*\prime\prime}(\theta^* e)-\theta^* D^{*\prime}(\theta^* e)\right.$$

$$\left.-\frac{e_\theta\theta-e}{\theta^2}D'\left(\frac{e}{\theta}\right)\right],$$

$$\frac{\partial^2}{\partial\theta^{*2}}G^*(e(\theta,\theta^*),\theta,\theta^*) \tag{38}$$

$$= \frac{e\theta^* e_{\theta^*\theta^*} - \theta^* e_{\theta^*}^2 - e e_{\theta^*}}{(e\theta^*)^2}\left[D\left(\frac{e}{\theta}\right) - \frac{e}{\theta}(\theta^* - 1)D'\left(\frac{e}{\theta}\right)\right]$$

$$+ \frac{e_{\theta^*}}{e\theta^*}\left[\frac{e_{\theta^*}}{\theta}D'\left(\frac{e}{\theta}\right) - \frac{(1-\theta^*)e_{\theta^*} - e}{\theta}D'\left(\frac{e}{\theta}\right)\right.$$

$$\left. - \frac{(1-\theta^*)e_{\theta^*}e}{\theta^2}D''\left(\frac{e}{\theta}\right)\right]$$

$$= \frac{e\theta^* e_{\theta^*\theta^*} - \theta^* e_{\theta^*}^2 - e e_{\theta^*}}{(e\theta^*)^2}\left[D\left(\frac{e}{\theta}\right) - \frac{e}{\theta}(\theta^* - 1)D'\left(\frac{e}{\theta}\right)\right]$$

$$+ \frac{e_{\theta^*}}{e\theta^*}\left[\frac{\theta^* e_{\theta^*} + e}{\theta}D'\left(\frac{e}{\theta}\right) - \frac{(1-\theta^*)e_{\theta^*}e}{\theta^2}D''\left(\frac{e}{\theta}\right)\right].$$

Assuming that the system (1.10), (1.11) has a solution and $xD(x) \to 0$ as $x \to \infty$, we get

$$xD\left(\frac{x}{\theta}\right) - D^*(\theta^* x) > 0, if\ x < e,$$
$$xD\left(\frac{x}{\theta}\right) - D^*(\theta^* x) = 0, if\ x = e,$$
$$xD\left(\frac{x}{\theta}\right) - D^*(\theta^* x) < 0, if\ x > e.$$

Then

$$\frac{d}{dx}|_{x=e}\left[xD\left(\frac{x}{\theta}\right) - D^*(\theta^* x)\right] = D\left(\frac{e}{\theta}\right) + \frac{e}{\theta}D'\left(\frac{e}{\theta}\right) - \theta^* D^{*\prime}(\theta^* e) < 0,$$

and from (33), (34) follows that $e_\theta > 0, e_{\theta^*} < 0$. Since the first summands of (37) and (38) are zero, the conditions

$$\frac{\partial^2}{\partial\theta^2}G(e(\theta,\theta^*),\theta,\theta^*) < 0, \frac{\partial^2}{\partial\theta^{*2}}G^*(e(\theta,\theta^*),\theta,\theta^*) < 0$$

provide (1.16), (1.17).

Appendix B

Differentiating the given demand functions

$$D(x) = \boldsymbol{exp}(-\delta x), \quad D^*(x) = \alpha x \boldsymbol{exp}(\beta x) \tag{39}$$

gives

$$D'(x) = -\delta \boldsymbol{exp}(-\delta x), \;\; D^{*\prime}(x) = (\alpha\beta x + \alpha)\, \boldsymbol{exp}(\beta x), \tag{40}$$

$$D'(x) = -\delta^2 \boldsymbol{exp}(-\delta x), \;\; D^{*\prime\prime}(x) = (\alpha\beta^2 x + 2\alpha\beta)\, \boldsymbol{exp}(\beta x). \tag{41}$$

The equilibrium exchange rate is found from (1.5) as follows

$$e\boldsymbol{exp}\left(-\delta \frac{e}{\theta}\right) = \alpha\theta^* e \boldsymbol{exp}(\beta\theta^* e), \tag{42}$$

$$-\delta \frac{e}{\theta} = \ln(\alpha\theta^*) + \beta\theta^* e,$$

$$e(\theta\theta^*\beta + \delta) = -\theta \ln(\alpha\theta^*),$$

$$e = \frac{-\theta \ln{(\alpha\theta^*)}}{\theta\theta^*\beta + \delta}. \tag{43}$$

Here in (42), e is eliminated and logarithms are taken from both sides. From (1.11) we find θ

$$\boldsymbol{exp}\left(-\delta \frac{e}{\theta}\right) = \frac{e}{\theta}(\theta^* - 1)\left(-\delta \boldsymbol{exp}\left(-\delta \frac{e}{\theta}\right)\right),$$

cancelling $\boldsymbol{exp}\left(-\delta \frac{e}{\theta}\right)$ from both sides gives

$$1 = \frac{e}{\theta}(1 - \theta^*)\delta,$$

substituting e from (43) gives

$$1 = \frac{(\theta^* - 1)\delta \ln{(\alpha\theta^*)}}{\theta\theta^*\beta + \delta},$$

now θ can be expressed in terms of θ^* solely as

$$\theta = \frac{(\theta^* - 1)\delta \ln{(\alpha\theta^*)} - \delta}{\theta^*\beta}. \tag{44}$$

Putting (43), (44) into (1.10) leads to the solution of θ^*. Specifically, redefining (1.10) in terms of (39) gives

$$\boldsymbol{exp}\left(-\delta\frac{e}{\theta}\right) = \theta^*(1-\theta)\alpha\boldsymbol{exp}(\beta\theta^* e)(\beta\theta^* e + 1),$$

putting (43) into this equation results in the following expression

$$\boldsymbol{exp}\left(\delta\frac{ln(\alpha\theta^*)}{\theta\theta^*\beta+\delta}\right)$$
$$= \theta^*(1-\theta)\alpha\boldsymbol{exp}\left(\beta\theta^*\left[\frac{-\theta\ln(\alpha\theta^*)}{\theta\theta^*\beta+\delta}\right]\right)\left(\delta\theta^*\left[\frac{-\theta\ln(\alpha\theta^*)}{\theta\theta^*\beta+\delta}\right]+1\right),$$

replacing θ with its definition from (44)

$$\exp\left(\frac{\delta\ln(\alpha\theta^*)}{(\theta^*-1)\delta\ln(\alpha\theta^*)-\delta+\delta}\right)$$
$$= \theta^*\frac{\theta^*\beta-(\theta^*-1)\delta\ln(\alpha\theta^*)+\delta}{\theta^*\beta}\alpha\boldsymbol{exp}\left(\frac{1-(\theta^*-1)\ln(\alpha\theta^*)}{\theta^*-1}\right)\left(\frac{1-(\theta^*-1)\ln(\alpha\theta^*)}{\theta^*-1}\right),$$

eliminating and rearranging some terms gives a simplified equation

$$\exp\left(\frac{1}{\theta^*-1}\right)$$
$$= \frac{\theta^*\beta(\theta^*-1)\delta\ln(\alpha\theta^*)+\delta}{\beta}\alpha\boldsymbol{exp}\left(\frac{1-(\theta^*-1)\ln(\alpha\theta^*)}{\theta^*-1}\right)\frac{\theta^*-(\theta^*-1)\ln(\alpha\theta^*)}{\theta^*-1},$$

combining the exponents gives

$$\boldsymbol{exp}\left(\frac{1}{\theta^*-1}-\frac{1-(\theta^*-1)\ln(\alpha\theta^*)}{\theta^*-1}\right)$$
$$= \frac{\theta^*\beta-(\theta^*-1)\delta\ln(\alpha\theta^*)+\delta}{\beta}\alpha\frac{\theta^*-(\theta^*-1)\ln(\alpha\theta^*)}{\theta^*-1},$$

simplifying the power of the exponent yields

$$\alpha\theta^* = \frac{\theta^*\beta-(\theta^*-1)\delta\ln(\alpha\theta^*)+\delta}{\beta}\alpha\frac{\theta^*-(\theta^*-1)\ln(\alpha\theta^*)}{\theta^*-1},$$

finally, we obtain the equation involving only θ^* to solve for

$$\beta\theta^*(\theta^*-1) = (\theta^*\beta - (\theta^*-1)\delta\ln(\alpha\theta^*) + \delta)(\theta^* - (\theta^*-1)\ln(\alpha\theta^*)). \quad (45)$$

This equation cannot be explicitly solved for θ^* but it can be computed approximately. Putting $\alpha = 0.01, \beta = 2, \delta = 2.5$ into (45) gives $\theta^* = 0.73$, putting this value into (44) gives $\theta = 0.54$, and ultimately the equilibrium exchange rate is obtained by putting these values in (43) which gives $e = 0.81$. So, the equilibrium triple is $(\hat{e}, \hat{\theta}, \hat{\theta}^*) = (0.81, 0.54, 0.73)$. The derivatives of the exchange rate function with respect to θ^* and θ are

$$e_{\theta^*} = \frac{\beta\theta^2\theta^*\ln(\alpha\theta^*) - \theta(\theta\theta^*\beta + \delta)}{\theta^*(\theta\theta^*\beta + \delta)^2} = -0.49,$$

$$e_\theta = \frac{-\delta\ln(\alpha\theta^*)}{(\theta\theta^*\beta + \delta)^2} = 1.13.$$

Using (40)-(42), inequalities (1.16), (1.17) take the form

$$\theta^*(1-\theta)e_\theta(\theta^*e\beta^2 + 2\beta) - 1 + \frac{\delta}{\theta^2}(e_\theta\theta - e) < 0,$$

$$\theta(\theta^*e_{\theta^*} + e) + \delta(1-\theta^*)ee_{\theta^*} < 0.$$

For $(\hat{e}, \hat{e}_\theta, \hat{e}_{\theta^*}, \hat{\theta}, \hat{\theta}^*) = (0.81, 1.13, -0.49, 0.54, 0.73)$, these inequalities can be verified.

Appendix C

In order to ensure that the equilibrium point $\left(\hat{e},\hat{\theta},\hat{\theta}^*\right)=\left(e\left(\hat{\theta},\hat{\theta}^*\right),\hat{\theta},\hat{\theta}^*\right)$ obtained by solving the system of equations (1.13), (1.14), (1.15), really gives the maximums of the gain functions, we proceed by checking the second derivatives

$$\frac{\partial^2}{\partial\theta^2}G(e,\theta,\theta^*) \tag{46}$$
$$=\frac{\partial}{\partial\theta}\left[-\frac{e^2}{\theta^3}D'\left(\frac{e}{\theta}\right)\right.$$
$$\left.+\left(\frac{e}{\theta^2}D'\left(\frac{e}{\theta}\right)-\theta^*D^{*\prime}(\theta^*e)\right)\frac{\frac{e^2}{\theta^2}D'\left(\frac{e}{\theta}\right)}{D\left(\frac{e}{\theta}\right)+\frac{e}{\theta D'\left(\frac{e}{\theta}\right)}-\theta^*D^{*\prime}(\theta^*e)}\right]$$
$$=\frac{\partial}{\partial\theta}\Big[\frac{e^2}{\theta^3}\delta\boldsymbol{exp}\left(-\delta\frac{e}{\theta}\right)$$
$$+\left(\frac{e}{\theta^2}\delta\boldsymbol{exp}\left(-\delta\frac{e}{\theta}\right)\right.$$
$$-\theta^*\alpha\boldsymbol{exp}(\beta\theta^*e)(\beta\theta^*e$$
$$\left.+1)\right)\frac{\frac{e^2}{\theta^2}\delta\boldsymbol{exp}\left(-\delta\frac{e}{\theta}\right)}{\boldsymbol{exp}\left(-\delta\frac{e}{\theta}\right)-\frac{e}{\theta}\delta\boldsymbol{exp}\left(-\delta\frac{e}{\theta}\right)-\theta^*\alpha\boldsymbol{exp}(\beta\theta^*e)(\beta\theta^*e+1)}\Big]$$

where e is a shorthand notation for $e(\theta,\theta^*)$. Here we substituted $D\left(\frac{e}{\theta}\right), D'\left(\frac{e}{\theta}\right), D^{*\prime}(\theta^*e)$ from (40). Since (46) involves several similar terms, let us introduce the following notations for more convenience

$$f(e,\theta)=\boldsymbol{exp}\left(-\delta\frac{e}{\theta}\right), \tag{47}$$
$$g(e)=\alpha\boldsymbol{exp}(\beta\theta^*e)(\beta\theta^*e+1)$$

then their derivatives with respect to θ are

$$\frac{\partial}{\partial\theta}f(e,\theta)=\frac{\delta}{\theta^2}\boldsymbol{exp}\left(-\delta\frac{e}{\theta}\right)(e-\theta e_\theta), \tag{48}$$

$$\frac{\partial}{\partial\theta}g(e)=\alpha\beta\theta^* e_\theta\boldsymbol{exp}(\beta\theta^* e)(2+\beta\theta^* e)$$

and the derivative of the exchange rate function (43) with respect to θ is

$$\frac{\partial e}{\partial\theta}=\frac{-\delta\ln(\alpha\theta^*)}{(\theta\theta^*\beta+\delta)^2},$$

applying these notations, (46) now becomes

$$\frac{\partial^2}{\partial\theta^2}G(e,\theta,\theta^*) \tag{49}$$

$$=\frac{\partial}{\partial\theta}\left[\frac{e^2}{\theta^3}\delta f(e,\theta)\right.$$

$$\left.+\left(\frac{e}{\theta^2}\delta f(e,\theta)+\theta^* g(e)\right)\frac{\frac{e^2}{\theta^2}\delta f(e,\theta)}{f(e,\theta)-\frac{e}{\theta}\delta f(e,\theta)-\theta^* g(e)}\right]$$

$$=\frac{e^2}{\theta^3}\delta\frac{\partial}{\partial\theta}f(e,\theta)+\delta f(e,\theta)\frac{2ee_\theta\theta^3-3\theta^2e^2}{\theta^4}$$

$$+\left(\frac{e}{\theta^2}\delta f(e,\theta)\right.$$

$$\left.+\theta^* g(e)\right)\frac{\left(f(e,\theta)-\frac{e}{\theta}\delta f(e,\theta)-\theta^* g(e)\right)\frac{e^2}{\theta^2}\delta\frac{\partial}{\partial\theta}f(e,\theta)+\delta f(e,\theta)\frac{2ee}{}}{\left(\frac{e}{\theta^2}\delta f(e,\theta)+\theta^* g(e)\right)^2}$$

$$-\frac{\frac{e^2}{\theta^2}\delta f(e,\theta)\left(\frac{\partial}{\partial\theta}f(e,\theta)-\frac{e}{\theta}\delta f'(e,\theta)-\delta f(e,\theta)\frac{\theta e\theta-e}{\theta^2}-\theta^*\frac{\partial}{\partial\theta}g(e)\right)}{\left(\frac{e}{\theta^2}\delta f(e,\theta)+\theta^* g(e)\right)^2}$$

$$+\frac{\frac{e^2}{\theta^2}\delta f(e,\theta)}{f(e,\theta)-\frac{e}{\theta}\delta f(e,\theta)-\theta^* g(e)}\left(\frac{e}{\theta^2}\delta\frac{\partial}{\partial\theta}f(e,\theta)+\delta f(e,\theta)\frac{\theta^2e_\theta-2\theta e}{\theta^4}\right.$$

$$+\theta^*\frac{\partial}{\partial\theta}g(e).$$

Putting the equilibrium values from Appendix A, $(\hat{e},\hat{\theta},\hat{\theta}^*)=(0.81,0.54,0.73)$ in (49) gives

$$G''(0.81, 0.54, 0.73) = -0.24 < 0.$$

Second derivative of $G^*(e, \theta, \theta^*)$ is

$$\frac{\partial^2}{\partial \theta^2} G^*(e, \theta, \theta^*) \tag{50}$$
$$= \frac{\partial}{\partial \theta^*} [\frac{1}{\theta^*} D^{*\prime}(\theta^* e)$$
$$- \left(\frac{1}{e} D^{*\prime}(\theta^* e) - \frac{1}{\theta} D'\left(\frac{e}{\theta}\right)\right) \frac{e D^{*\prime}(\theta^* e)}{f(e,\theta) - \frac{e}{\theta}\delta f(e,\theta) - \theta^* g(e)}$$
$$= \frac{\partial}{\partial \theta^*} [\frac{1}{\theta^*} \alpha \boldsymbol{exp}(\beta\theta^* e)(\beta\theta^* e + 1) - \frac{1}{e} \alpha \boldsymbol{exp}(\beta\theta^* e)(\beta\theta^* e + 1)$$
$$+ \frac{\delta}{\theta} \boldsymbol{exp}\left(-\delta \frac{e}{\theta}\right) \frac{e\alpha \boldsymbol{exp}(\beta\theta^* e)(\beta\theta^* e + 1)}{\boldsymbol{exp}\left(-\delta \frac{e}{\theta}\right) - \frac{e}{\theta}\delta \boldsymbol{exp}\left(-\delta \frac{e}{\theta}\right) - \theta^* \alpha \boldsymbol{exp}(\beta\theta^* e)(\beta\theta^* e + 1}$$

Similarly to (46), here we also substituted $D\left(\frac{e}{\theta}\right), D'\left(\frac{e}{\theta}\right), D^{*\prime}(\theta^* e)$ from (40). Introduce the functions

$$f(e, \theta^*) = \alpha \boldsymbol{exp}(\beta\theta^* e)(\beta\theta^* e + 1), \tag{51}$$
$$g(e) = \boldsymbol{exp}\ (-\delta \frac{e}{\theta})$$

with their derivatives

$$\frac{\partial}{\partial \theta} f(e, \theta^*) = \alpha\beta \boldsymbol{exp}(\beta\theta^* e)(\theta^* e_{\theta^*} + e)(2 + \beta\theta^* e), \tag{52}$$
$$\frac{\partial}{\partial \theta^*} g(e) = -\delta \frac{e}{\theta} e_{\theta^*} \boldsymbol{exp}\left(-\delta \frac{e}{\theta}\right).$$

Derivative of the exchange rate function (43) with respect to θ^* is

$$\frac{\partial e}{\partial \theta^*} = \frac{\beta\theta^2\theta^* \ln(\alpha\theta^*) - \theta(\theta\theta^*\beta + \delta)}{\theta^*(\theta\theta^*\beta + \delta)^2}.$$

Putting these values in (50) yields

$$\frac{\partial^2}{\partial\theta^{*2}}G^*(e,\theta,\theta^*) = \frac{\partial}{\partial\theta^*}\left[\frac{1}{\theta^*}f(e,\theta^*) - \left(\frac{1}{e}f(e,\theta^*) + \frac{\delta}{\theta}g(e)\right)\frac{ef(e,\theta^*)}{g(e) - \frac{e}{\theta}\delta g(e) - \theta^* f(e,\theta^*)}\right]$$

$$= \frac{f(e,\theta^*)}{\theta^*} - \frac{f(e,\theta^*)}{\theta^{*2}}$$

$$-\left(\frac{1}{e}f(e,\theta^*) + \frac{\delta}{\theta}g(e)\right)\frac{(g(e) - \frac{e}{\theta}\delta g(e) - \theta^* f(e,\theta^*))(e\frac{\partial}{\partial\theta^*}f(e,\theta^*) + e_{\theta^*}f(e,\theta^*) - ef(e,\theta^*)(\frac{\partial}{\partial\theta^*}g(e) - \frac{e}{\theta}\delta\frac{\partial}{\partial\theta^*}g(e)}{\left(g(e) - \frac{e}{\theta}\delta g(e) - \theta^* f\left(e,\theta^*\right)\right)^2}$$

$$+\frac{ef\left(e,\theta^*\right)}{g(e) - \frac{e}{\theta}\delta g(e) - \theta^* f\left(e,\theta^*\right)}\left(\frac{1}{e}\frac{\partial}{\partial\theta^*}f\left(e,\theta^*\right) - \frac{e_{\theta^*}}{e^2}f\left(e,\theta^*\right) + \frac{\partial}{\theta}\frac{\partial}{\partial\theta^*}g(e)\right)$$

from which we obtain

$$G^{*''}(0.81, 0.54, 0.73) = -0.02 < 0$$

by putting the equilibrium values.

So we can conclude that since for our example $G^{*''}(0.81, 0.54, 0.73) < 0$ and $G''(0.81, 0.54, 0.73) < 0$, $\left(e\left(\hat{\theta}, \hat{\theta}^*\right), \hat{\theta}, \hat{\theta}^*\right) = (0.81, 0.54, 0.73)$ represents the Nash equilibrium. Imposing these tariffs on imported commodities ensures maximum gains from trade for both nations.

Appendix D

Solution to (2.4): The goal is to express h, f, h^*, f^* in terms of t and t^*.

$$\frac{\partial \pi}{\partial h} = a - f^* - 2h = 0 => \hat{h} = \frac{a - f^*}{2}, \tag{53}$$

$$\frac{\partial \pi}{\partial f} = a - h^* - 2f - t^* = 0 => \hat{f} = \frac{a - h^* - t^*}{2}, \tag{54}$$

$$\frac{\partial \pi}{\partial h^*} = a - f - 2h^* = 0 => \hat{h}^* = \frac{a - f}{2}, \tag{55}$$

$$\frac{\partial \pi}{\partial f^*} = a - h - 2f^* - t = 0 => \hat{f}^* = \frac{a - h - t}{2}, \tag{56}$$

where $a > f^*, a > h^* + t^*, a > f, a > h + t$. Putting (56) in (53) gives

$$\hat{h} = \frac{a + t}{3}, \tag{57}$$

putting (55) into (54) yields

$$\hat{f} = \frac{a - 2t^*}{3}, \tag{58}$$

and putting (58) into (55) gives

$$\hat{h}^* = \frac{a + t^*}{3}, \tag{59}$$

while putting (57) into (56) gives

$$\hat{f}^* = \frac{a - 2t}{3}. \tag{60}$$

Solution to (2.6):

$$\begin{aligned}
\frac{\partial}{\partial t}\,\widehat{W}(t, t^*) &= \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{a+t}{3} + \frac{a-2t}{3}\right)^2 + \left(a - \frac{a+t}{3} - \frac{a-2t}{3}\right)\frac{a+t}{3}\right. \\
&\quad \left. + \left(a - \frac{a+t^*}{3} - \frac{a-2t}{3}\right)\frac{a-2t^*}{3} - t^*\frac{a-2t^*}{3} + t\frac{a-2t}{3}\right] \\
&= \frac{\partial}{\partial t}\left[\frac{1}{2}\frac{2a-t^2}{3} + \left(\frac{a+t}{3}\right)^2\right] \\
&= -\frac{1}{9}(2a-t) + \frac{2}{9}(a+t) + \frac{1}{3}a - \frac{4}{3}t = 0.
\end{aligned} \tag{61}$$

Solving for t gives $t = \frac{a}{3}$. Similarly

$$\frac{\partial}{\partial t^*}\widehat{W}^*(t,t^*) = \frac{\partial}{\partial t^*}\left[\frac{1}{2}\left(\frac{a+t^*}{3}+\frac{a-2t}{3}\right)^2 + \left(a-\frac{a+t^*}{3}-\frac{a-2t^*}{3}\right)\frac{a+t^*}{3}\right.$$
$$\left. + \left(a-\frac{a+t}{3}-\frac{a-2t}{3}\right)\frac{a-2t}{3} - t\frac{a-2t}{3} + t^*\frac{a-2t^*}{3}\right]$$
$$= \frac{\partial}{\partial t^*}\left[\frac{1}{2}\left(\frac{2a-t^*}{3}\right)^2 + \left(\frac{a+t^*}{3}\right)^2 + \frac{1}{3}\left(t^*a-2t^{*2}\right)^2\right]$$
$$= -\frac{1}{9}(2a-t^*) + \frac{2}{9}(a+t^*) + \frac{a}{3} - \frac{4}{3t^*} = 0$$

Solving for t^* gives $t^* = a/3$. Solution to (2.9): $\frac{\partial \pi}{\partial h}$ and $\frac{\partial \pi}{\partial h}$ are the same as (53) and (54), while h and f correspond to (57) and (58) respectively. Differentiating π^* from (2.7) with respect to h^* and f^* gives

$$\frac{\partial \pi^*}{\partial h^*} = e(a-2ah^*-f) = 0 => \hat{h}^* = \frac{a-f}{2}, \tag{62}$$

putting (58) into (62) gives

$$h^* = \frac{a+t^*}{3}. \tag{63}$$

Since $\pi^*(t,t^*,h,f,h^*,f^*) = \frac{pf^*}{p^*f}[(a-h^*-f)h^* + (a-h-f^*)f^* - tf^*]$, we have

$$\frac{\partial \pi^*}{\partial f^*} = -\frac{p}{p^*f}\left[3f^{*2} + 2(t+h-a)f^* - (a-h^*-f)h^*\right] = 0, \tag{64}$$

solving (64) results in

$$f^* = \frac{-t-h+a+\sqrt{(t+h-a)^2+3(a-h^*-f)h^*}}{3} \tag{65}$$

Appendix E

Table E1 illustrates a historical data for Vanguard Total Stock Market ETF (VTI) and iShares 7-10 Year Treasury Bond ETF (EIF) with the daily returns computed. (Source: www.nasdaq.com) Based on which we optimize the portfolio in three dimensions. Spot prices are given on a daily bases for four months. $T = 30$.

Date	VTI	IEF	R_1	R_2
4/18/2019	148.27	105.32	0.001	0.002
4/17/2019	148.05	105.07	-0.004	0.000
4/16/2019	148.60	105.05	0.000	-0.003
4/15/2019	148.56	105.41	-0.001	0.001
4/12/2019	148.68	105.27	0.007	-0.005
4/11/2019	147.69	105.76	0.000	-0.002
4/10/2019	147.70	106.00	0.005	0.002
4/9/2019	147.02	105.75	-0.006	0.002
4/8/2019	147.91	105.56	0.001	-0.002
4/5/2019	147.78	105.73	0.005	0.000
4/4/2019	147.04	105.68	0.002	0.001
4/3/2019	146.73	105.57	0.002	-0.003
4/2/2019	146.38	105.90	0.000	0.002
4/1/2019	146.39	105.72	0.012	-0.009
3/29/2019	144.71	106.67	0.007	-0.002
3/28/2019	143.76	106.84	0.004	0.000
3/27/2019	143.13	106.83	-0.005	0.003
3/26/2019	143.79	106.52	0.008	-0.001
3/25/2019	142.69	106.61	-0.006	0.003
3/22/2019	143.56	106.34	-0.021	0.007
3/21/2019	146.62	105.56	0.012	0.000

Table E1: Historical Data for VTI and EIF, Daily Returns

Computations of $\bar{\mu}$ and $\bar{\sigma}_i$ for $j = 1,2$ from (3.8) and (3.9) for various combinations of weights is illustrated in Table E2. The last columns indicate $\tilde{\sigma}$ from (3.12) and $P(\tau_m \leq T)$ from (3.17).

q_1	q_2	σ_1	σ_2	$\tilde{\mu}$	$\tilde{\sigma}$	$P(\tau_m \leq T)$
0.1	0.9	0.0006	0.0028	0.0004	0.0028	0.6410
0.2	0.8	0.0013	0.0024	0.0005	0.0028	0.6418
0.3	0.7	0.0020	0.0021	0.0006	0.0029	0.6397
0.4	0.6	0.0026	0.0018	0.0007	0.0032	0.6210
0.5	0.5	0.0033	0.0015	0.0008	0.0036	0.5953
0.6	0.4	0.0040	0.0012	0.0009	0.0041	0.5734
0.7	0.3	0.0046	0.0009	0.0010	0.0047	0.5579
0.8	0.2	0.0052	0.0006	0.0011	0.0053	0.5478
0.9	0.1	0.0059	0.0003	0.0012	0.0060	0.5414
1.0	0.0	0.0066	-8.9E-06	0.0013	0.0066	0.5376

Table E2: Portfolios of Two Assets

The quantities from (3.25) are given in Table E3 below

q_1	q_2	m	$E[R_p]$	$E[\tau_m \wedge T]$
0.1	0.9	-0.0038	0.0117	14
0.2	0.8	-0.0031	0.0148	13
0.3	0.7	-0.0030	0.0179	13
0.4	0.6	-0.0033	0.0209	14
0.5	0.5	-0.0040	0.0239	14
0.6	0.4	-0.0049	0.0269	15
0.7	0.3	-0.0060	0.0299	16
0.8	0.2	-0.0071	0.0329	16
0.9	0.1	-0.0083	0.0359	16
1.0	0.0	-0.0095	0.0388	17

Table E3: Three Dimensions for Two Asset Portfolios

Table E4 illustrates the annual spot prices from 2009 to 2019 (Source: www.nasdaq.com). The annual spot prices of four common stocks are given with their rates of returns – Travelzoo (TZOO), AXT Inc. (AXTI), Universal Forest Products (UFPI), Advanced Micro Devices (AMD). The investment horizon is taken to be 10 years. Unlike the previous example, here we take the annual spot prices. This example intends to illustrate $E[\tau_m \wedge T]$ within the investment horizon [0,T].

Date	TZOO	AXTI	UFPI	AMD	R_1	R_2	R_3	R_4
16:00	18.26	5.55	36.65	27.88	0.8576	0.2759	0.4118	0.5103
12/31/2018	9.83	4.35	25.96	18.46	0.5240	-0.500	-0.310	0.7957
12/29/2017	6.45	8.7	37.62	10.28	-0.314	0.813	0.1045	-0.094
12/30/2016	9.4	4.8	34.06	11.34	0.123	0.9355	0.4945	2.9512
12/31/2015	8.37	2.48	22.79	2.87	-0.337	-0.114	0.2852	0.0749
12/31/2014	12.62	2.8	17.33	2.67	-0.408	0.0728	0.0203	-0.310
12/31/2013	21.32	2.61	17.38	3.87	0.1227	-0.071	0.3707	0.6125
12/31/2012	18.99	2.81	12.68	2.4	-0.227	-0.326	0.2323	-0.556
12/30/2011	24.58	4.17	10.29	5.4	-0.406	-0.601	-0.206	-0.340
12/31/2010	41.375	10.44	12.97	8.18	2.367	2.2123	0.0568	-0.155
12/31/2009	12.29	3.25	12.27	9.68				

Table E4: Historical Data for TZOO, AXTI, UFPI, AMD, annual returns

Computations of $\bar{\mu}$ and $\bar{\sigma}_i$ for $j = 1,2,3,4$ from (3.8) and (3.9) for various combinations of weights are shown in E5. The last column contains computations of $\tilde{\sigma}$ from (3.12).

q_1	q_2	q_3	q_4	σ_1	σ_2	σ_3	σ_4	$\tilde{\mu}$	$\tilde{\sigma}$
0.0	0.0	0.6	0.4	0.0200	0.0905	0.0807	0.4374	0.0296	0.4544
0.0	0.0	0.8	0.2	0.0095	0.0722	0.0717	0.2726	0.0958	0.2911
0.0	0.1	0.6	0.3	0.0626	0.1411	0.0753	0.3588	0.0159	0.3978
0.0	0.1	0.7	0.2	0.0573	0.1320	0.0708	0.2764	0.0551	0.3195
0.0	0.1	0.8	0.1	0.0520	0.1228	0.0663	0.1940	0.084	0.2446
0.0	0.1	0.9	0.0	0.0467	0.1137	0.0618	0.1116	0.1026	0.1771
0.0	0.2	0.6	0.2	0.1051	0.1918	0.0670	0.2801	0.0021	0.3622
0.0	0.2	0.7	0.1	0.0998	0.1826	0.0654	0.1977	0.0422	0.2944
0.0	0.2	0.8	0.0	0.0945	0.1735	0.0609	0.1153	0.0720	0.2367
0.1	0.0	0.6	0.3	0.0819	0.1236	0.0699	0.3494	0.0201	0.3860
0.1	0.0	0.7	0.2	0.0767	0.1145	0.0653	0.2670	0.0580	0.3075
0.1	0.0	0.8	0.1	0.0714	0.1053	0.0608	0.1846	0.0860	0.2323
0.1	0.1	0.6	0.2	0.1244	0.1743	0.0645	0.2708	0.0064	0.3512
0.1	0.1	0.7	0.1	0.1192	0.1651	0.0600	0.1884	0.045	0.2838
0.1	0.1	0.8	0.0	0.1139	0.1560	0.0554	0.1060	0.0738	0.2271
0.1	0.2	0.7	0.0	0.1617	0.2160	0.0546	0.1097	0.0324	0.2962
0.2	0.0	0.7	0.1	0.1385	0.1476	0.0545	0.1790	0.0481	0.2757
0.2	0.0	0.8	0.0	0.1333	0.1385	0.0500	0.0966	0.0753	0.2208
0.3	0.0	0.06	0.1	0.2057	0.1899	0.0481	0.1734	0.0011	0.3328
0.3	0.0	0.7	0.0	0.2004	0.1807	0.0436	0.0910	0.0382	0.2881

Table E5: Portfolios of Multi Assets

Ultimately the three dimensions are generated in Table E6. This table corresponds to the efficient surface similar to Figure 3.4. Since there are too many combinations of weights constructing different portfolios, Table E6 illustrates all combinations of weights of four assets. Detailed results are provided in the accompanying spreadsheet.

q_1	q_2	q_3	q_4	m	$E[R_p]$	$E[\tau_m \wedge T]$
0.0	0.0	0.6	0.4	-0.5503	2.2320	0.2956
0.0	0.0	0.8	0.2	-0.3636	0.4641	0.9584
0.0	0.1	0.6	0.3	-0.4528	2.3620	0.1585
0.0	0.1	0.7	0.2	-0.3662	1.4853	0.5505
0.0	0.1	0.8	0.1	-0.3002	0.5336	0.8398
0.0	0.1	0.9	0.0	-0.2674	-0.2307	1.0263
0.0	0.2	0.6	0.2	-0.3961	2.5076	0.0206
0.0	0.2	0.7	0.1	-0.3346	1.7371	0.4218
0.0	0.2	0.8	0.0	-0.3044	0.8344	0.7203
0.1	0.0	0.6	0.3	-0.4321	2.2494	0.2011
0.1	0.0	0.7	0.2	-0.3426	1.3555	0.5799
0.1	0.0	0.8	0.1	-0.2735	0.4101	0.8560
0.1	0.1	0.6	0.2	-0.3688	2.3515	0.0643
0.1	0.1	0.7	0.1	-0.3043	1.5633	0.4524
0.1	0.1	0.8	0.0	-0.2726	0.6937	0.7377
0.1	0.2	0.7	0.0	-0.3345	1.9615	0.3241
0.2	0.0	0.6	0.2	-0.3578	2.2738	0.1064
0.2	0.0	0.7	0.1	-0.2928	1.4642	0.4813
0.2	0.0	0.8	0.2	-0.2617	0.6079	0.7534
0.3	0.0	0.6	0.1	-0.3488	2.4285	0.0115
0.3	0.0	0.7	0.0	-0.3240	1.8054	0.3824

Table E6: Three Dimensions for Multi Asset Portfolios

Table E7 shows the monthly spot prices of three common stocks – Apple Inc (AAPL), JPMorgan Chase Co. (JPM) and Walmart Inc. (WMT) from April the 28th, 2017 to 1st of April, 2019. T = 30 months. (Source www.nasdaq.com)

Date	AAPL	JPM	WMT	R_1	R_2	R_3
16:00	203.86	113.46	103.18	0.0732	0.1208	0.0579
3/29/2019	189.95	101.23	97.53	0.0970	-0.0300	-0.015
2/28/2019	173.15	104.36	98.99	0.0403	0.0083	0.0330
1/31/2019	166.44	103.5	95.83	0.0552	0.0602	0.0288
12/31/2018	157.74	97.62	93.15	-0.1167	-0.1220	-0.0461
11/30/2018	178.58	111.19	97.65	-0.1840	0.0199	-0.0262
10/31/2018	218.86	109.02	100.28	-0.0304	-0.0339	0.0678
9/28/2018	225.74	112.84	93.91	-0.0083	-0.0152	-0.0203
8/31/2018	227.63	114.58	95.86	0.1962	-0.0032	0.0743
7/31/2018	190.29	114.95	89.23	0.0280	0.1032	0.0418
6/29/2018	185.11	104.2	85.65	-0.0094	-0.0263	0.0377
5/31/2018	186.87	107.01	82.54	0.1308	-0.0163	-0.067
4/30/2018	165.26	108.78	88.46	-0.0150	-0.0108	-0.0057
3/29/2018	167.78	109.97	88.97	-0.0581	-0.0479	-0.0116
2/28/2018	178.12	115.5	90.01	0.06388	-0.0015	-0.1556
1/31/2018	167.43	115.67	106.6	-0.0106	0.0816	0.0795
12/29/2017	169.23	106.94	98.75	-0.0153	0.0232	0.0156
11/30/2017	171.85	104.52	97.23	0.0166	0.0389	0.1136
10/31/2017	169.04	100.61	87.31	0.0968	0.0534	0.1174
9/29/2017	154.12	95.51	78.14	-0.0602	0.0508	0.0009
8/31/2017	164	90.89	78.07	0.1027	-0.0100	-0.0240
7/31/2017	148.73	91.8	79.99	0.0327	0.0044	0.0570
6/30/2017	144.02	91.4	75.68	-0.0572	0.1126	-0.0372
5/31/2017	152.76	82.15	78.6	0.0634	-0.0558	0.0455

Table E7: Historical Data for AAPL, JPM and WMT, Daily Returns

Computations of $\bar{\mu}$ and $\bar{\sigma}_j$ for $j = 1,2,3$ from (3.8) and (3.9} for various combinations of weights are shown in Table E8. We also compute $\tilde{\sigma}$ from (3.12) illustrated in the last column. First three columns denote the weights, columns 4-6 denote the $\bar{\sigma}_j$ and the last columns denote $\tilde{\mu}$ and $\tilde{\sigma}$.

q_1	q_2	q_3	σ_1	σ_2	σ_3	$\tilde{\mu}$	$\tilde{\sigma}$
0.00	0.00	1.00	0.0009	0.00010	0.0036	0.0151	0.0741
0.00	0.10	0.90	0.0008	0.0013	0.0033	0.0149	0.0735
0.00	0.20	0.08	0.0008	0.0014	0.0031	0.0146	0.0729
0.00	0.30	0.70	0.0007	0.0017	0.0028	0.0144	0.0722
0.00	0.40	0.60	0.0007	0.0019	0.0026	0.0141	0.0716
0.00	0.50	0.50	0.0007	0.0019	0.0026	0.0141	0.0716
0.00	0.60	0.40	0.0006	0.0023	0.0020	0.0137	0.0704
0.00	0.70	0.30	0.0006	0.0025	0.0018	0.0134	0.0697
0.00	0.80	0.20	0.0005	0.0027	0.0015	0.0131	0.0691
0.00	0.90	0.10	0.0005	0.0030	0.0013	0.0129	0.0684
0.00	1.00	0.00	0.0004	0.0032	0.0010	0.0127	0.0677
0.10	0.00	0.90	0.0015	0.0010	0.0033	0.0154	0.0757
0.10	0.10	0.80	0.0014	0.0012	0.0031	0.0151	0.0751
0.10	0.20	0.70	0.0014	0.0014	0.0028	0.0149	0.0745
0.10	0.30	0.60	0.0013	0.0016	0.0025	0.0147	0.0739
0.10	0.40	0.50	0.0013	0.0018	0.0023	0.0144	0.0732
0.10	0.50	0.40	0.0012	0.0020	0.0020	0.0142	0.0726
0.10	0.60	0.30	0.0012	0.0022	0.0018	0.0139	0.0720
0.10	0.70	0.20	0.0011	0.0025	0.0015	0.0137	0.00714
0.10	0.80	0.10	0.0011	0.0027	0.0013	0.0135	0.0707
0.10	0.90	0.00	0.0010	0.0029	0.0010	0.0132	0.0701

Table E8: Portfolios of Three Assets

Table E9 illustrates the ultimate result of combinations m, $E[R_T], E[\tau_m \wedge T]$ from (3.25)

q_1	q_2	q_3	m	$E[R_p]$	$E[\tau_m \wedge T]$
0.00	0.00	1.00	-0.0854	0.4527	14.2704
0.00	0.10	0.90	-0.0789	0.4455	13.5044
0.00	0.20	0.80	-0.0734	0.4383	12.8292
0.00	0.30	0.70	-0.0690	0.4311	12.2883
0.00	0.40	0.60	-0.0661	0.4239	11.9260
0.00	0.50	0.50	-0.0648	0.4167	11.7800
0.00	0.60	0.40	-0.0652	0.4095	11.8718
0.00	0.70	0.30	-0.0672	0.4023	12.2005
0.00	0.80	0.20	-0.0708	0.3951	12.7428
0.00	0.90	0.10	-0.0758	0.3878	13.4597
0.00	1.00	0.00	-0.0819	0.3806	14.3059
0.10	0.00	0.90	-0.0786	0.4613	13.2018
0.10	0.10	0.80	-0.0722	0.4541	12.4184
0.10	0.20	0.70	-0.0670	0.4470	11.7459
0.10	0.30	0.60	-0.0630	0.4398	11.2339
0.10	0.40	0.50	-0.0606	0.4325	10.9307
0.10	0.50	0.40	-0.0600	0.4253	10.8727
0.10	0.60	0.30	-0.0611	0.4181	11.0741
0.10	0.70	0.20	-0.0640	0.4109	11.5217
0.10	0.80	0.10	-0.0684	0.4037	12.1791
0.10	0.90	0.00	-0.0742	0.3965	12.9981

Table E9: Three Dimensions for Three Asset Portfolios

q_1	q_2	q_3	m	$E[R_p]$	$E[\tau_m \wedge T]$
0.20	0.00	0.80	-0.0739	0.4700	12.4007
0.20	0.10	0.70	-0.0678	0.4628	11.6412
0.20	0.20	0.60	-0.0630	0.4556	11.0151
0.20	0.30	0.50	-0.0597	0.4484	10.5739
0.20	0.40	0.40	-0.0580	0.4412	10.3642
0.20	0.50	0.30	-0.0581	0.4340	10.4152
0.20	0.60	0.20	-0.0601	0.4268	10.7289
0.20	0.70	0.10	-0.0638	0.4195	11.2796
0.20	0.80	0.00	-0.0689	0.4123	12.0223
0.30	0.00	0.70	-0.0717	0.4786	11.9286
0.30	0.10	0.60	-0.0662	0.4714	11.2386
0.30	0.20	0.50	-0.0620	0.4642	10.7021
0.30	0.30	0.40	-0.0595	0.4570	10.3672
0.30	0.40	0.30	-0.0586	0.4498	10.2716
0.30	0.50	0.20	-0.0596	0.4426	10.4322
0.30	0.60	0.10	-0.0622	0.4354	10.8389
0.30	0.70	0.00	-0.0665	0.4282	11.4581
0.40	0.00	0.60	-0.0721	0.4872	11.8105
0.40	0.10	0.50	-0.0674	0.4800	11.2274
0.40	0.20	0.40	-0.0641	0.4728	10.8102
0.40	0.30	0.30	-0.0624	0.4656	10.5976
0.40	0.40	0.20	-0.0623	0.4584	10.6146

Table E9: Three Dimensions for Three Asset Portfolios

q_1	q_2	q_3	m	$E[R_p]$	$E[\tau_m \wedge T]$
0.40	0.50	0.10	-0.0639	0.4512	10.8655
0.40	0.60	0.00	-0.0672	0.4440	11.3321
0.50	0.00	0.50	-0.0751	0.4958	12.0248
0.50	0.10	0.40	-0.0713	0.4886	11.5678
0.50	0.20	0.30	-0.0689	0.4814	11.2775
0.50	0.30	0.20	-0.0680	0.4742	11.1812
0.50	0.40	0.10	-0.0686	0.4670	11.2914
0.50	0.50	0.00	-0.0708	0.4598	11.6033
0.60	0.00	0.40	-0.0805	0.5044	12.5119
0.60	0.10	0.30	-0.0776	0.4972	12.1798
0.60	0.20	0.20	-0.0759	0.4900	12.0057
0.60	0.30	0.10	-0.0759	0.4828	12.0052
0.60	0.40	0.00	-0.0770	0.4756	12.1823
0.70	0.10	0.20	-0.0857	0.5058	12.9739
0.70	0.20	0.10	-0.0848	0.4986	12.8948
0.70	0.30	0.00	-0.0852	0.4914	12.9658
0.80	0.00	0.20	-0.0967	0.5215	14.0020
0.80	0.10	0.10	-0.0953	0.5143	13.8710
0.80	0.20	0.00	-0.0950	0.5071	13.8643
0.90	0.00	0.10	-0.1068	0.5301	14.8721
0.90	0.10	0.00	-0.1060	0.5229	14.8114
1.00	0.00	0.00	-0.1179	0.5386	15.7629

Table E9: Three Dimensions for Three Asset Portfolios

As a comparison, Table E10 illustrates the portfolio with the same assets optimized under the Mean-Variance framework. This table corresponds to the efficient frontier given in the example.

q_1	q_2	q_3	$E[R]$	σ^2	σ
0.00	0.00	1.00	0.0151	0.0036	0.0598
0.00	0.10	0.90	0.0149	0.0031	0.0558
0.00	0.20	0.80	0.0146	0.0027	0.0524
0.00	0.30	0.70	0.0144	0.0025	0.0496
0.00	0.40	0.60	0.0141	0.0023	0.0478
0.00	0.50	0.50	0.0139	0.0022	0.0468
0.00	0.60	0.40	0.0137	0.0022	0.0469
0.00	0.70	0.30	0.0134	0.0023	0.0480
0.00	0.80	0.20	0.0132	0.0025	0.0500
0.00	0.90	0.10	0.0129	0.0028	0.0528
0.00	1.00	0.00	0.0127	0.0032	0.0563
0.10	0.00	0.90	0.0154	0.0031	0.0559
0.10	0.10	0.80	0.0151	0.0027	0.0520
0.10	0.20	0.70	0.0149	0.0024	0.0487
0.10	0.30	0.60	0.0147	0.0021	0.0462
0.10	0.40	0.50	0.0144	0.0020	0.0447
0.10	0.50	0.40	0.0142	0.0019	0.0441
0.10	0.60	0.30	0.0139	0.0020	0.0447
0.10	0.70	0.20	0.0137	0.0021	0.0462
0.10	0.80	0.10	0.0135	0.0024	0.0487
0.10	0.90	0.00	0.0132	0.0027	0.0520
0.20	0.00	0.80	0.0157	0.0028	0.0533

Table E10: Mean-Variance Pairs

Table E11 illustrates the daily spot prices from March the 25th, 2019 to 23rd of April, 2019 (Source: www.nasdaq.com) for three common stocks – YUMA Energy Inc (YUMA), Immunic Inc. (IMUX), Savara Inc (SVRA) with their rates of returns. T=30. Unlike the previous example, here we take the daily spot prices.

Date	YUMA	IMUX	SVRA	R_1	R_2	R_3
4/23/2019	0.355	17.35	10.7	0.6136	0.2133	0.1088
4/22/2019	0.22	14.3	9.65	0.2557	-0.1222	0.0354
4/18/2019	0.1752	16.29	9.32	-0.1314	0.1352	0.0344
4/17/2019	0.2017	14.35	9.01	0.3447	-0.1196	-0.008
4/16/2019	0.15	16.3	9.08	-0.1892	-0.3242	-0.0011
4/15/2019	0.185	24.12	9.09	-0.0537	-0.2735	-0.0401
4/12/2019	0.1955	33.2	9.47	0.0736	0.0778	-0.0094
4/11/2019	0.1821	30.804	9.56	-0.2251	0.3144	0.0063
4/10/2019	0.235	23.436	9.5	0.0000	-0.1849	0.0556
4/9/2019	0.235	28.752	9	0.3824	-0.0133	-0.0033
4/8/2019	0.17	29.14	9.03	0.4167	0.4173	0.0261
4/5/2019	0.12	20.56	8.8	0.0000	1.4582	-0.0112
4/4/2019	0.12	8.364	8.9	-0.0400	0.0034	0.0023
4/3/2019	0.125	8.336	8.88	0.0000	0.0589	0.0559
4/2/2019	0.125	7.872	8.41	-0.1554	-0.0386	0.0646
4/1/2019	0.148	8.188	7.9	0.2437	0.0386	0.0719
3/29/2019	0.119	7.884	7.37	0.0540	0.0144	-0.016
3/28/2019	0.1129	7.772	7.49	0.0000	-0.0122	-0.0730
3/27/2019	0.1129	7.868	8.08	-0.0053	-0.0475	0.0215
3/26/2019	0.1135	8.26	7.91	-0.0291	-0.0282	-0.0247
3/25/2019	0.1169	8.5	8.11			

Table E11: Historical Data for YUMA, IMUX and SVRA, Daily Returns

We compute $\bar{\mu}$ and $\bar{\sigma}_j$ for $j = 1,2,3$ from (3.8) and (3.9) for various combinations of weighs. We also compute $\tilde{\sigma}$ from (3.12). This information is summarized in Table E12.

q_1	q_2	q_3	σ_1	σ_2	σ_3	$\tilde{\mu}$	$\tilde{\sigma}$
0.40	0.60	0.00	0.0227	0.0803	0.0013	0.0727	0.0835
0.50	0.00	0.50	0.0256	0.0029	0.0024	0.0458	0.0259
0.50	0.10	0.40	0.0258	0.0159	0.0023	0.0517	0.0304
0.50	0.20	0.30	0.0261	0.0289	0.0021	0.0574	0.0390
0.50	0.30	0.20	0.0264	0.0419	0.0019	0.0629	0.0496
0.50	0.40	0.10	0.0266	0.0549	0.0018	0.0682	0.0610
0.50	0.50	0.00	0.0269	0.0679	0.0016	0.0734	0.0730
0.60	0.00	0.40	0.0301	0.0035	0.0026	0.0519	0.0304
0.60	0.10	0.30	0.0303	0.0165	0.0024	0.0577	0.0346
0.60	0.20	0.20	0.0306	0.0295	0.0022	0.0633	0.0426
0.60	0.30	0.10	0.0309	0.0425	0.0021	0.0688	0.0525
0.60	0.40	0.00	0.0311	0.0555	0.0019	0.0741	0.0636
0.70	0.00	0.30	0.0346	0.0041	0.0027	0.0580	0.0349
0.70	0.10	0.20	0.0348	0.0171	0.0025	0.0637	0.0389
0.70	0.20	0.10	0.0351	0.0301	0.0024	0.0693	0.0463
0.70	0.30	0.00	0.0354	0.0430	0.0022	0.0747	0.0557
0.80	0.00	0.20	0.0390	0.0046	0.0029	0.0641	0.0394
0.80	0.10	0.10	0.0393	0.0176	0.0027	0.0697	0.0432
0.80	0.20	0.00	0.0396	0.0306	0.0025	0.0752	0.0501
0.90	0.00	0.10	0.0435	0.0052	0.0030	0.0701	0.0440
0.90	0.10	0.00	0.0438	0.0182	0.0028	0.0757	0.0475
1.00	0.00	0.00	0.0480	0.0058	0.0031	0.0761	0.0485

Table E12: Portfolios of Three Assets, Large FPTs

Ultimately the three dimensions are generated in Table E13 according to (3.25). All $E[\tau_m \wedge T]$ values coincide with T.

YUMA	IMUX	SVRA	m	$E[R_p]$	$E[\tau_m \wedge T]$
0.00	0.00	1.00	-0.0542	0.4438	30
0.00	0.10	0.90	-0.0654	0.6299	30
0.00	0.20	0.80	-0.1043	0.8109	30
0.00	0.30	0.70	-0.1516	0.9868	30
0.00	0.40	0.60	-0.2018	1.1575	30
0.00	0.50	0.50	-0.2532	1.3231	30
0.00	0.60	0.40	-0.3053	1.4835	30
0.00	0.70	0.30	-0.3577	1.6388	30
0.00	0.80	0.20	-0.4104	1.7890	30
0.00	0.90	0.10	-0.4632	1.9340	30
0.00	1.00	0.00	-0.5162	2.0739	30
0.10	0.00	0.90	-0.0610	0.6315	30
0.10	0.10	0.80	-0.0711	0.8156	30
0.10	0.20	0.70	-0.1071	0.9945	30
0.10	0.30	0.60	-0.1526	1.1684	30
0.10	0.40	0.50	-0.2017	1.3370	30
0.10	0.50	0.40	-0.2524	1.5006	30
0.10	0.60	0.30	-0.3040	1.6590	30
0.10	0.70	0.20	-0.3560	1.8122	30
0.10	0.80	0.10	-0.4084	1.9604	30
0.10	0.90	0.00	-0.4610	2.1033	30
0.20	0.00	0.80	-0.0780	0.8184	30

Table E13: Three Dimensions for Three Asset Portfolios, Large FPTs

YUMA	IMUX	SVRA	m	$E[R_p]$	$E[\tau_m \wedge T]$
0.20	0.10	0.70	-0.0860	1.0004	30
0.20	0.20	0.60	-0.1171	1.1773	30
0.20	0.30	0.50	-0.1591	1.3491	30
0.20	0.40	0.40	-0.2060	1.5157	30
0.20	0.50	0.30	-0.2552	1.6772	30
0.20	0.60	0.20	-0.3057	1.8336	30
0.20	0.70	0.10	-0.3570	1.9848	30
0.20	0.80	0.00	-0.4087	2.1309	30
0.30	0.00	0.70	-0.0999	1.0044	30
0.30	0.10	0.60	-0.1060	1.1844	30
0.30	0.20	0.50	-0.1324	1.3593	30
0.30	0.30	0.40	-0.1703	1.5290	30
0.30	0.40	0.30	-0.2142	1.6936	30
0.30	0.50	0.20	-0.2614	1.8531	30
0.30	0.60	0.10	-0.3103	2.0074	30
0.30	0.70	0.00	-0.3605	2.1566	30
0.40	0.00	0.60	-0.1243	1.1897	30
0.40	0.10	0.50	-0.1289	1.3676	30
0.40	0.20	0.40	-0.1512	1.5405	30
0.40	0.30	0.30	-0.1851	1.7082	30
0.40	0.40	0.20	-0.2259	1.8707	30

Table E13: Three Dimensions for Three Asset Portfolios, Large FPTs

YUMA	IMUX	SVRA	m	$E[R_p]$	$E[\tau_m \wedge T]$
0.40	0.50	0.10	-0.2706	2.0281	30
0.40	0.60	0.00	-0.3177	2.1804	30
0.50	0.00	0.50	-0.1499	1.3741	30
0.50	0.10	0.40	-0.1534	1.5500	30
0.50	0.20	0.30	-0.1725	1.7208	30
0.50	0.30	0.20	-0.2028	1.8865	30
0.50	0.40	0.10	-0.2402	2.0470	30
0.50	0.50	0.00	-0.2824	0.2024	30
0.60	0.00	0.40	-0.1762	1.5577	30
0.60	0.10	0.30	-0.1789	1.7316	30
0.60	0.20	0.20	-0.1954	1.9004	30
0.60	0.30	0.10	-0.2225	2.0640	30
0.60	0.40	0.00	-0.2570	2.2225	30
0.70	0.00	0.30	-0.2031	1.7405	30
0.70	0.10	0.20	-0.2051	1.9124	30
0.70	0.20	0.10	-0.2195	2.0791	30
0.70	0.30	0.00	-0.2438	2.2407	30
0.80	0.00	0.20	-0.2302	1.9226	30
0.80	0.10	0.10	-0.2318	2.0924	30
0.80	0.20	0.00	-0.2444	2.2570	30
0.90	0.00	0.10	-0.2576	2.1037	30
0.90	0.10	0.00	-0.2587	2.2715	30
1.00	0.00	0.00	-0.2851	2.2841	30

Table E13: Three Dimensions for Three Asset Portfolios, Large FPTs

The values of $E[\tau_m \wedge T]$ for all weights in Tables E13 are equal to the investment horizon T. So the expected bounded first passage time component in (3.25) can be dropped and efficient surface can be replaced by the efficient frontier in two dimensions.